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Optimal Decentralized Dynamic Quantizers for Discrete-Valued Input Control: A Closed Form Solution and Experimental Evaluation

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Abstract— **This paper discusses how to design decentralized dynamic quantizers for feedback control systems with discrete-valued input constraints. First, we analytically derive an optimal decentralized dynamic quantizer such that the systems subject to discrete-valued input constraints optimally approximate the behavior of those without such constraints. Next, the effectiveness of such decentralized dynamic quantizers is demonstrated by an experiment with a cart-seesaw system.**

I. Introduction

In order to control dynamical systems including discrete-valued devices such as discrete-level (on/off level) actuators, D/A converters, and communication encoders, one of the promising approaches is to adopt quantizers which appropriately convert continuous-valued signals to discrete-valued ones. Thus the design problem of quantizers has been actively discussed, and various results have been reported so far. For instance, the (coarsest) quantizers for networked control have been derived in [1]–[6], and various quantizers for control with discrete-level actuators have been proposed: the locational optimization based quantizer [7], the receding horizon quantizer [8], and the $\Delta\Sigma$ modulator [9].

The authors also have considered a quantizer design problem [10], [11]. There the following problem is considered: when a plant P and a controller K are given in the system in Fig. 1 (a), find a "dynamic" quantizer Q such that the system in Fig. 1 (a) *optimally* approximates the ideal system in Fig. 1 (b) in the sense of the input-output relation. To this problem, we have derived a closed form solution and clarified an optimal quantization structure. On the other hand, unlike the system in Fig. 1 (a), it is often necessary to have a decentralized structure in dynamic quantizers. For example, when two quantizers have to be separately embedded in systems i.e., quantization is necessary for both actuator and sensor signals as shown in Fig. 1 (c), the quantizer should be decentralized. Therefore, in practice, it is crucial to design the dynamic quantizers of decentralized structure.

Motivated by the above, this paper addresses an optimization problem of a class of decentralized dynamic quantizers for feedback control systems with discrete-valued signal constraints, i.e., for systems whose input/state signals take values only on a fixed discrete set. More concretely, we consider the decentralized version of our former problem in [10]: when a linear plant P and a controller K are given in Fig. 1 (c), find a *decentralized* dynamic quantizer Q such

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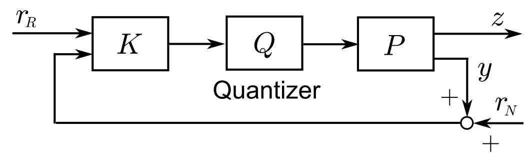
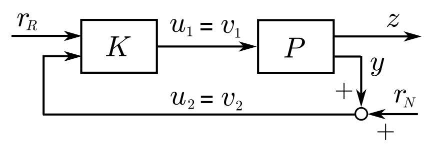
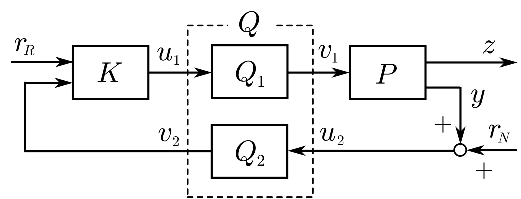


(a) Feedback system with a quantizer and a discrete-valued input plant.



(b) Feedback system with a continuous-valued input plant (ideal case).



(c) Feedback system with I/O quantizers.

Fig. 1. Three feedback systems.

that the system in Fig. 1 (c) optimally approximates the ideal system in Fig. 1 (b) in the sense of the input-output relation. Since this is the design problem of dynamic quantizers with decentralized information structure, it is more challenging than the previous problem.

To this problem, the following contributions are obtained.

First, we derive an optimal decentralized dynamic quantizer in a closed form in terms of the plant/controller parameters. Since the optimal control design subject to decentralized structure is known to be difficult (or NP hard) [12], [13], this result is not trivial at all. A key to obtain the closed form solution is to exploit the special decentralized structures of dynamic quantizers in order to divide the original problem into several tractable sub-problems.

Second, we verify the effectiveness of the proposed quantizer by an experiment with a cart-seesaw system. Note that most of the existing studies on the quantizer design have focused on theoretical aspects, and therefore few results on experimental validation can be found. In particular, to the best of our knowledge, the experimental results which achieve satisfactory control performance in case of decentralized dynamic quantizers have never reported so far. Hence, this is a valuable contribution from the practical view point.

It should be remarked that, although the existing studies

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for discrete-valued input control [7]–[11] have not focused on decentralized quantizers, some results on design of decentralized quantizers for the networked control have been derived (e.g., [5] and [6]). However, these results are different from ours. In fact, when we regard the quantizer as a mapping from the (continuous-valued) domain into the (discrete-valued) range, the existing results [5], [6] have focused on the design of the domain and/or range rather than the mapping. While we are interested in choosing the (non-static) mapping with the fixed range. Since we mainly aim at controlling the plants with discrete-valued actuators, we have no flexibility in choosing the range of quantizers for such plants.

Notation: Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ denote the real number field, the set of positive real numbers, and the set of positive integers, respectively. We use I, 0, and $\mathbf{1}$ to express the identity matrix, the zero matrix, and the vector whose all elements are one. For the matrix $M := \{M_{ij}\}$, let $\mathrm{abs}(M)$ be the matrix composed of the absolute values of the elements, i.e., $\mathrm{abs}(M) := \{|M_{ij}|\}$, and let $M^\dagger$ be the pseudo-inverse. We denote the $nm \times nm$ block diagonal matrix whose diagonal elements are $M_1, M_2, \ldots, M_m \in \mathbb{R}^{n\times n}$ by $\mathrm{diag}(M_1, M_2, \ldots, M_m)$. For the vector x, $\mathrm{sign}(x)$ expresses the vector obtained by elementwisely applying the signum function to x. Finally, for the vector x, the matrix M, and the sequence of vectors $X := \{x_1, x_2, \ldots\}$, the symbols $\|x\|$, $\|M\|$, and $\|X\|$ express their ∞-norms.

II. Problem Formulation

Consider the feedback system Σ_Q shown in Fig. 2, which is composed of the discrete-time linear system G and the quantizer Q. The system G is given by

$$G : \begin{cases} x(k+1) &= Ax(k) + B_1 r(k) + B_2 v(k), \\ z(k) &= C_1 x(k) + D_1 r(k), \\ u(k) &= C_2 x(k) + D_2 r(k) \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, $r \in \mathbb{R}^p$ and $v \in \mathbb{R}^m$ are the inputs, $z \in \mathbb{R}^l$ and $u \in \mathbb{R}^m$ are the outputs, $k \in \{0\} \cup \mathbb{N}$ is the time, and $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times p}$, $B_2 \in \mathbb{R}^{n\times m}$, $C_1 \in \mathbb{R}^{l\times n}$, $C_2 \in \mathbb{R}^{m\times n}$, $D_1 \in \mathbb{R}^{l\times p}$, $D_2 \in \mathbb{R}^{m\times p}$ are constant matrices. The signals r and z correspond to the external input and the controlled output, respectively.

On the other hand, Q is the decentralized dynamic quantizer composed of s sub-quantizers

$$Q_i : \begin{cases} \xi_i(k+1) &= \mathcal{A}_i \xi_i(k) + \mathcal{B}_{1i} u_i(k) + \mathcal{B}_{2i} v_i(k), \\ v_i(k) &= \mathrm{q}[\, \mathcal{C}_i \xi_i(k) + u_i(k)\,], \end{cases} \quad (2)$$

where $i \in \{1, 2, \ldots, s\}$, $\xi_i \in \mathbb{R}^{\mathcal{N}_i}$ is the state, $u_i \in \mathbb{R}^{m_i}$ is the input, $v_i \in \mathbb{V}^{m_i}$ is the output, $\mathbb{V}^{m_i} \subset \mathbb{R}^{m_i}$ is the discrete set given by $\mathbb{V}^{m_i} := \{0, \pm d, \pm 2d, \ldots\}^{m_i}$ for the quantization interval $d \in \mathbb{R}_+$, and $\mathcal{A}_i \in \mathbb{R}^{\mathcal{N}_i \times \mathcal{N}_i}$, $\mathcal{B}_{1i}$, $\mathcal{B}_{2i} \in \mathbb{R}^{\mathcal{N}_i \times m_i}$, $\mathcal{C}_i \in \mathbb{R}^{m_i \times \mathcal{N}_i}$ are constant matrices. The function $\mathrm{q} : \mathbb{R}^{m_i} \to \mathbb{V}^{m_i}$ is the nearest-neighbor static quantizer toward $-\infty$ [1]. For the static quantizer q, the relation

$$\mathrm{abs}(\mathrm{q}[\mu] - \mu) = \mathrm{abs}(\mu - \mathrm{q}[\mu]) \leq \frac{d}{2}\mathbf{1} \quad (\forall \mu \in \mathbb{R}^{m_i}) \quad (3)$$

holds, which will be used in this paper. The initial state is given by $\xi_i(0) = 0$ to guarantee that $v_i(k) = 0$ for $u_i(k) = 0$ ($k = 0, 1, \ldots$), i.e., Q_i is drift-free. Each sub-quantizer Q_i is the same as those considered in [10], [11] as a class of the simplest and most fundamental dynamic quantizers.

Using different fonts, we distinguish the symbols $(\mathcal{N}, \mathcal{A}, \mathcal{B}, \mathcal{C})$ used for Q from (n, A, B, C) for G. Note here that the signal r includes the reference, the disturbance, and the noise, and the signals v and u of G are respectively composed of $v_i \in \mathbb{R}^{m_i}$ ($i = 1, 2, \ldots, s$) and $u_i \in \mathbb{R}^{m_i}$ ($i = 1, 2, \ldots, s$), i.e., $v = [v_1^\top \ v_2^\top \ \cdots \ v_s^\top]^\top$ and $u = [u_1^\top \ u_2^\top \ \cdots \ u_s^\top]^\top$.

It should be noticed that various types of feedback systems, which include sub-quantizers in the form of (2), can be expressed as the system Σ_Q in Fig. 2. For instance, the I/O quantized system in Fig. 1 (c) corresponds to Σ_Q when the signals r, u, and v are defined as $r := [r_R^\top \ \ r_N^\top]^\top$, $u := [u_1^\top \ u_2^\top]^\top$, and $v := [v_1^\top \ v_2^\top]^\top$, respectively. So the following discussion holds not only for the system in Fig. 1 (c) but also for other types of quantized systems.

Before formulating the problem discussed here, some symbols are introduced. For $i = 1, 2, \ldots, s$, let $B_{2i} \in \mathbb{R}^{n\times m_i}$ denote the i-th block column of B_2, i.e., $B_2 = [B_{21} \ \ B_{22} \ \cdots \ B_{2s}]$. Moreover, we denote the smallest

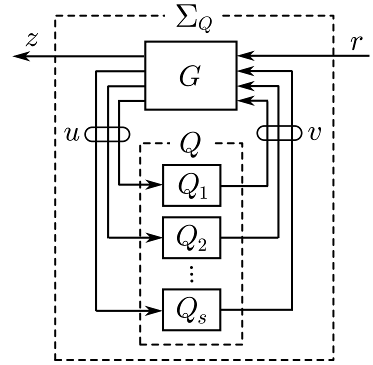


Fig. 2. General feedback system with multiple quantizers.

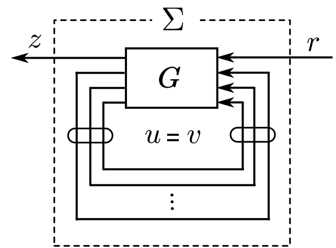


Fig. 3. Ideal system with continuous-valued v.

[1]Note that the value of $\mathrm{q}[\mu]$ is uniquely determined for every $\mu \in \mathbb{R}^{m_i}$, since $\mathrm{q}[\mu]$ is given as the smallest vector (in the sense of the sum of the all elements) of the solutions to $\min_{v_i \in \mathbb{V}^{m_i}} (v_i - \mu)^\top (v_i - \mu)$. For example, $\mathrm{q}[\mu] = [0 \ \ 0]^\top$ for $\mu := [d/2 \ \ d/2]^\top$ and $\mathrm{q}[\mu] = [0 \ \ -d]^\top$ for $\mu := [d/2 \ \ -d/2]^\top$.

integer $k \in \{0\} \cup \mathbb{N}$ satisfying $C_1(A+B_2C_2)^k B_{2i} \neq 0$ by τ_i. Then, in this paper, we make the following two assumptions for the system G.

- **(A1)** The matrix D_2 is full row rank.
- **(A2)** The matrix $C_1(A+B_2C_2)^{\tau_i} B_{2i}$ is full row rank for each $i \in \{1,2,\ldots,s\}$.

Under **(A1)** and **(A2)**, the pseudo-inverse matrices of D_2 and $F_i := C_1(A+B_2C_2)^{\tau_i} B_{2i}$ are respectively expressed as $D_2^\dagger = D_2^\top (D_2 D_2^\top)^{-1}$ and $F_i^\dagger = F_i^\top (F_i F_i^\top)^{-1}$. Note that **(A2)** is essential in this paper, since an optimal dynamic quantizer will be derived with the pseudo-inverse matrices $F_i^\dagger$ $(i=1,2,\ldots,s)$. The case that **(A1)** does not hold will be mentioned in Section III-B.

Now we define the cost function. For the system Σ_Q with the initial state $x(0)=x_0 \in \mathbb{R}^n$ and the external input sequence $R := \{r_0, r_1, \ldots\} \in \ell_\infty^p$, let $Z_Q(x_0,R)$ be the controlled output sequence for $k=1,2,\ldots$, and $z_Q(k,x_0,R)$ be the output at the k-th time. In addition, let Σ denote the ideal system in Fig. 3 (i.e., the system without quantizers), for which the output sequence and the output at k-th time are similarly defined by $Z(x_0,R)$ and $z(k,x_0,R)$, respectively. Then for given $x_0 \in \mathbb{R}^n$ and $R \in \ell_\infty^p$, the output difference is expressed as

$$\begin{aligned}\|Z_Q(x_0,R) - Z(x_0,R)\| \\ := \sup_{k\in\mathbb{N}} \|z_Q(k,x_0,R) - z(k,x_0,R)\|,\end{aligned} \tag{4}$$

and we employ the following performance index.

$$E(Q) := \sup_{(x_0,R)\in\mathbb{R}^n\times\ell_\infty^p} \|Z_Q(x_0,R) - Z(x_0,R)\|. \tag{5}$$

In this paper, the following problem is considered.

Problem 1: For the system Σ_Q, assume **(A1)** and **(A2)**. Then, find a decentralized dynamic quantizer $Q := (Q_1, Q_2, \ldots, Q_s)$ (i.e., find dimensions $\mathcal{N}_i$ and matrices $\mathcal{A}_i$, $\mathcal{B}_{1i}$, $\mathcal{B}_{2i}$, $\mathcal{C}_i$ $(i=1,2,\ldots,s)$) minimizing $E(Q)$, and determine the minimum value of $E(Q)$. ■

The performance index $E(Q)$ represents the difference between Σ and Σ_Q in terms of the input-output (r-z) relation. For a good decentralized quantizer Q in the sense of $E(Q)$, the output behavior of the system Σ is almost preserved in Σ_Q. For example, by using a quantizer $Q := (Q_1, Q_2)$ with small enough $E(Q)$ for the system in Fig. 1 (c), the output response of the system is similar to that of the ideal system in Fig. 1 (b) when the same external inputs (r_R and r_N) are applied to both systems. This implies that the optimal quantizer Q allows us to use a controller K designed for the ideal plant in Fig. 1 (b). Thus we can use the existing controller design methods for the system in Fig. 1 (c).

III. Optimal decentralized dynamic quantizer

This section shows how to obtain a closed-form solution to Problem 1.

A. An optimal decentralized quantizer

The following lemma provided in [10] is the fundamental.

Lemma 1: For the system Σ_Q, assume **(A1)** and suppose that $d \in \mathbb{R}_+$ is given. If

$$\bar{C}\bar{A}^k\bar{B}_1 = 0 \quad (k=0,1,\ldots) \tag{6}$$

holds, then

$$E(Q) = \left\| \sum_{k=0}^{\infty} \mathrm{abs}(\bar{C}\bar{A}^k\bar{B}_2) \right\| \frac{d}{2}; \tag{7}$$

otherwise

$$E(Q) = \infty, \tag{8}$$

where

$$\bar{A} := \begin{bmatrix} \widetilde{A} & B_2\mathcal{C} \\ 0 & \mathcal{A}+\mathcal{B}_2\mathcal{C} \end{bmatrix},\ \bar{B}_1 := \begin{bmatrix} 0 \\ \mathcal{B}_1+\mathcal{B}_2 \end{bmatrix},\ \bar{B}_2 := \begin{bmatrix} B_2 \\ \mathcal{B}_2 \end{bmatrix},$$

$$\bar{C} := [\ C_1 \quad 0\ ],\quad \widetilde{A} := A + B_2C_2,$$

$$\mathcal{A} := \begin{bmatrix} \mathcal{A}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{A}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{A}_s \end{bmatrix},\ \mathcal{B}_1 := \begin{bmatrix} \mathcal{B}_{11} & 0 & \cdots & 0 \\ 0 & \mathcal{B}_{12} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{B}_{1s} \end{bmatrix},$$

$$\mathcal{B}_2 := \begin{bmatrix} \mathcal{B}_{21} & 0 & \cdots & 0 \\ 0 & \mathcal{B}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{B}_{2s} \end{bmatrix},\ \mathcal{C} := \begin{bmatrix} \mathcal{C}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{C}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{C}_s \end{bmatrix}.$$

■

Lemma 1 provides a closed form expression of the performance index $E(Q)$. The result is explained as follows. We introduce the new variable $w \in [-d/2\ d/2]^m$:

$$w(k) := \mathrm{q}[\mathcal{C}\xi(k) + u(k)] - (\mathcal{C}\xi(k) + u(k)) \tag{9}$$

with $\xi := [\ \xi_1^\top\ \xi_2^\top\ \cdots\ \xi_s^\top\ ]^\top$ (note that w corresponds to the quantization error generated by the static quantizer q in Q). Using this variable, the error system for Σ_Q and Σ, which is shown in Fig. 4, can be represented as a linear system whose output is $z_Q - z$ and inputs are r and w [10]. Then, $\bar{C}\bar{A}^k\bar{B}_1$ in (6) corresponds to the impulse response matrices from r to $z_Q - z$, and $\bar{C}\bar{A}^k\bar{B}_2$ in (7) is those from w to $z_Q - z$. By considering these facts, it follows that

$$z_Q(T,x_0,R) - z(T,x_0,R) = \sum_{k=0}^{T-1} \bar{C}\bar{A}^{(T-1)-k}\bar{B}_2 w(k), \tag{10}$$

subject to (6). This leads to (7). On the other hand, if (6) does not hold, then $z_Q - z$ depends on r, which gives (8).

Now let us derive a solution to Problem 1 by using Lemma 1. For this purpose, we divide the original design problem into several easier problems as follows. If **(A1)** and (6) hold for Σ_Q, the value of $E(Q)$ is given by the right hand side of (7), where the matrices $\bar{A}$, $\bar{B}_2$, and $\bar{C}$ in (7)

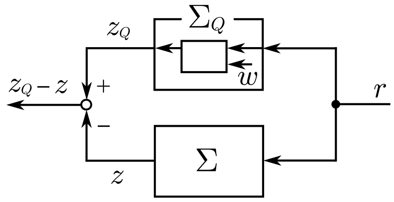


Fig. 4. Error system for Σ_Q and Σ.

can be represented by

$$\bar{A} = \begin{bmatrix} \widetilde{A} & B_{21}\mathcal{C}_1 & B_{22}\mathcal{C}_2 & \cdots & B_{2s}\mathcal{C}_s \\ 0 & \mathcal{A}_1 + \mathcal{B}_{21}\mathcal{C}_1 & 0 & \cdots & 0 \\ 0 & 0 & \mathcal{A}_2 + \mathcal{B}_{22}\mathcal{C}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \mathcal{A}_s + \mathcal{B}_{2s}\mathcal{C}_s \end{bmatrix},$$

$$\bar{B}_2 = \begin{bmatrix} B_{21} & B_{22} & \cdots & B_{2s} \\ \mathcal{B}_{21} & 0 & \cdots & 0 \\ 0 & \mathcal{B}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{B}_{2s} \end{bmatrix}, \ \bar{C} = [\ C_1 \ 0 \ \ 0 \ \cdots \ 0\ ].$$

Then, by letting $\bar{B}_{2i} \in \mathbb{R}^{(n+\mathcal{N}_i)\times m_i}$ be the i-th block column of $\bar{B}_2$, i.e., $\bar{B}_2 = [\bar{B}_{21} \ \bar{B}_{22} \ \cdots \ \bar{B}_{2s}]$, we obtain

$$\begin{aligned} E(Q) &= \left\| \sum_{k=0}^{\infty} \mathrm{abs}\left(\bar{C}\bar{A}^k[\ \bar{B}_{21} \ \ \bar{B}_{22} \ \cdots \ \bar{B}_{2s}\ ]\right) \right\| \frac{d}{2} \\ &= \left\| \left[\ \sum_{k=0}^{\infty} \mathrm{abs}\left(\bar{C}\bar{A}^k\bar{B}_{21}\right) \ \ \sum_{k=0}^{\infty} \mathrm{abs}\left(\bar{C}\bar{A}^k\bar{B}_{22}\right) \right.\right. \\ &\qquad\qquad \left.\left. \cdots \ \sum_{k=0}^{\infty} \mathrm{abs}\left(\bar{C}\bar{A}^k\bar{B}_{2s}\right)\ \right] \right\| \frac{d}{2}. \end{aligned}$$

The matrix $\bar{C}\bar{A}^k\bar{B}_{2i}$ in the norm can be expressed as

$$\bar{C}\bar{A}^k\bar{B}_{2i} = [\ C_1 \ \ 0\ ] \begin{bmatrix} \widetilde{A} & B_{2i}\mathcal{C}_i \\ 0 & \mathcal{A}_i + \mathcal{B}_{2i}\mathcal{C}_i \end{bmatrix}^k \begin{bmatrix} B_{2i} \\ \mathcal{B}_{2i} \end{bmatrix}, \tag{11}$$

where the relation $\bar{C}\bar{A}^k\bar{B}_{2i} = 0$ holds for $k \leq \tau_i - 1$, and $\bar{C}\bar{A}^k\bar{B}_{2i} = C_1\widetilde{A}^{\tau_i}B_{2i}$ for $k = \tau_i$. Therefore it follows that

$$\begin{aligned} E(Q) = \Bigg\| & \Big[\ \mathrm{abs}(C_1\widetilde{A}^{\tau_1}B_{21}) \ \ \mathrm{abs}(C_1\widetilde{A}^{\tau_2}B_{22}) \\ & \qquad\qquad \cdots \ \mathrm{abs}(C_1\widetilde{A}^{\tau_s}\bar{B}_{2s})\ \Big] \\ & + \Big[\ \sum_{k=\tau_1+1}^{\infty} \mathrm{abs}\left(\bar{C}\bar{A}^k\bar{B}_{21}\right) \ \ \sum_{k=\tau_2+1}^{\infty} \mathrm{abs}\left(\bar{C}\bar{A}^k\bar{B}_{22}\right) \\ & \qquad\qquad \cdots \ \sum_{k=\tau_s+1}^{\infty} \mathrm{abs}\left(\bar{C}\bar{A}^k\bar{B}_{2s}\right)\ \Big] \Bigg\| \frac{d}{2}. \end{aligned} \tag{12}$$

In the norm of the equation, the first term depends only on G and the second term depends on both G and Q. In addition, each element of the matrices $\mathrm{abs}(\bar{C}\bar{A}^k\bar{B}_{2i})$ $(i = 1, 2, \ldots, s)$ is nonnegative for every $k \in \{0\} \cup \mathbb{N}$. Thus, if there exists a decentralized quantizer Q satisfying (6) and

$$\bar{C}\bar{A}^k\bar{B}_{2i} = 0 \quad (k = \tau_i+1, \tau_i+2, \ldots, \quad i = 1, 2, \ldots, s), \tag{13}$$

such a quantizer Q is a solution to Problem 1. Thus, we obtain the following result.

Theorem 1: For the system Σ_Q, assume **(A1)** and **(A2)**. Then an optimal decentralized dynamic quantizer is given by $Q^\star = (Q_1^\star, Q_2^\star, \ldots, Q_s^\star)$ with

$$Q_i^\star : \begin{cases} \xi_i(k+1) = \widetilde{A}\xi_i(k) - B_{2i}u_i(k) + B_{2i}v_i(k) \\ v_i(k) = \mathrm{q}[\,-(C_1\widetilde{A}^{\tau_i}B_{2i})^\dagger C_1\widetilde{A}^{\tau_i+1}\xi_i(k) + u_i(k)\,] \end{cases} \tag{14}$$

where $\mathcal{N}_i := n$ $(i = 1, 2, \ldots, s)$, and the minimum value of the performance is given by

$$E(Q^\star) = \left\| \mathrm{abs}\left(C_1[\widetilde{A}^{\tau_1}B_{21} \ \ \widetilde{A}^{\tau_2}B_{22} \ \cdots \ \widetilde{A}^{\tau_s}B_{2s}] \right) \right\| \frac{d}{2}. \tag{15}$$

Proof: As mentioned above, a decentralized quantizer Q satisfying (6) and (13) is optimal under **(A1)**. In fact, (6) and (13) always hold for $Q^\star$ in Theorem 1. This is verified by using the relation $(C_1\widetilde{A}^{\tau_i}B_{2i})^\dagger = (C_1\widetilde{A}^{\tau_i}B_{2i})^\top((C_1\widetilde{A}^{\tau_i}B_{2i})(C_1\widetilde{A}^{\tau_i}B_{2i})^\top)^{-1}$ under **(A2)**. ■

Theorem 1 gives an optimal decentralized dynamic quantizer and the minimum value of $E(Q)$, and then the latter means the performance limitation of the decentralized quantizer composed of s sub-quantizers in the form of (2). The key points of obtaining the analytical solution to Problem 1 are that the parameters of i-th sub-quantizer Q_i are included only in the matrices $\bar{C}\bar{A}^k\bar{B}_{2i}$ $(k = \tau_i+1, \tau_i+2, \ldots)$ in (12) (see (11)), and that the relation (13) holds for i-th optimal quantizer $Q_i^\star$ in (14).

B. Extensions to more general cases

This section extends Theorem 1 to more general cases.

1) Case that (A1) does not hold: If **(A1)** does not hold, the performance index $E(Q)$ cannot be expressed as (7) and (8) but $E(Q)$ is estimated as

$$E(Q) \leq \left\| \sum_{k=0}^{\infty} \mathrm{abs}(\bar{C}\bar{A}^k\bar{B}_2) \right\| \frac{d}{2} \tag{16}$$

and (8). Therefore we cannot directly derive an optimal quantizer in a similar way to Section III-A.

However, we can prove that $Q^\star$ given in Theorem 1 is optimal under **(A2)** and the following condition (the proof is omitted in order to save space).

> **(A1')** The matrix $[C_2 \ D_2]$ is full row rank, and the set $\mathbb{I}$ is empty or the matrix $\hat{D}_2$ is full row rank.

The set $\mathbb{I}$ is defined by $\mathbb{I} := \{i \in \{1, 2, \ldots, s\} : \tau_i \neq \tau_{\max}\}$ for $\tau_{\max} := \max_{i\in\{1,2,\ldots,s\}} \tau_i$. In addition, let $D_{2i} \in \mathbb{R}^{m_i \times p}$ $(i = 1, 2, \ldots, s)$ denote the i-th block row of $D_2 \in \mathbb{R}^{m\times p}$, i.e., $D_2 = [D_{21}^\top \ \ D_{22}^\top \ \cdots \ D_{2s}^\top]^\top$. Then, the matrix $\hat{D}_2$ is composed of the matrices D_{2i} $(i \in \mathbb{I})$[2] if $\mathbb{I}$ is not empty.

On the other hand, if **(A1')** is not satisfied, the dynamic quantizer $Q^\star$ in Theorem 1 is not always optimal, while the quantizer minimizes the upper bound of $E(Q)$, i.e., the right

[2]For example, when $\mathbb{I} := \{1, 2, \ldots, s-1\}$, $\hat{D}_2$ is defined as $\hat{D}_2 := [D_{21}^\top \ \ D_{22}^\top \ \cdots \ D_{2(s-1)}^\top]^\top$.

hand side of (16). In this sense, $Q^\star$ is a practical quantizer in such a case.

2) Case that the quantization intervals are different in each sub-quantizers: Even if the quantization intervals of sub-quantizers $Q_1, Q_2, \ldots, Q_s$ are different each other as $d_1, d_2, \ldots, d_s \in \mathbb{R}_+$, similar results to Lemma 1 can be obtained by replacing the matrix $\bar{B}_2$ with the scaled matrix $\bar{B}_2^\sharp := \bar{B}_2 \operatorname{diag}(I_{m_1}, (d_2/d_1)I_{m_2}, (d_3/d_1)I_{m_3}, \ldots, (d_s/d_1)I_{m_s})$. Note that I_{m_i} is the $m_i \times m_i$ identity matrix. Therefore, an optimal decentralized quantizer is given by $Q^\star = (Q_1^\star, Q_2^\star, \ldots, Q_s^\star)$ in Theorem 1, where the static quantizer q in (14) is replaced by q_i with the interval d_i. In addition the minimum performance is given by

$$E(Q^\star) = \left\| \operatorname{abs}\left(C_1 [\widetilde{A}^{\tau_1} B_{21} \;\; \widetilde{A}^{\tau_2} B_{22} \;\; \cdots \;\; \widetilde{A}^{\tau_s} B_{2s}] \right) \cdot \operatorname{diag}\left(I_{m_1}, \frac{d_2}{d_1} I_{m_2}, \frac{d_3}{d_1} I_{m_3}, \ldots, \frac{d_s}{d_1} I_{m_s} \right) \right\| \frac{d_1}{2}. \tag{17}$$

This will be used in Section IV.

IV. Application to a cart-seesaw system

We evaluate the effectiveness of the proposed decentralized quantizer by an experiment using a cart-seesaw system.

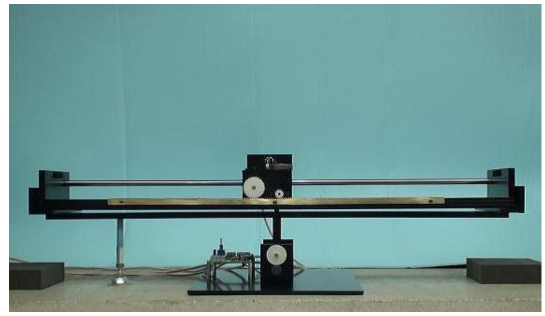
A. Description of experimental setup

We consider the cart-seesaw system shown in Fig. 5. The system is composed of the cart and the seesaw, and it is $0.8(\mathrm{L}) \times 0.3(\mathrm{W}) \times 0.3(\mathrm{H})$ meters long. The mass of the cart is 0.57 [kg] and that of the seesaw is 2.29 [kg]. The cart moves along the seesaw rail by applying a voltage onto the DC motor (0.023 [Nm/A]), and the seesaw rotates only in the vertical plane. The position of the cart and the angle of the seesaw are measured by the potentiometers. The computer (using MATLAB Real-time Workshop and Simulink) is connected to the cart-seesaw system. Then the feedback system in Fig. 1 (c) is constructed, where a controller K and a decentralized dynamic quantizer $Q := (Q_1, Q_2)$ are implemented in the computer. The control objective is to achieve the desired seesaw angle by the coarse discrete-valued control input on $\{-8, 0, +8\}$ [V] and the feedback signals (i.e., the signals for calculating the control input) on $\{0, \pm 0.02, \pm 0.04, \ldots, \pm 0.06\}$.

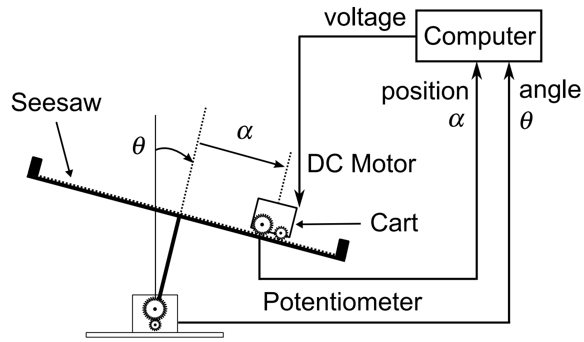
The continuous-time model of the cart-seesaw system is given by

$$\begin{cases} \dot{x}_P(t) = \begin{bmatrix} 0 & 1.00 & 0 & 0 \\ -1.92 & -4.15 & 5.38 & 0 \\ 0 & 0 & 0 & 1.00 \\ 25.63 & 1.27 & 9.91 & 0 \end{bmatrix} x_P(t) + \begin{bmatrix} 0 \\ 0.51 \\ 0 \\ -0.15 \end{bmatrix} v_1(t), \\ z(t) = [\, 0.5 \;\; 0.45 \;\; 1.0 \;\; 1.0 \,] x_P(t), \\ y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_P(t) \end{cases} \tag{18}$$

for the state variable $x_P := [\, \alpha \;\; \dot{\alpha} \;\; \theta \;\; \dot{\theta} \,]^\top \in \mathbb{R}^4$, where α is the position of the cart, θ is the angle of seesaw, and v_1 is the voltage applied to the motor as a control input.



(a) Overview of cart-seesaw system



(b) Diagram of control system

Fig. 5. Experimental device of cart-seesaw system

The controlled output z is chosen so as to reflect the relative importance of θ and $\dot{\theta}$ compared to α and $\dot{\alpha}$ while satisfying **(A2)**. The measured output y is of the position α and the angle θ.

Then P is the discrete-time model obtained from (18) and the sampling period $h := 0.01$ [s], and K is the observer based integral servo controller:

$$\begin{cases} A_K := \begin{bmatrix} 0.769 & 0.007 & -0.008 & 0.001 & 0 \\ -2.371 & 0.741 & -0.664 & -0.296 & 0 \\ -0.017 & 0 & 0.737 & 0.009 & 0 \\ -0.079 & 0.074 & -1.959 & 1.079 & 0 \\ 0 & 0 & 0 & 0 & 0.99 \end{bmatrix}, \\ [B_{1K} \;\; B_{2K}] := \begin{bmatrix} 0 & 0.222 & 0.003 \\ 0 & 0.764 & -0.451 \\ 0 & 0.021 & 0.264 \\ 0 & 0.821 & 2.416 \\ 0.01 & 0 & -0.01 \end{bmatrix}, \\ C_K := [\, -330.6 \;\; -44.1 \;\; -244.8 \;\; -62.2 \;\; 57.59 \,], \\ [D_{1K} \;\; D_{2K}] := [\; 0.01 \quad 0 \;\; 0 \;], \end{cases}$$

which achieves the control objective for the ideal system in Fig. 1 (b). For G defined by the above P and K, **(A1)** and **(A2)** are satisfied, and so $Q^\star = (Q_1^\star, Q_2^\star)$ in Theorem 1 can be used. Here, by considering that the inputs of P and K are restricted to taking values on the aforementioned discrete sets, the quantization intervals d_1 of Q_1 and d_2 of Q_2 are respectively defined as $d_1 := 8$ and $d_2 := 0.02$. Note that as mentioned in Section III-B. 2), even if $d_1 \neq d_2$, we can use $Q^\star$ in Theorem 1 by replacing q in Q_i with q_i.

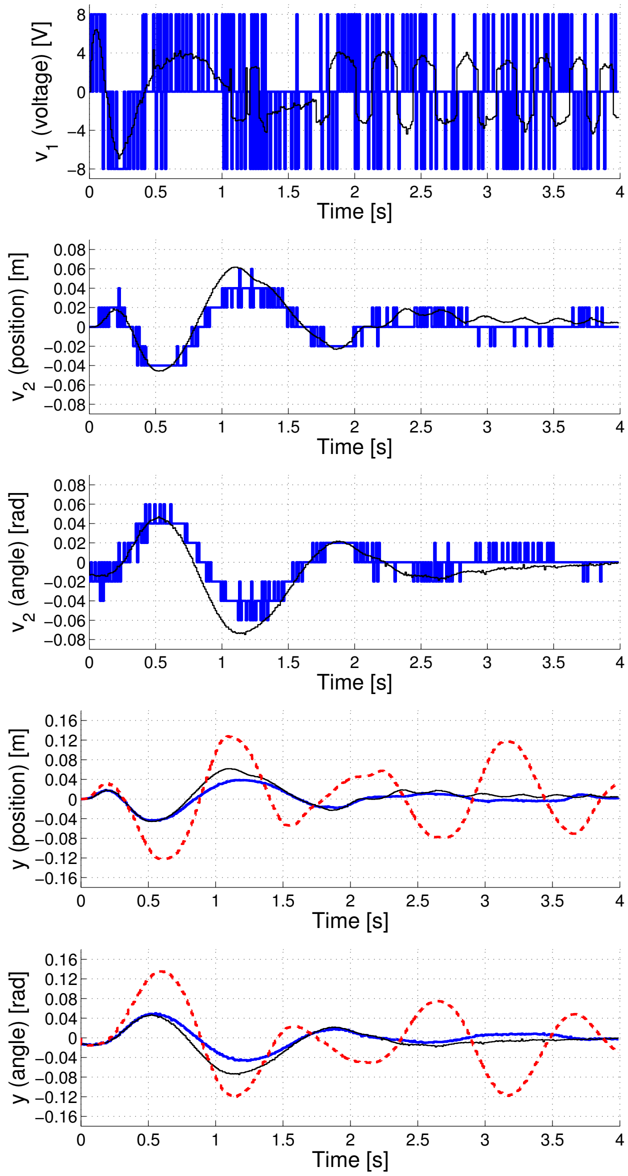


Fig. 6. Experimental result of the system in Fig. 1 (c) with $Q_i := Q_i^\star$ (thick lines), the ideal system in Fig. 1 (b) (thin line) and the system in Fig. 1 (c) with the static quantizer $Q_i := \mathrm{q}_i$ (dotted line).

B. *Experimental result*

Fig. 6 shows by thick lines the experimental result where the initial state of P and K are given by $x_P(0) := [0\ \ 0\ \ -0.013\ \ 0]^\top$ and $x_K(0) := [0\ \ 0\ \ 0\ \ 0\ \ 0]^\top$, and we assume that $r_R(k) \equiv 0$. Also in this figure, the experimental result of the ideal system in Fig. 1 (b) is depicted by the thin lines. In spite of the very coarse input signals of P and K as shown in the first, second, and third figures, the position α and the angle θ of the system in Fig. 1 (c) are close to those of the ideal system in Fig. 1 (b). Moreover, for comparison, the result of the static quantizer case ($Q_i := \mathrm{q}_i$) is shown in the forth and fifth figures by the dotted lines. The response of the static quantizer case is worse. This means that $Q^\star$ provides a satisfactory control performance under such severe discrete-valued signal constraints.

V. Conclusion

This paper has discussed an optimization problem of a class of decentralized dynamic quantizers in feedback control. For the problem, we have derived an optimal quantizer in an analytical way. The result is based on the formulation with the LFT (linear fraction transformation) representation of a linear system and a decentralized dynamic quantizer, and thus our framework will be a fundamental tool in decentralized quantizer design for various systems with discrete-valued input. Finally, the validity of the proposed optimal decentralized quantizer has been demonstrated by an experiment with a cart-seesaw system.

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