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Decentralized Output-Feedback Control of Large-Scale Nonlinear Systems Based on High-Gain Multiple Time Scaling

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***Abstract*— A globally stabilizing decentralized output-feedback controller is proposed for a general class of nonlinear interconnected large-scale systems. The subsystem interconnections and the dynamics of each subsystem feature both parametric and functional uncertainty. The controller design is based on a general high-gain scaling technique that utilizes arbitrary powers (instead of requiring successive powers) of the high-gain parameter with the powers chosen to satisfy certain inequalities depending on system nonlinearities. The scaling induces a weak-Cascading Upper Diagonal Dominance (w-CUDD) structure on the dynamics and allows relaxation of the cascading dominance assumption on upper diagonal terms. Disturbance attenuation properties of the proposed decentralized controller are also investigated.**

I. Introduction

Large-scale systems occurring in several application domains (including, as a very short representative list, power systems, multi-robot systems, communication/transportation networks, supply chains, etc.) can be profitably viewed as interconnections of multiple subsystems. In this general context, the development of control algorithms for interconnected large-scale systems has attracted considerable research interest. Interest in decentralized control designs has been significantly renewed in recent years due to noteworthy extensions promised by the application of new results emerging in nonlinear robust and adaptive output-feedback control. In this vein, we leverage our recent results from [11] and address the design of a decentralized output-feedback controller for global stabilization and disturbance attenuation of interconnected nonlinear systems of the class shown in (1). The results here follow the general direction in the decentralized literature of attempting to generalize the form of the dynamics subsystems and simultaneously weaken the assumptions on subsystem interconnections. Early results in decentralized control focused on linear systems [1], [2] and linearly bounded interconnections [3], [4]. In [5], higher order (i.e., polynomial type) interconnections were considered for large-scale systems assuming matching conditions. Backstepping-based robust decentralized controllers were designed in [6], [7] for systems of output-feedback canonical form including uncertain parameters and polynomially bounded uncertainties. Decentralized output-feedback robust disturbance attenuation for systems in output-feedback canonical form with appended stable linear dynamics with interconnections bounded by nonlinear functions of the outputs was addressed in [8]. Using the Cascading Upper Diagonal Dominance (CUDD) based technique in [9], a decentralized output-feedback disturbance attenuation scheme was proposed in [10] for interconnected large-scale systems with each subsystem being in the generalized output-feedback canonical form [9] and with nonlinear appended dynamics. The dual high-gain scaling technique from [11] was applied to design a decentralized controller in [12].

We consider a class of interconnected large-scale systems wherein each subsystem is of the form

$$\begin{aligned}
\dot{z}_m &= q_m(z,x,u,t,\varpi)\\
\dot{x}_{(i,m)} &= \phi_{(i,m)}(z,x,u,t,\varpi)\\
&\quad +\phi_{(i,i+1,m)}(x_{(1,m)})x_{(i+1,m)}\ ,\ i=1,\ldots,s_m-1\\
\dot{x}_{(s_m+i,m)} &= \phi_{(s_m+i,m)}(z,x,u,t,\varpi)\\
&\quad +\phi_{(s_m+i,s_m+i+1,m)}(x_{(1,m)})x_{(s_m+i+1,m)}\\
&\quad +\mu_{(i,m)}(x_{(1,m)})u_m\ ,\ i=0,\ldots,n_m-s_m-1\\
&\ \vdots\\
\dot{x}_{(n_m,m)} &= \phi_{(n_m,m)}(z,x,u,t,\varpi)\\
&\quad +\mu_{(n_m-s_m,m)}(x_{(1,m)})u_m\\
y_m &= x_{(1,m)}
\end{aligned} \tag{1}$$

where $x_m = [x_{(1,m)},\ldots,x_{(n_m,m)}]^T \in \mathcal{R}^{n_m}$ is the state, $u_m \in \mathcal{R}$ is the input, $y_m \in \mathcal{R}$ is the output, and $z_m \in \mathcal{R}^{n_{z_m}}$ is the state of the appended dynamics of the m^{th} subsystem. M is the number of subsystems, $x = [x_1^T,\ldots,x_M^T]^T$, $u=[u_1,\ldots,u_M]^T$, and $z=[z_1^T,\ldots,z_M^T]^T$. $\phi_{(i,i+1,m)}, i = 1,\ldots,n_m-1$ and $\mu_{(i,m)}$ are known continuous scalar real-valued functions. q_m is an uncertain continuous function. s_m is the relative degree of the m^{th} subsystem. $\varpi \in \mathcal{R}^{n_\varpi}$ is the exogenous disturbance input. $\phi_{(i,m)}, i=1,\ldots,n_m$ and q_m are continuous scalar real-valued uncertain functions.

The design is fundamentally based on our earlier result in [11] where a single subsystem of form (1) (i.e., $M = m = 1$), but without appended dynamics z_1 and with slightly stronger assumptions than used here on bounds on functions $\phi_{(i,1)}$, was considered and an output-feedback controller was proposed. The design in [11] was based on the dynamic high-gain scaling paradigm [13]–[15] but introduced a multiple time scaling through the use of arbitrary (not necessarily successive) powers of a dynamic high-gain scaling parameter r enabled through a new result on coupled parameter-dependent Lyapunov inequalities [11], [16], [17]. The utilization of non-successive powers of the dynamic high-gain scaling parameter in [11] allowed the removal of the cascading dominance assumption on upper diagonal terms (i.e., the assumption that the ratios $\phi_{(i,i+1,1)}/\phi_{(i-1,i,1)}$ and $\phi_{(i-1,i,1)}/\phi_{(i,i+1,1)}$ for $i = 2,\ldots,n_1-1$ are bounded) which was central in the earlier results [12]–[15]. The construction in [11] resulted in the cascading dominance being induced in the scaled system when r was of an appropriate size; the dynamics of r were then designed to achieve the required properties of the signal $r(t)$. In contrast with [11], we consider here an appended dynamics z_m, uncertain parameters in the bounds on $\phi_{(i,m)}$, a disturbance input ϖ, and introduction of multiple subsystems with nonlinear interconnections. The decentralized extension of the technique from [11] proves

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particularly challenging due to the fact that the observer gains in this approach are designed as functions of the high-gain scaling parameter and thus tend to amplify subsystem cross-coupling arising from $\phi_{(1,m)}$ as seen in the stability analysis. The design in this paper yields decentralized output-feedback control results for a significantly wider class of systems than available from prior results.

II. Assumptions

The design is carried out under the Assumptions A1-A6, each of which is required to hold for all $m \in \{1, \ldots, M\}$.

Assumption A1: A constant $\sigma_m > 0$ exists such that

$$\begin{aligned} |\phi_{(i,i+1,m)}(x_{(1,m)})| &\geq \sigma_m > 0\,,\, 1 \leq i \leq n_m - 1 \\ |\mu_{(0,m)}(x_{(1,m)})| &\geq \sigma_m > 0 \end{aligned} \quad (2)$$

for all $x_{(1,m)} \in \mathcal{R}$. Furthermore, the sign of each $\phi_{(i,i+1,m)}, i = 1, \ldots, n_m - 1$, is independent of its argument.

Assumption A2: The *inverse dynamics* of (1) satisfies the Bounded-Input-Bounded-State (BIBS) condition that the system given by $\dot{\Upsilon}_m = \Omega_m(x_{(1,m)})\Upsilon_m + v_m$ is BIBS stable with $[x_{(1,m)}, v_m^T]^T \in \mathcal{R}^{n_m - s_m + 1}$ considered the input and $\Upsilon_m \in \mathcal{R}^{n_m - s_m}$ being the state where the $(i,j)^{th}$ element of the $(n_m - s_m) \times (n_m - s_m)$ matrix $\Omega_m(x_{(1,m)})$ defined as

$$\Omega_{m(i,i+1)}(x_{(1,m)}) = \phi_{(s_m+i, s_m+i+1,m)} - \frac{\mu_{(i,m)}}{\mu_{(0,m)}}\phi_{(s_m, s_m+1,m)}$$

for $i = 1, \ldots, n_m - s_m - 1$ with zeros elsewhere.

Assumption A3: Continuous functions $\hat{\phi}_{(i,m)}(t, x_{(1,m)}, \ldots, x_{(i,m)})$ and nonnegative functions $\phi_{(i,j,m)}$, $\tilde{\Lambda}_{(m,k)}, k = 1, \ldots, M$, $\Gamma_{(m,k)}, k = 1, \ldots, M$, and $\Gamma_{(m,\varpi)}$ are known such that

$$|\phi_{(1,m)}(z,x,u,t,\varpi)| \leq \theta_m \sum_{k=1}^{M} \Big[\Gamma_{(m,k)}(x_{(1,k)})|x_{(1,k)}| + \Lambda_{(m,k)}(|z_k|)\Big] + \Gamma_{(m,\varpi)}(|\varpi|) \quad (3)$$

$$\begin{aligned} &\Big|\hat{\phi}_{(i,m)}(t, x_{(1,m)}, \hat{x}_{(2,m)}, \ldots, \hat{x}_{(i,m)}) - \phi_{(i,m)}(z,x,u,t,\varpi)\Big| \\ &\leq \theta_m \sum_{k=1}^{M}\Big[\Gamma_{(m,k)}(x_{(1,k)})|x_{(1,k)}| + \Lambda_{(m,k)}(|z_{(k)}|)\Big] \\ &\quad + \sum_{j=2}^{i} \phi_{(i,j,m)}(x_{(1,m)})|\hat{x}_{(j,m)} - x_{(j,m)}| \\ &\quad + \Gamma_{(m,\varpi)}(|\varpi|)\ ,\ 2 \leq i \leq n_m \end{aligned} \quad (4)$$

for all $t \geq 0$, $x_m \in \mathcal{R}^{n_m}, m = 1 \ldots, M$, $z_m \in \mathcal{R}^{n_{z_m}}$, $m = 1, \ldots, M$, $u \in \mathcal{R}^M$, and $\varpi \in \mathcal{R}^{n_\varpi}$, with θ_m being an unknown non-negative constant.

Assumption A4: The z_m subsystem is ISpS with ISpS Lyapunov function V_{z_m} satisfying

$$\dot{V}_{z_m} \leq -\alpha_{z_m}(|z_m|) + \theta_m \sum_{k=1}^{M} \beta_{(z_m,k)}(|x_{(1,k)}|) + \beta_{(z_m,\varpi)}(|\varpi|) \quad (5)$$

where θ_m is an unknown non-negative constant, α_{z_m} is a known class K_∞ function, and $\beta_{(z_m,k)}, k = 1, \ldots, M$ and $\beta_{(z_m,\varpi)}$ are known continuous non-negative functions. The following local order estimates hold as $\pi \to 0^+$: (a) $\sum_{k=1}^{M} \Lambda^2_{(k,m)}(\pi) = O[\alpha_{z_m}(\pi)]$, (b) $\sum_{k=1}^{M} \Lambda_{(k,m)}(\pi) = O[\pi]$, (c) $\sum_{k=1}^{M} \beta_{(z_k,m)}(\pi) = O[\pi^2]$.

Remark 1: Unlike [12], we do not require the cascading dominance on upper diagonal terms in the current approach. Instead, the observer-context cascading dominance will be induced through a generalized scaling. The price, however, that one must pay to relax the cascading dominance assumption is that the assumption on the functions $\phi_{(i,m)}$ needs to be stronger than in [12] since a high-gain controller cannot be used due to the fact that the cascading dominance of upper diagonal terms required in observer and controller contexts are dual, i.e., the observer-context cascading dominance condition requires ratios $|\phi_{(i,i+1)}|/|\phi_{(i-1,i)}|$ to be upper bounded while the controller-context cascading dominance condition requires ratios $|\phi_{(i-1,i)}|/|\phi_{(i,i+1)}|$ to be upper bounded. Hence, the high-gain observer design requires upper diagonal terms nearer to the output to be larger while the high-gain controller design requires upper diagonal terms closer to the input to be larger. Therefore, it is not, in general, possible to design a high-gain observer and high-gain controller using the generalized scaling technique since either observer-context or controller-context cascading dominance can be induced by the scaling, but not both. The output-feedback design in this paper uses the generalized scaling technique for the observer which is then coupled with a backstepping controller. This constrains the functions $\phi_{(i,m)}$ to be incrementally linear in unmeasured states and prevents them from having the more general bound which can be handled using the results in [15]: For $i = 2, \ldots, n$,

$$\begin{aligned} &\Big|\hat{\phi}_{(i,m)}(t, x_{(1,m)}, \hat{x}_{(2,m)}, \ldots, \hat{x}_{(i,m)}) - \phi_{(i,m)}(z,x,u,t,\varpi)\Big| \\ &\leq \theta_m \sum_{k=1}^{M}\Big[\Gamma_{(m,k)}(x_{(1,k)})|x_{(1,k)}| + \Lambda_{(m,k)}(|z_{(k)}|)\Big] \\ &\quad + \theta_m \sum_{j=1}^{i} \phi_{(i,j,m)}(x_{(1,m)})[|\hat{x}_{(j,m)}| + |\hat{x}_{(j,m)} - x_{(j,m)}|] \\ &\quad + \Gamma_{(m,\varpi)}(|\varpi|)\ ,\ 2 \leq i \leq n_m. \end{aligned} \quad (6)$$

III. Observer Design

A reduced-order observer for the m^{th} subsystem of the interconnected large-scale system (1) is given by[1]

$$\begin{aligned} \dot{\hat{x}}_{(i,m)} = &\hat{\phi}_{(i,m)}(t, x_{(1,m)}, \hat{x}_{(2,m)} + f_{(2,m)}(r_m, x_{(1,m)}), \\ &\ldots, \hat{x}_{(i,m)} + f_{(i,m)}(r_m, x_{(1,m)})) \\ &+ \phi_{(i,i+1,m)}(x_{(1,m)})[\hat{x}_{(i+1,m)} + f_{(i+1,m)}(r_m, x_{(1,m)})] \\ &+ g_{(i,m)}(r_m, x_{(1,m)})[\hat{x}_{(2,m)} + f_{(2,m)}(r_m, x_{(1,m)})] \\ &+ \mu_{(i-s_m,m)}(x_{(1,m)})u_m - \dot{r}_m h_{(i,m)}(r_m, x_{(1,m)}), 2 \leq i \leq n_m \end{aligned} \quad (7)$$

where r_m is a dynamic high-gain scaling parameter, $f_{(i,m)}(r_m, x_{(1,m)})$ are design functions of $x_{(1,m)}$ which will be picked during the stability analysis, and

$$\begin{aligned} g_{(i,m)}(r_m, x_{(1,m)}) &= -\phi_{(1,2,m)}(x_{(1,m)}) \frac{\partial f_{(i,m)}(r_m, x_{(1,m)})}{\partial x_{(1,m)}} \\ h_{(i,m)}(r_m, x_{(1,m)}) &= \frac{\partial f_{(i,m)}(r_m, x_{(1,m)})}{\partial r_m}. \end{aligned} \quad (8)$$

The dynamics of the high-gain scaling parameter r_m will be designed to be of the form $\dot{r}_m = w_m(r_m, x_{(1,m)})$ with w_m being $(s_m - 2)$-times continuously differentiable. r_m is initialized greater than 1. The dynamics of r_m designed during the stability analysis will ensure that r_m is non-decreasing. Defining the observer errors

$$e_{(i,m)} = \hat{x}_{(i,m)} + f_{(i,m)}(r_m, x_{(1,m)}) - x_{(i,m)}\ ,\ 2 \leq i \leq n_m, \quad (9)$$

the observer error dynamics are, $2 \leq i \leq n_m$,

$$\begin{aligned} \dot{e}_{(i,m)} = &\tilde{\phi}_{(i,m)} - \phi_{(i,m)} + \phi_{(i,i+1,m)}(x_{(1,m)})e_{(i+1,m)} \\ &- g_{(i,m)}(r_m, x_{(1,m)}) \frac{\phi_{(1,m)}}{\phi_{(1,2,m)}(x_{(1,m)})} \\ &+ g_{(i,m)}(r_m, x_{(1,m)})e_{(2,m)} \end{aligned} \quad (10)$$

[1]For simplicity of notation, we introduce the dummy variables $\phi_{(n_m,n_m+1,m)} = \hat{x}_{(n_m+1,m)} = f_{(n_m+1,m)} = g_{(n_m+1)} = 0$ and $\mu_{(i,m)} \equiv 0$ for $i < 0$. For notational clarity, we also drop arguments of functions when no confusion will result.

with $e_{(n_m+1,m)} = 0$ being a dummy variable where, for notational convenience, we have introduced

$$\tilde{\phi}_{(i,m)} = \hat{\phi}_{(i,m)}(t, x_{(1,m)}, \hat{x}_{(2,m)} + f_{(2,m)}(r_m, x_{(1,m)}), \ldots, \hat{x}_{(i,m)} + f_{(i,m)}(r_m, x_{(1,m)})) \,,\ i = 2, \ldots, n_m. \quad (11)$$

Hence, the dynamics of $e_m = [e_{(2,m)}, \ldots, e_{(n,m)}]^T$ are

$$\dot{e}_m = \tilde{\Phi}_m + [A_{(o,m)} + G_m C_m] e_m \quad (12)$$

where $C_m = [1, 0, \ldots, 0]$, $G_m(r_m, x_{(1,m)}) = [g_{(2,m)}(r_m, x_{(1,m)}), \ldots, g_{(n_m,m)}(r_m, x_{(1,m)})]^T$, and

$$\tilde{\Phi}_m = [\tilde{\Phi}_{(2,m)}, \ldots, \tilde{\Phi}_{(n_m,m)}]^T \quad (13)$$

$$\tilde{\Phi}_{(i,m)} = \tilde{\phi}_{(i,m)} - \phi_{(i,m)} - g_{(i,m)}(r_m, x_{(1,m)}) \frac{\phi_{(1,m)}}{\phi_{(1,2,m)}} \quad (14)$$

$$A_{(o,m)} = \begin{bmatrix} 0 & \phi_{(2,3,m)} & 0 & \cdots & 0 \\ 0 & 0 & \phi_{(3,4,m)} & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & & & & \phi_{(n_m-1,n_m,m)} \\ 0 & 0 & \cdots & & 0 \end{bmatrix}$$

IV. Controller Design

Define $\xi_{(i,m)} = \hat{x}_{(i,m)} + f_{(i,m)}(r_m, x_{(1,m)})$, $i = 2, \ldots, s_m$. The controller for the m^{th} subsystem is designed through backstepping [18] using the subsystem with states $(x_{(1,m)}, \xi_{(2,m)}, \ldots, \xi_{(s_m,m)})$ whose dynamics are

$$\begin{aligned}
\dot{x}_{(1,m)} &= -\phi_{(1,2,m)}(x_{(1,m)}) e_{(2,m)} + \phi_{(1,m)} + \phi_{(1,2,m)}(x_{(1,m)}) \xi_{(2,m)} \\
\dot{\xi}_{(i,m)} &= g_{(i,m)}(r_m, x_{(1,m)}) e_{(2,m)} \\
&\quad + \hat{\phi}_{(i,m)}(t, x_{(1,m)}, \xi_{(2,m)}, \ldots, \xi_{(i,m)}) \\
&\quad - g_{(i,m)}(r_m, x_{(1,m)}) \frac{\phi_{(1,m)}}{\phi_{(1,2,m)}(x_{(1,m)})} \\
&\quad + \phi_{(i,i+1,m)}(x_{(1,m)}) \xi_{(i+1,m)} \ ,\ i = 2, \ldots, s_m - 1 \\
\dot{\xi}_{(s_m,m)} &= g_{(s_m,m)}(r_m, x_{(1,m)}) e_{(2,m)} \\
&\quad + \hat{\phi}_{(s_m,m)}(t, x_{(1,m)}, \xi_{(2,m)}, \ldots, \xi_{(s_m,m)}) \\
&\quad - g_{(s_m,m)}(r_m, x_{(1,m)}) \frac{\phi_{(1,m)}}{\phi_{(1,2,m)}(x_{(1,m)})} \\
&\quad + \phi_{(s_m,s_m+1,m)}(x_{(1,m)}) \Big[\hat{x}_{(s_m+1,m)} \\
&\quad + f_{(s_m+1,m)}(r_m, x_{(1,m)}) \Big] + \mu_{(0,m)}(x_{(1,m)}) u_m. \quad (15)
\end{aligned}$$

The backstepping-based controller design follows similar lines as in [11] except for the introduction of the adaptation parameter $\hat{\theta}_m$, and more importantly, the introduction of a parameter $\overline{\theta}_m$ used at a key point in the stability analysis.

Step 1: The backstepping is commenced using the Lyapunov function $V_{(1,m)} = \frac{1}{2}\eta^2_{(1,m)}$ with $\eta_{(1,m)} = x_{(1,m)}$ yielding

$$\begin{aligned}
\dot{V}_{(1,m)} &= -x_{(1,m)}\phi_{(1,2,m)}[e_{(2,m)} - \xi_{(2,m)}] + x_{(1,m)}\phi_{(1,m)} \\
&\leq -\alpha_m(r_m, x_{(1,m)}) x^2_{(1,m)} + \phi_{(1,2,m)}\eta_{(1,m)}\eta_{(2,m)} \\
&\quad + \frac{e^2_{(2,m)}}{4r_m^2\overline{\theta}_m} + \frac{1}{4\overline{\theta}_m}\phi^2_{(1,m)} + \psi_{(1,m)}(\dot{\hat{\theta}}_m + \gamma_{2m}\hat{\theta}_m) \\
&\quad + (\overline{\theta}_m - \hat{\theta}_m - r_{(1,m)}\psi_{(1,m)})\tau_{(1,m)} \quad (16)
\end{aligned}$$

where $\zeta_{(1,m)}$ is an arbitrary positive constant, α_m is a smooth nonnegative function to be picked during stability analysis, $\hat{\theta}_m$ is an adaptation parameter, $\overline{\theta}_m$ is an (unknown) positive constant (depending on θ_m) which will be specified during stability analysis, and

$$\eta_{(2,m)} = \xi_{(2,m)} - \xi^*_{(2,m)}(r_m, x_{(1,m)}) \quad (17)$$

$$\begin{aligned}
\xi^*_{(2,m)}(r_m, x_{(1,m)}) &= -\frac{1}{\phi_{(1,2,m)}(x_{(1,m)})} \\
&\quad \times \Big[\theta_m r_m^2 x_{(1,m)} \phi^2_{(1,2,m)}(x_{(1,m)}) \\
&\quad + \hat{\theta}_m x_{(1,m)} + \alpha_m(r_m, x_{(1,m)}) x_{(1,m)} \Big] \quad (18)
\end{aligned}$$

$$\psi_{(1,m)} = 0\,;\ \tau_{(1,m)} = x^2_{(1,m)} + r^2 x^2_{(1,m)}\phi^2_{(1,2)} \quad (19)$$

Step i ($2 \leq i \leq s_m - 1$): Assume that at step $(i-1)$, a Lyapunov function $V_{(i-1,m)}$ has been designed such that

$$\begin{aligned}
\dot{V}_{(i-1,m)} &\leq -\alpha_m(r_m, x_{(1,m)}) x^2_{(1,m)} - \sum_{j=2}^{i-1} \zeta_{(j,m)}\eta^2_{(j,m)} \\
&\quad + \phi_{(i-1,i,m)}\eta_{(i-1,m)}\eta_{(i,m)} + \frac{(i-1)e^2_{(2,m)}}{4r_m^2\overline{\theta}_m} \\
&\quad + \frac{i-1}{4\overline{\theta}_m}\phi^2_{(1,m)} + (\overline{\theta}_m - \hat{\theta}_m - \gamma_{(1,m)}\psi_{(i-1,m)})\tau_{(i-1,m)} \\
&\quad + \psi_{(i-1,m)}[\dot{\hat{\theta}}_m + \gamma_{(2,m)}\hat{\theta}_m] \quad (20)
\end{aligned}$$

where for $j = 2, \ldots, i$,
$\eta_{(j,m)} = \xi_{(j,m)} - \xi^*_{(j,m)}(t, r_m, x_{(1,m)}, \xi_{(2,m)}, \ldots, \xi_{(j-1,m)})$,
with $\xi^*_{(j,m)}$ being functions designed in the previous steps of backstepping. Defining $V_{(i,m)} = V_{(i-1,m)} + \frac{1}{2}\eta^2_{(i,m)}$, and differentiating,

$$\begin{aligned}
\dot{V}_{(i,m)} &\leq -\alpha_m(r_m, x_{(1,m)}) x^2_{(1,m)} - \sum_{j=2}^{i} \zeta_{(j,m)}\eta^2_{(j,m)} \\
&\quad + \phi_{(i,i+1,m)}\eta_{(i,m)}\eta_{(i+1,m)} + \frac{i e^2_{(2,m)}}{4r_m^2\overline{\theta}_m} \\
&\quad + \frac{i}{4\overline{\theta}_m}\phi^2_{(1,m)} + (\overline{\theta}_m - \hat{\theta}_m - \gamma_{(1,m)}\psi_{(i,m)})\tau_{(i,m)} \\
&\quad + \psi_{(i,m)}[\dot{\hat{\theta}}_m + \gamma_{(2,m)}\hat{\theta}_m]
\end{aligned}$$

where

$$\eta_{(i+1,m)} = \xi_{(i+1,m)} - \xi^*_{(i+1,m)}(t, r_m, x_{(1,m)}, \xi_{(2,m)}, \ldots, \xi_{(i,m)})$$

$$\begin{aligned}
\xi^*_{(i+1,m)} &= -\frac{1}{\phi_{(i,i+1,m)}(x_{(1,m)})} \Bigg\{ \zeta_{(i,m)}\eta_{(i,m)} \\
&\quad + \phi_{(i-1,i,m)}(x_{(1,m)})\eta_{(i-1,m)} \\
&\quad + \hat{\phi}_{(i,m)}(t, x_{(1,m)}, \xi_{(2,m)}, \ldots, \xi_{(i,m)}) \\
&\quad - \frac{\partial \xi^*_{(i,m)}}{\partial t} - \frac{\partial \xi^*_{(i,m)}}{\partial r_m} w_m(r_m, x_{(1,m)}) \\
&\quad - \frac{\partial \xi^*_{(i,m)}}{\partial x_{(1,m)}} \phi_{(1,2,m)}(x_{(1,m)}) \xi_{(2,m)} \\
&\quad - \sum_{j=2}^{i-1} \frac{\partial \xi^*_{(i,m)}}{\partial \xi_{(j,m)}} \Big[\hat{\phi}_{(j,m)}(t, x_{(1,m)}, \xi_{(2,m)}, \ldots, \xi_{(j,m)}) \\
&\quad + \phi_{(j,j+1,m)}(x_{(1,m)}) \xi_{(j+1,m)} \Big] \\
&\quad + (\hat{\theta}_m + \gamma_{(1,m)}\psi_{(i,m)}) r_m^2 \eta_{(i,m)} \Big[g_{(i,m)}(r_m, x_{(1,m)}) \\
&\quad + \frac{\partial \xi^*_{(i,m)}}{\partial x_{(1,m)}} \phi_{(1,2,m)}(x_{(1,m)}) \\
&\quad - \sum_{j=2}^{i-1} \frac{\partial \xi^*_{(i,m)}}{\partial \xi_{(j,m)}} g_{(j,m)}(r_m, x_{(1,m)}) \Big]^2
\end{aligned}$$

$$+(\hat{\theta}_m + \gamma_{(1,m)}\psi_{(i,m)})\eta_{(i,m)}\Bigg[-\frac{g_{(i,m)}(r_m, x_{(1,m)})}{\phi_{(1,2,m)}(x_{(1,m)})} - \frac{\partial \xi^*_{(i,m)}}{\partial x_{(1,m)}} + \frac{\sum_{j=2}^{i-1} \frac{\partial \xi^*_{(i,m)}}{\partial \xi_{(j,m)}} g_{(j,m)}(r_m, x_{(1,m)})}{\phi_{(1,2,m)}(x_{(1,m)})}\Bigg]^2 - \gamma_{(2,m)}\frac{\partial \xi^*_{(i,m)}}{\partial \hat{\theta}_m}\theta_m - \frac{\partial \xi^*_{(i,m)}}{\partial \hat{\theta}_m}\gamma_{(1,m)}\tau_{(i-1,m)}\Bigg\} \quad (21)$$

$$\psi_{(i,m)} = \psi_{(i-1,m)} - \eta_{(1,m)}\frac{\partial \xi^*_{(i,m)}}{\partial \hat{\theta}_m} \quad (22)$$

$$\tau_{(i,m)} = \tau_{(i-1,m)} + \eta^2_{(i,m)}\Bigg[r_m^2 \Bigg[g_{(i,m)}(r_m, x_{(1,m)}) + \frac{\partial \xi^*_{(i,m)}}{\partial x_{(1,m)}}\phi_{(1,2,m)}(x_{(1,m)}) - \sum_{j=2}^{i-1}\frac{\partial \xi^*_{(i,m)}}{\partial \xi_{(j,m)}} g_{(j,m)}(r_m, x_{(1,m)})\Bigg]^2 + \Bigg[\frac{g_{(i,m)}(r_m, x_{(1,m)})}{\phi_{(1,2,m)}(x_{(1,m)})} + \frac{\partial \xi^*_{(i,m)}}{\partial x_{(1,m)}} + \frac{\sum_{j=2}^{i-1} \frac{\partial \xi^*_{(i,m)}}{\partial \xi_{(j,m)}} g_{(j,m)}(r_m, x_{(1,m)})}{\phi_{(1,2,m)}(x_{(1,m)})}\Bigg]^2\Bigg] \quad (23)$$

where $\zeta_{(i,m)}$ is any positive constant.

Step s_m: At this step, the control input u_m is designed as

$$u_m = \xi^*_{(s_m+1,m)}(t, r_m, x_{(1,m)}, \xi_{(2,m)}, \ldots, \xi_{(s_m,m)}) - \phi_{(s_m,s_m+1,m)}(x_{(1,m)})\left[\hat{x}_{(s_m+1,m)} + f_{(s_m+1,m)}(r_m, x_{(1,m)})\right]$$

where $\xi^*_{(s_m+1,m)}$ is defined analogously to (21) with i substituted to be s_m and with $\mu_{(0,m)}(x_{(1,m)})$ in the denominator of the first term rather than $\phi_{(i,i+1,m)}(x_{(1,m)})$. $\psi_{(s_m,m)}$ and $\tau_{(s_m,m)}$ are defined as in (22) and (23), respectively, with i substituted to be s_m. The Lyapunov function $V_{(s_m,m)} = \frac{1}{2}\sum_{i=1}^{s_m}\eta^2_{(i,m)}$ satisfies

$$\begin{aligned} \dot{V}_{(s_m,m)} \leq & -\alpha_m(r_m, x_{(1,m)})x^2_{(1,m)} - \sum_{j=2}^{s_m}\zeta_{(j,m)}\eta^2_{(j,m)} \\ & + \frac{s_m e^2_{(2,m)}}{4r_m^2\overline{\theta}_m} + \frac{s_m}{4\overline{\theta}_m}\phi^2_{(1,m)} \\ & + (\overline{\theta}_m - \hat{\theta}_m - \gamma_{(1,m)}\psi_{(s_m,m)})\tau_{(s_m,m)} \\ & + \psi_{(s_m,m)}[\dot{\hat{\theta}}_m + \gamma_{(2,m)}\hat{\theta}_m]. \end{aligned} \quad (24)$$

Designing the dynamics of the adaptation parameter $\hat{\theta}_m$ as

$$\dot{\hat{\theta}}_m = -\gamma_{(2,m)}\hat{\theta}_m + \gamma_{(1,m)}\tau_{(s_m,m)}, \quad (25)$$

and defining

$$\overline{V}_{(s_m,m)} = V_{(s_m,m)} + \frac{1}{2\gamma_{(1,m)}}(\hat{\theta}_m - \overline{\theta}_m)^2, \quad (26)$$

we have

$$\dot{\overline{V}}_{(s_m,m)} \leq -\alpha_m(r_m, x_{(1,m)})x^2_{(1,m)} - \sum_{j=2}^{s_m}\zeta_{(j,m)}\eta^2_{(j,m)} + \frac{s_m e^2_{(2,m)}}{4r_m^2\overline{\theta}_m} + \frac{s_m\phi^2_{(1,m)}}{4\overline{\theta}_m} - \frac{\gamma_{(2,m)}}{2\gamma_{(1,m)}}(\hat{\theta}_m - \overline{\theta}_m)^2 + \frac{\gamma_{(2,m)}}{2\gamma_{(1,m)}}\overline{\theta}^2_m \quad (27)$$

The design freedoms in the controller for the m^{th} subsystem are the function $\alpha_m(r_m, x_{(1,m)})$, and the constants $\zeta_{(2,m)}, \ldots, \zeta_{(s_m,m)}$, $\gamma_{(1,m)}$, and $\gamma_{(2,m)}$. The constants $\zeta_{(2,m)}, \ldots, \zeta_{(s_m,m)}$, $\gamma_{(1,m)}$, and $\gamma_{(2,m)}$ can be picked to be arbitrary positive constants and the function α_m must be chosen to satisfy a lower bound to be specified during stability analysis in Section V. Note that by picking the dynamics of r_m to be of form $\dot{r}_m = w_m(r_m, x_{(1,m)})$, the functions $\xi^*_{(2,m)}, \ldots, \xi^*_{(s_m+1,m)}$ are well-defined and continuous.

V. Stability Analysis

Define $M_m = [M_{(2,m)}, \ldots, M_{(n_m,m)}]^T$ where

$$M_{(i,m)} = \phi_{(i,1,m)}(x_{(1,m)}) + |g_{(i,m)}(r_m, x_{(1,m)})|\frac{\phi_{(1,1,m)}(x_{(1,m)})}{|\phi_{(1,2,m)}(x_{(1,m)})|}$$

and $\overline{\Phi}_m$ is the $(n_m-1)\times(n_m-1)$ matrix with $(i,j)^{th}$ entry

$$\begin{aligned} \overline{\Phi}_{m(i,j)} &= \phi_{(i+1,j+1,m)}\ ,\ i = 1, \ldots, n_m - 1\ ,\ j = 1, \ldots, i \\ \overline{\Phi}_{m(i,j)} &= 0\ ,\ i = 1, \ldots, n_m - 2\ ,\ j = i+1, \ldots, n_m - 1. \end{aligned} \quad (28)$$

The matrix $A_{(o,m)} + \overline{\Phi}_m$ satisfies the assumptions of Theorem 1 in [11]. Hence, given any positive constant ρ_m, nonnegative constants $q_{(1,m)}, \ldots, q_{(n_m-1,m)}$, and a positive function $R_m(x_{(1,m)}) \geq 1$ exist such that $T_m(r_m)[A_{(o,m)} + \overline{\Phi}_m]T_m^{-1}(r_m)$ is w-CUDD(ρ_m) for all $r_m \geq R_m(x_{(1,m)})$ where $T_m(r_m) = [\text{diag}(r_m^{q_{(1,m)}}, \ldots, r_m^{q_{(n_m-1,m)}})]^{-1}$. The w-CUDD property was defined in [17] and shown to be central in solvability of coupled Lyapunov inequalities [16]. From the construction in the proof of Theorem 2 in [11], $q_{(1,m)}$ can be taken to be 1 and[2] $q_{(n_m-1,m)} - q_{(n_m-2,m)} = 1$. Using Theorem 1 from [11], a $(n_m-1)\times 1$ vector $\tilde{G}_m(r_m, x_{(1,m)})$, a symmetric positive-definite matrix $P_{(o,m)}$, and positive constants $\nu_{(o,m)}, \underline{\nu}_{(o,m)}$, and $\overline{\nu}_{(o,m)}$ exist such that for all $r_m \geq R_m(x_{(1,m)})$ and all $x_{(1,m)} \in \mathcal{R}$

$$\begin{aligned} & P_{(o,m)}\Big\{T_m(r_m)[A_{(o,m)} + Q_{(1,m)}\overline{\Phi}_m Q_{(2,m)}]T_m^{-1}(r_m) + \tilde{G}_m C_m\Big\} \\ & + \Big\{T_m(r_m)[A_{(o,m)} + Q_{(1,m)}\overline{\Phi}_m Q_{(2,m)}]T_m^{-1}(r_m) + \tilde{G}_m C_m\Big\}^T P_{(o,m)} \\ & \leq -\frac{\nu_{(o,m)}}{r_m^{q_{(n_m-2,m)} - q_{(n_m-1,m)}}}|\phi_{(n_m-1,n_m,m)}|I \\ & \underline{\nu}_{(o,m)}I \leq P_{(o,m)}D_{(o,m)} + D_{(o,m)}P_{(o,m)} \leq \overline{\nu}_{(o,m)}I \end{aligned} \quad (29)$$

where $D_{(o,m)} = \text{diag}(q_{(1,m)}, \ldots, q_{(n_m-1,m)})$ and $Q_{(1,m)}$ and $Q_{(2,m)}$ are arbitrary diagonal matrices of dimension $(n_m - 1) \times (n_m - 1)$ with each diagonal entry $+1$ or -1. By Theorem 1 in [11], the choice of $\tilde{G}_m$ does not need to depend on $Q_{(1,m)}$ and $Q_{(2,m)}$. $G_m(r_m, x_{(1,m)}) = [g_{(2,m)}(r_m, x_{(1,m)}), \ldots, g_{(n,m)}(r_m, x_{(1,m)})]^T$ is defined as $G_m(r_m, x_{(1,m)}) = r_m^{-q_{(1,m)}}T_m^{-1}(r_m)\tilde{G}_m(r_m, x_{(1,m)})$ so that $\tilde{G}_m C_m = T_m(r_m)G_m C_m T_m^{-1}(r_m)$. $f_{(i,m)}, i = 2, \ldots, n_m$, are obtained as

$$f_{(i,m)}(r_m, x_{(1,m)}) = -\int_0^{x_{(1,m)}}\frac{g_{(i,m)}(r_m, \pi)}{\phi_{(1,2,m)}(\pi)}d\pi. \quad (30)$$

The dynamics of $\epsilon_m \stackrel{\triangle}{=} T_m(r_m)e_m$ are

$$\dot{\epsilon}_m = T_m\tilde{\Phi}_m + T_m[A_{(o,m)} + G_m C_m]T_m^{-1}\epsilon_m - \frac{\dot{r}_m}{r_m}D_o\epsilon_m. \quad (31)$$

The derivative of the Lyapunov function $V_{(o,m)} = \epsilon_m^T P_{(o,m)}\epsilon_m$ satisfies

$$\begin{aligned} \dot{V}_{(o,m)} = & 2\epsilon_m^T P_{(o,m)}T_m(r_m)\tilde{\Phi}_m \\ & + \epsilon_m^T\{P_{(o,m)}T_m(r_m)[A_{(o,m)} + G_m C_m]T_m^{-1}(r_m) \\ & + T_m^{-1}(r_m)[A_{(o,m)} + G_m C_m]^T T_m(r_m)P_{(o,m)}\}\epsilon \\ & - \frac{\dot{r}_m}{r_m}\epsilon_m^T[P_{(o,m)}D_{(o,m)} + D_{(o,m)}P_{(o,m)}]\epsilon_m. \end{aligned} \quad (32)$$

The scaling $\epsilon_m = T_m(r_m)e_m$ which comprises a scaling with non-successive powers $q_{(1,m)}, \ldots, q_{(n_m-1,m)}$ of the

[2] Note that since the observer used is a reduced-order observer, $A_{(o,m)}$ is of dimension $(n_m - 1) \times (n_m - 1)$. Hence, by the construction in the proof of Theorem 2 in [11], $q_{(i,m)} = 1 + (n_m - 2)(n_m - 1)/2 - (n_m - i - 1)(n_m - i)/2, i = 1, \ldots, n_m - 1$.

scaling parameter r_m essentially yields a multiple time scaling and is the key ingredient in allowing the removal of the cascading dominance assumption by using the Theorems 1 and 2 in [11]. In common with [15], the dynamics of the high-gain parameter are designed as

$$\dot{r}_m = q_m(R_m - r_m)\Delta_m(r_m, x_{(1,m)}) \tag{33}$$

with initial value $r_m(0) \geq 1$ and with Δ_m being an appropriately designed function. q_m is chosen to be any nonnegative $(s_m - 2)$-times continuously differentiable function such that $q_m(b) = 1$ if $b > 0$ and $q_m(b) = 0$ if $b < -\epsilon_r$ with ϵ_r being a positive constant. In contrast with the design in [11] where a single subsystem of form (1) was considered without appended dynamics, the function Δ_m should be chosen through a careful bounding of the term $2\epsilon_m^T P_{(o,m)} T_m(r_m)\tilde{\Phi}_m$ since this term will generate cross-products of the form $g_{(i,m)}(r_m, x_{(1,m)})x_{(1,k)}$ which cannot be handled in the composite Lyapunov function framework. To see the origin of such cross-products, note that[3]

$$|\tilde{\Phi}_m|_e \leq_e \overline{\Phi}_m |e_m|_e + \mathcal{Q}_m \tag{34}$$

$$\begin{aligned} 2\epsilon_m^T P_{(o,m)} T_m(r_m)\tilde{\Phi}_m \leq\ & 2\epsilon_m^T P_{(o,m)} T_m(r_m) \\ & \times Q_{(1,m)}\overline{\Phi}_m Q_{(2,m)} T_m^{-1}(r_m)\epsilon_m \\ & + 2|\epsilon_m^T P_{(o,m)}|_e T_m(r_m)\mathcal{Q}_m \end{aligned} \tag{35}$$

and observe that the bound on $\phi_{(1,m)}$ arising from Assumption A3 involves $x_{(1,k)}$, z_k, and ϖ. In (34)-(35),

$$\begin{aligned} \mathcal{Q}_{(i,m)} = \theta_m \sum_{k=1}^{M} [\Gamma_{(m,k)}(|x_{(1,k)}|)|x_{(1,k)}| + \Lambda_{(m,k)}(|z_k|)] \\ + \Gamma_{(m,\varpi)}(|\varpi|) + \frac{|g_{(i,m)}|}{|\phi_{(1,2,m)}|}|\phi_{(1,m)}|\,,\ i = 2, \ldots, n_m \end{aligned} \tag{36}$$

$$\mathcal{Q}_m = [\mathcal{Q}_{(2,m)}, \ldots, \mathcal{Q}_{(n_m,m)}]^T, \tag{37}$$

and $Q_{(1,m)}$ and $Q_{(2,m)}$ are diagonal matrices with each diagonal entry $+1$ or -1 such that $|P_{(o,m)}\epsilon_m|_e = Q_{(1,m)}P_{(o,m)}\epsilon_m$ and $|\epsilon_m|_e = Q_{(2,m)}\epsilon_m$. To handle the bounding of the term $2\epsilon_m^T P_{(o,m)} T_m(r_m)\tilde{\Phi}_m$, consider two cases: Case $\mathcal{A}$: $r > R_m(x_{(1,m)})$, and Case $\mathcal{B}$: $r \leq R_m(x_{(1,m)})$. Under Case $\mathcal{A}$, it is inferred from Theorem 1 in [11] and the construction in the proof of Theorem 3 in [16] that a positive constant $\overline{G}$ exists such that $|g_{(i,m)}| \leq \overline{G} r^{q_{(i-1,m)}}|\phi_{(1,2,m)}|$ for $i = 2, \ldots, n_m$. Also, since (29) holds under Case $\mathcal{A}$, it follows that

$$\dot{V}_{(o,m)} \leq -\frac{\nu_{(o,m)}\sigma_m}{2}|\epsilon_m|^2 + \frac{2}{\nu_{(o,m)}\sigma_m}\lambda_{max}^2(P_{(o,m)})|\tilde{\mathcal{Q}}_m|^2 \tag{38}$$

where $\tilde{\mathcal{Q}}_m = [\tilde{\mathcal{Q}}_{(2,m)}, \ldots, \tilde{\mathcal{Q}}_{(n_m,m)}]^T$ with

$$\begin{aligned} \tilde{\mathcal{Q}}_{(i,m)} = \theta_m \sum_{k=1}^{M} [\Gamma_{(m,k)}(|x_{(1,k)}|)|x_{(1,k)}| + \Lambda_{(m,k)}(|z_k|)] \\ + \Gamma_{(m,\varpi)}(|\varpi|) + \overline{G}|\phi_{(1,m)}| \quad i = 2, \ldots, n_m. \end{aligned} \tag{39}$$

Under Case $\mathcal{B}$, it follows from (33) that $\dot{r} = \Delta_m(r_m, x_{(1,m)})$. The term $2\epsilon_m^T P_{(o,m)} T_m(r_m)\tilde{\Phi}_m$ is bounded as

$$\begin{aligned} 2\epsilon_m^T P_{(o,m)} T_m(r_m)\tilde{\Phi}_m \leq\ & 2\epsilon_m^T P_{(o,m)} T_m(r_m) \\ & \times Q_{(1,m)}\overline{\Phi}_m Q_{(2,m)} T_m^{-1}(r_m)\epsilon_m \\ & + \frac{\nu_{(o,m)}\sigma_m}{2}|\epsilon_m|^2 \\ & \times \max\left(\frac{\max\left\{g_{(i,m)}^2 | i = 2, \ldots, n_m\right\}}{\overline{G}^2\phi_{(1,2,m)}^2}, 1\right) \\ & + \frac{2}{\nu_{(o,m)}\sigma_m}\lambda_{max}^2(P_{(o,m)})|\tilde{\mathcal{Q}}_m|^2. \end{aligned} \tag{40}$$

[3] $|\beta|_e$ denotes a matrix of the same dimension as β with each element replaced by its absolute value. $\leq_e$ denotes an element-wise inequality between two matrices of equal dimension. $\lambda_{max}(P)$ with P being a square symmetric matrix denotes the maximum eigenvalue of P.

Hence, designing $\Delta_m(r_m, x_{(1,m)})$ to be

$$\begin{aligned} \Delta_m(r_m, x_{(1,m)}) \geq \frac{r_m}{\underline{\nu}_{(o,m)}}\Bigg\{ r_m^* \\ + 2\lambda_{max}(P_{(o,m)})\frac{r_m^{q_{n_m}-1}}{r_m^{q_1}}\Big[||A_{(o,m)} + G_m C_m|| + ||\overline{\Phi}_m||\Big] \\ + \frac{\nu_{(o,m)}\sigma_m}{2}\max\left(\frac{\max\left\{g_{(i,m)}^2 | i = 2, \ldots, n_m\right\}}{\overline{G}^2\phi_{(1,2,m)}^2}, 1\right)\Bigg\} \end{aligned} \tag{41}$$

with r_m^* being any positive constant, it follows that

$$\dot{V}_{(o,m)} \leq -r_m^*|\epsilon_m|^2 + \frac{2}{\nu_{(o,m)}\sigma_m}\lambda_{max}^2(P_{(o,m)})|\tilde{\mathcal{Q}}_m|^2. \tag{42}$$

Therefore, in either Case $\mathcal{A}$ or Case $\mathcal{B}$, the inequality

$$\begin{aligned} \dot{V}_{(o,m)} \leq\ & -\min\left(\frac{\nu_{(o,m)}\sigma_m}{2}, r_m^*\right)|\epsilon_m|^2 \\ & + \frac{2}{\nu_{(o,m)}\sigma_m}\lambda_{max}^2(P_{(o,m)})|\tilde{\mathcal{Q}}_m|^2 \end{aligned} \tag{43}$$

holds. By Assumption A4, $\sum_{k=1}^{M}\Lambda_{(k,m)}^2(\pi) = O[\alpha_{z_m}(\pi)]$ as $\pi \to 0^+$. Using a reasoning similar to that used in the proof of Theorem 2 in [19], it is seen that this local order estimate implies the existence of a new Lyapunov function $\tilde{V}_{z_m}$, class K_∞ functions $\tilde{\alpha}_{z_m}$ and $\tilde{\beta}_{(z_m,k)}$, and continuous non-negative functions $\alpha_{\theta m}$ and $\tilde{\beta}_{(z_m,\varpi)}$ such that

$$\begin{aligned} \dot{\tilde{V}}_{z_m} \leq -\tilde{\alpha}_{z_m}(|z_m|) + \alpha_{\theta m}(\theta_m)\sum_{k=1}^{M}\tilde{\beta}_{(z_m,k)}(|x_{(1,k)}|) \\ + \tilde{\beta}_{(z_m,\varpi)}(|\varpi|) \end{aligned} \tag{44}$$

with $\tilde{\alpha}_{z_m}(\pi) = O[\alpha_{z_m}(\pi)]$ as $\pi \to 0^+$, $\tilde{\alpha}_{z_m}(|z_m|) \geq \sum_{k=1}^{M}\Lambda_{(k,m)}^2(|z_{(1,m)}|)\forall z_m \in \mathcal{R}^{n_{z_m}}$, $\tilde{\beta}_{(z_m,k)}$ independent of θ_m, and $\tilde{\beta}_{(z_m,k)}(\pi) = O[\beta_{(z_m,k)}(\pi)]$ as $\pi \to 0^+$. Hence, a continuous non-negative function $\overline{\beta}_{(z_m,k)}$ exists such that $\tilde{\beta}_{(z_m,k)}(|x_{(1,k)}|) \leq x_{(1,k)}^2\overline{\beta}_{(z_m,k)}(x_{(1,k)})$. Defining

$$V_{x_m} = \overline{V}_{(s_m,m)} + \frac{s_m}{2\overline{\theta}_m\min(\frac{\nu_{(o,m)}\sigma_m}{2}, r_m^*)}V_{(o,m)}, \tag{45}$$

and using (43) and (27), we obtain

$$\begin{aligned} \dot{V}_{x_m} = & -\alpha_m(r_m, x_{(1,m)})x_{(1,m)}^2 - \sum_{j=2}^{s_m}\zeta_{(j,m)}\eta_{(j,m)}^2 \\ & -\frac{1}{2}s_m^*\min\left(\frac{\nu_{(o,m)}\sigma_m}{2}, r_m^*\right)|\epsilon_m|^2 \\ & -\frac{\gamma_{(2,m)}}{2\gamma_{(1,m)}}(\hat{\theta}_m - \overline{\theta}_m)^2 + \frac{\gamma_{(2,m)}}{2\gamma_{(1,m)}}\overline{\theta}_m^2. \\ & + \frac{Q_{m0}}{\overline{\theta}_m}\Bigg\{M\theta_m^2\sum_{k=1}^{M}\Gamma_{(m,k)}^2(|x_{(1,k)}|)x_{(1,k)}^2 \\ & + M\theta_m^2\sum_{k=1}^{M}\Lambda_{(m,k)}^2(|z_k|) + \Gamma_{(m,\varpi)}^2(|\varpi|)\Bigg\} \end{aligned} \tag{46}$$

where Q_{m0} is a constant given by

$$Q_{m0} = \frac{s_m}{4} + \frac{3\lambda_{max}^2(P_{(o,m)})s_m}{\nu_{(o,m)}\sigma_m\min(\frac{\nu_{(o,m)}\sigma_m}{2}, r_m^*)}[3M\overline{G}^2 + 2(n_m - 1)].$$

Note that Q_{m0} does not depend on $\overline{\theta}_m$. At this point, we can choose the (unknown) constants $\overline{\theta}_1, \ldots, \overline{\theta}_M$ to be $\overline{\theta}$ where

$$\begin{aligned} \overline{\theta} = \max\Big\{ & \max\{Q_{k0}|k = 1, \ldots, M\}, \\ & M\max\left\{Q_{k0}\theta_k^2 | k = 1, \ldots, M\right\}\max\left\{\alpha_{\theta_k}(\theta_k), 1\right\}\Big\}. \end{aligned} \tag{47}$$

Note that $\overline{\theta}_m$ is a constant used only in stability analysis and does not enter anywhere into the observer or controller equations. The overall composite Lyapunov function of the large-scale interconnected system is picked to be

$$V = \sum_{m=1}^{M}\left[V_{x_m} + 2\Big(\sum_{k=1}^{M}\frac{Q_{k0}M\theta_k^2}{\overline{\theta}_k}\Big)\tilde{V}_{z_m}\right]. \tag{48}$$

We obtain

$$\dot{V} = -\sum_{m=1}^{M}\Bigg[\alpha_m(r_m, x_{(1,m)})x_{(1,m)}^2 + \sum_{j=2}^{s_m}\zeta_{(j,m)}\eta_{(j,m)}^2 + \frac{1}{2}s_m^*\min\left(\frac{\nu_{(o,m)}\sigma_m}{2}, r_m^*\right)|\epsilon_m|^2 + \frac{\gamma_{(2,m)}}{2\gamma_{(1,m)}}(\hat{\theta}_m - \overline{\theta}_m)^2 + \left(\sum_{k=1}^{M}\frac{Q_{k0}M\theta_k^2}{\overline{\theta}_k}\right)\tilde{\alpha}_{z_m}(|z_m|) - \frac{\gamma_{(2,m)}}{2\gamma_{(1,m)}}\overline{\theta}_m^2 - \overline{\Gamma}_m(|x_{(1,m)}|)x_{(1,m)}^2\Bigg] + \overline{\Gamma}_{\varpi}(|\varpi|) \quad (49)$$

where

$$\overline{\Gamma}_m(|x_{(1,m)}|) = \sum_{k=1}^{M}[\Gamma_{(k,m)}^2(|x_{(1,m)}|) + \overline{\beta}_{(z_k,m)}(|x_{(1,m)}|)] \quad (50)$$

$$\overline{\Gamma}_{\varpi}(|\varpi|) = \sum_{m=1}^{M}[\tilde{\beta}_{(z_m,\varpi)}(|\varpi|) + \Gamma_{(m,\varpi)}^2(|\varpi|)]. \quad (51)$$

Picking the design function α_m to be

$$\alpha_m(r_m, x_{(1,m)}) = \zeta_{(1,m)} + \overline{\Gamma}_m(|x_{(1,m)}|), \quad (52)$$

where $\zeta_{(1,m)}, m = 1, \ldots, M$ are any positive constants, (49) reduces to

$$\dot{V} = -\sum_{m=1}^{M}\Bigg[\sum_{j=1}^{s_m}\zeta_{(j,m)}\eta_{(j,m)}^2 + \frac{1}{2}s_m^*\min\left(\frac{\nu_{(o,m)}\sigma_m}{2}, r_m^*\right)|\epsilon_m|^2 + \frac{\gamma_{(2,m)}}{2\gamma_{(1,m)}}(\hat{\theta}_m - \overline{\theta}_m)^2 + \left(\sum_{k=1}^{M}\frac{Q_{k0}M\theta_k^2}{\overline{\theta}_k}\right)\tilde{\alpha}_{z_m}(|z_m|) - \frac{\gamma_{(2,m)}}{2\gamma_{(1,m)}}\overline{\theta}_m^2\Bigg] + \overline{\Gamma}_{\varpi}(|\varpi|) \quad (53)$$

By straightforward signal chasing and using the BIBS assumption in A2, it can be shown that all closed-loop signals remain bounded on the maximal interval of existence $[0, t_f)$ implying that $t_f = \infty$ and solutions exist for all time. Furthermore, using standard Lyapunov arguments, Theorems 1-3 follow from (53). Also, note that $\overline{\Gamma}_{\varpi}$ is a linear combination of $\Gamma_{(m,\varpi)}^2$ and $\tilde{\beta}_{(z_m,\varpi)}$. This implies that if ϖ enters into each subsystem in a linear fashion, i.e., if $\Gamma_{(m,\varpi)}(|\varpi|)$ is linear in $|\varpi|$ and $\beta_{(z_m,\varpi)}(|\varpi|)$ is quadratic in $|\varpi|$, then it can be inferred from (53) that $\mathcal{L}_2$ disturbance attenuation is achieved by picking controller parameters appropriately.

Theorem 1: Under Assumptions A1-A4, the designed dynamic compensator given by achieves Bounded-Input-Bounded-State (BIBS) stability and Input-to-Output practical Stability (IOpS) of the closed-loop system with $(x, z, r_1, \ldots, r_M, \hat{\theta}_1, \ldots, \hat{\theta}_M, \hat{x}_1, \ldots, \hat{x}_M)$ considered to be the state of the closed-loop system (where $\hat{x}_m = [\hat{x}_{(2,m)}, \ldots, \hat{x}_{(n_m,m)}]^T$), ϖ the input, and $(x_{(1,1)}, \ldots, x_{(1,M)}, z_1, \ldots, z_M)$ the output. Furthermore, practical regulation of $x_{(1,m)}, m = 1 \ldots, m$ to zero is achieved in the presence of bounded disturbances, i.e., $\sum_{m=1}^{M}|x_{(1,m)}(t)|$ can be asymptotically regulated to within as small a value as desired by appropriately tuning the controller parameters.

Theorem 2: Under Assumptions A1-A4, given any initial conditions $(x(0), z(0))$ for the overall plant state and $(r_m(0), \hat{\theta}_m(0), \hat{x}_m(0)), m = 1, \ldots, M$, for the controller states with $r_m(0) \geq 1, m = 1, \ldots, M$, if the disturbance input terms go to zero asymptotically, i.e., if $\sum_{m=1}^{M}\left[\Gamma_{(m,\varpi)}(\varpi(t)) + \beta_{(z_m,\varpi)}(\varpi(t))\right] \to 0$ as $t \to \infty$, then the signals $\overline{x}(t), z(t), e_1(t), \ldots, \hat{e}_M(t)$ where $\overline{x} = [\overline{x}_1^T, \ldots, \overline{x}_N^T]^T$ with $\overline{x}_m = [x_{(1,m)}, \ldots, x_{(s_m,m)}]^T$ go to zero asymptotically as $t \to \infty$ if the controller parameters $\gamma_{\theta m}, m = 1, \ldots, M$, are picked to be zero. Furthermore, if the BIBS Assumption A2 is strengthened to a minimum phase assumption, then $x(t), z(t), \hat{x}_1(t), \ldots, \hat{x}_M(t)$ go to zero asymptotically as $t \to \infty$.

Theorem 3: Under Assumptions A1-A4 and the additional Assumption A5 below, given any initial conditions $(x(0), z(0))$ for the overall plant state and $(r_m(0), \hat{\theta}_m(0), \hat{x}_m(0)), m = 1, \ldots, M$, for the controller states with $r_m(0) \geq 1, m = 1, \ldots, M$, the designed dynamic controller achieves boundedness of all closed-loop states.

Assumption A5: The values of $\int_0^\infty \Gamma_{(m,\varpi)}^2(|\varpi(t)|)dt$ and $\int_0^\infty \beta_{(z_m,\varpi)}(|\varpi(t)|)dt$ are finite for all $m = 1, \ldots, M$.

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