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High-Gain-Observer Tracking Performance in the Presence of Measurement Noise

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Abstract— **High-gain observers are commonly utilized to estimate the states used in constructing the output feedback control, when such states are derivatives of the output. In the presence of measurement noise, the estimation error can be noticeably compromised if the observer gain is chosen too large. However, the effect of measurement noise on the tracking error is less significant. This phenomenon is demonstrated with a nonlinear motivating example, and further explored in the context of linear systems from a transfer function perspective.**

I. INTRODUCTION

In the absence of measurement noise, high-gain observers are able to concurrently reject modeling uncertainty and rapidly estimate the system states. Although these objectives are still achieved in the presence of noise, the bound on the estimation error is altered. In [1], it is shown that the state estimation error contains one term (related to the modeling uncertainty) proportional to the observer parameter ε and another containing the noise inversely proportional to ε^{n-1}, where n is the observer dimension. A bound on the differentiation error was also estimated in [2]. The effect of measurement noise on the system states and control was briefly considered in a motivating example found in [3]. Thus, it has been shown that noise can have a large impact on the state estimation and control. Yet, simulation studies have suggested that the effect of the measurement noise on the tracking error is significantly less than the effect manifested in the estimation error.

Given the complexity of nonlinear systems and the viability of transfer functions as an analysis tool in linear systems, the purpose of this note is to investigate the aforementioned phenomenon from a linear systems perspective. In order to explicitly show the importance and prevalence of this topic, Section II begins the discussion with a nonlinear example. Section III derives the transfer functions relevant to unveiling the effect that measurement noise has on the tracking performance. Ultimately, the tracking error is shown to be uniformly bounded in the observer parameter ε.

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II. A NONLINEAR MOTIVATION

Consider the example found in [4]

$$\dot{x}_1 = x_2 \quad (1)$$
$$\dot{x}_2 = x_2^3 + u \quad (2)$$
$$y = x_1 + v \quad (3)$$

where the x_i's are the the system states, y the output, u the control, and v the measurement noise. The control objective is to have the state x_1 track a sinusoid with an amplitude of 0.1 and a frequency of 0.3 rad/s. Using standard feedback linearization techniques, the controller is chosen as $u = -\hat{x}_2^3 - (\hat{x}_1 - r) - (\hat{x}_2 - \dot{r}) + \ddot{r}$. In order to prevent peaking in the plant during the transient period, the controller is saturated outside [-1, 1]. The estimates for the output feedback controller are obtained from the high-gain observer defined as

$$\dot{\hat{x}}_1 = \hat{x}_2 + \frac{2}{\varepsilon}(y - \hat{x}_1) \quad (4)$$
$$\dot{\hat{x}}_2 = \frac{1}{\varepsilon^2}(y - \hat{x}_1) \quad (5)$$

where two separate trials are run with $\varepsilon = 0.001$ and $\varepsilon = 0.0005$. The initial conditions are set at $x_1(0) = 0.1$, $x_2(0) = \hat{x}_1(0) = \hat{x}_2(0) = 0$. Note that $x_1(0)$ and $\hat{x}_1(0)$ are deliberately chosen to be unequal to induce peaking of the transient response. The measurement noise v is generated using the Simulink block "Uniform Random Number", where the magnitude is limited to [-0.00011, 0.00011] and the sampling time is set at 0.00005 seconds.

Fig. 1 shows the steady-state response of the estimation error $x_2 - \hat{x}_2$ for the linear observers. In particular, as the value of the observer parameter ε is decreased, the magnitude of the error significantly increases. However, the error in tracking the reference signal, displayed in Fig. 2, shows no appreciable change as ε is decreased. In fact, for values of $\varepsilon \in [0.0005, 0.01]$, the steady-state response of the tracking error is restricted to the range $[-0.000029, 0.00011]$. Hence, the tracking error is uniformly bounded in ε. The same phenomenon is exhibited in linear systems with measurement noise, and will be investigated in the subsequent section.

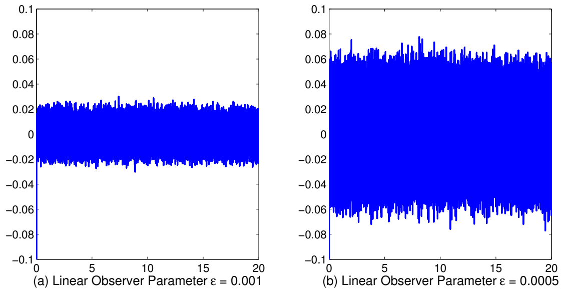

Fig. 1. Steady-state response of the error $(x_2 - \hat{x}_2)$ vs. time

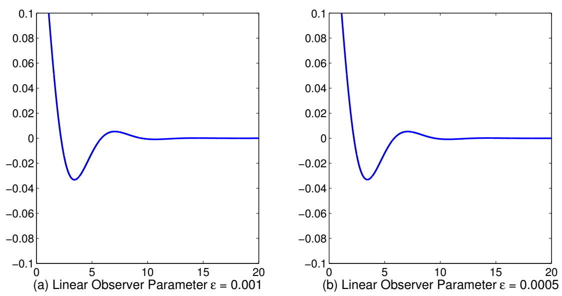

Fig. 2. Steady-state response of the tracking error $(x_1 - r)$ vs. time

III. LINEAR SYSTEM EXPLORATION

Unlike nonlinear representations, the transfer functions of a linear system can reveal how the measurement noise impacts the tracking error. Consider the system

$$\dot{x}_1 = x_2 \quad (6)$$
$$\dot{x}_2 = x_3 \quad (7)$$
$$\dot{x}_3 = a_1 x_1 + a_2 x_2 + a_3 x_3 + bu \quad (8)$$
$$y = x_1 + v \quad (9)$$

where the x_i's are the system states, y the output, and u the control. The variable v is the measurement noise. Tracking can be achieved by the state feedback controller

$$u = \frac{1}{b}[-a_1 x_1 - a_2 x_2 - a_3 x_3 + \dddot{r} - k_1(x_1 - r) - k_2(x_2 - \dot{r}) - k_3(x_3 - \ddot{r})] \quad (10)$$

where the coefficients k_1, k_2, and k_3 are chosen such that

$$s^3 + k_3 s^2 + k_2 s + k_1 \quad (11)$$

is Hurwitz. The state estimates for the output feedback controller are generated with the linear high-gain observer

$$\dot{\hat{x}}_1 = \hat{x}_2 + \frac{\alpha_1}{\varepsilon}(y - \hat{x}_1) \quad (12)$$
$$\dot{\hat{x}}_2 = \hat{x}_3 + \frac{\alpha_2}{\varepsilon^2}(y - \hat{x}_1) \quad (13)$$
$$\dot{\hat{x}}_3 = \frac{\alpha_3}{\varepsilon^3}(y - \hat{x}_1) \quad (14)$$

where the α_i's are designed such that

$$s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 \quad (15)$$

is Hurwitz. In this discussion the control is not saturated, as typically employed to avoid peaking in the plant during the transient response, because tracking performance is examined during steady-state where the saturation is not active.

The transfer functions from the noise to the tracking errors can be represented as

$$\frac{E_1(s)}{V(s)} = H_1(s,\varepsilon) \quad (16)$$
$$\frac{E_2(s)}{V(s)} = H_2(s,\varepsilon) \quad (17)$$
$$\frac{E_3(s)}{V(s)} = \frac{1}{\varepsilon}H_3(s,\varepsilon) \quad (18)$$

where in the time domain $e_1 = x_1 - r$, $e_2 = x_2 - \dot{r}$, and $e_3 = x_3 - \ddot{r}$. Realize that $H_1(s,\varepsilon)$, $H_2(s,\varepsilon)$, and $H_3(s,\varepsilon)$ are transfer functions composed from a two-time scale system. By applying the results of [5], it can be shown that the $\mathcal{H}_\infty$-norm of $H_i(s,\varepsilon)$ is of $\mathcal{O}(1)$. Therefore, the $\mathcal{H}_\infty$-norms of (16), (17), and (18) are of $\mathcal{O}(1)$, $\mathcal{O}(1)$, and $\mathcal{O}(\frac{1}{\varepsilon})$, respectively. Yet, the estimation error does not decrease as ε is made smaller [1]. In general, the $\mathcal{H}_\infty$-norm for an n-dimensional system can be expressed as

$$\left\| \frac{E_i(s)}{V(s)} \right\|_\infty = \mathcal{O}\left(\frac{1}{\varepsilon^{i-2}}\right) \quad (19)$$

where $2 < i \leq n$. In essence, the presence of measurement noise in the system affects the estimation error to a larger degree than the tracking error.

IV. CONCLUSION

When high-gain observers are employed in the presence of measurement noise, there exists a trade-off between fast state reconstruction and a reasonable state estimation error. However, this sort of compromise does not exist when the primary interest is in the system tracking error. It was argued by constructing the system transfer functions from the noise to the tracking error and its derivatives, that the error and its first derivative are bounded uniformly in ε. All subsequent tracking error derivatives are inversely proportional to some power of ε. Currently, the authors of this note are investigating the technical challenges behind showing a similar result for nonlinear systems. One aspect lies in revisiting the work of [3] to further generalize the results reported on the effect of measurement noise, by removing the restrictions pertaining to the existence of all noise derivatives.

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