2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

On the Generation of Nearly Optimal, Planar Paths of Bounded Curvature and Bounded Curvature Gradient

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Abstract— **We present a numerically efficient scheme to generate a family of path primitives that can be used to construct paths that take into consideration point-wise constraints on both the curvature and its derivative. The statement of the problem is a generalization of the Dubins problem to account for more realistic vehicle dynamics. The problem is solved by appropriate concatenations of line segments, circular arcs and pieces of clothoids, which are the path primitives in our scheme. Our analysis reveals that the use of clothoid segments, in addition to line segments and circular arcs, for path generation introduces significant changes on issues such as path admissibility and length minimality, when compared with the standard Dubins problem.**

I. Introduction

A path generation scheme for a vehicle operating in an obstacle-free environment has to meet two basic objectives. First, the generated path should be compatible to the vehicle dynamics, and second, the path should minimize an appropriate objective function. Typically, the objective function to be minimized is the total length of the path, whereas the vehicle's dynamics may be incorporated into the path generation scheme by considering point-wise constraints over the curvature of the path. In particular, curvature-constrained planar paths of minimal length, and with prescribed initial and final positions and tangents, for a vehicle that travels only forward, have been characterized by Dubins [1], and for a vehicle that moves both forward and backwards by Reeds and Shepp [2]. Sussmann and Tang [3] and Boissonnat et al [4] treated the same problem and provided more general and rigorous proofs using Pontryagin's Maximum Principle. The path synthesis problem for the Dubins vehicle was studied by Bui et al in [5] and for the Reeds-Shepp vehicle was addressed by Souères and Laumond in [6].

One drawback of the aforementioned path planning schemes is the discontinuous curvature profiles of the paths they typically generate. In practice, discontinuous curvature profiles induce poor tracking performance [7]. The main source of this poor performance is the latency associated to the steering command inputs of typical ground vehicles [8]. Boissonnat et al in [9], and Sussmann in [10] investigated the continuous-curvature, shortest path problem with a constraint on the derivative of the curvature of the path using optimal control techniques. Constraints on both the curvature and the derivative of the curvature of the path were taken into account for the continuous curvature extension of the Dubins vehicle in [11] and the Reeds-Shepp vehicle in [12]. A recent result for planar shortest paths with free terminal position under acceleration constraints is given in [13].

In this paper, we present a scheme to generate a sufficiently rich family of path primitives that can be used to synthesize near length-optimal paths with bounded curvature and bounded derivative of the curvature when the vehicle can move only forward. The developments in the paper are similar to those of Scheuer and Laugier [11], where the authors investigated the problem of a forward-only moving vehicle on curvature-constrained paths. In contrast to the approach of [11], however, the path concatenation scheme proposed in this paper is based on some recent analytic/computational geometry techniques on clothoid construction and, in particular, the work by Meek and Walton [14]. The use of these computational techniques results in a numerically more efficient scheme for constructing the concatenated path. In particular, it is reminded that a concatenation of a circular arc and a clothoid segment requires the solution of a nonlinear algebraic equation involving a Fresnel integral. The work of [14] provides a numerical technique to solve this algebraic equation with strong convergence properties. Furthermore, the analytic/computational geometry approach we adopt in this paper allows us to associate the path planning problem with the standard Dubins problem by means of a family of geometric transformations. This provides a natural and straightforward manner to construct feasible (albeit length-suboptimal) paths from the initial to the final configuration. Finally, we investigate the implications of imposing constraints on both the curvature and its derivative over the admissibility and length minimality of the path. Our analysis illustrates the intrinsic difficulties of the shortest path problem with curvature and curvature gradient bounds when compared with the standard Dubins problem.

We need to point out that the–much more interesting–synthesis problem for arbitrary initial and final conditions is not addressed in this paper and is left for future investigation. As far as we know this is still an open problem. Given the complexity of the synthesis problem even for the case of the standard Dubins vehicle [5] it is expected that the solution to the synthesis problem of the path planning problem addressed in this paper will most likely turn out to be rather involved.

II. Kinematic Model and Problem Formulation

In this paper we are interested in the solution of the planar, shortest-path problem with prescribed initial and final positions, tangents and curvatures, when both the curvature

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of the path and its derivative are bounded by explicit bounds known a priori. To address this problem, we employ a continuous curvature extension of the Dubins vehicle, which is described by the following set of equations [9]

$$\dot{x}(t) = \cos\theta(t), \tag{1}$$
$$\dot{y}(t) = \sin\theta(t), \tag{2}$$
$$\dot{\theta}(t) = \kappa(t), \tag{3}$$
$$\dot{\kappa}(t) = u(t), \tag{4}$$

where x, y are the cartesian coordinates of a reference point of the vehicle, θ is the vehicle's orientation (always tangent to the ensuing path), κ is the curvature of the ensuing path, and u (steering acceleration) is the control input. We assume that the set of admissible control inputs is given by

$$\mathcal{U} \triangleq \{u \in \mathfrak{U} \ : \ u(t) \in U, \ t \in [0, T_f]\}, \tag{5}$$

where $\mathfrak{U}$ is the set of all measurable functions defined over the interval $[0, T_f]$, $U \triangleq [-\gamma_{\max}, \gamma_{\max}]$, and $\gamma_{\max}$ is the maximum curvature gradient. We assume furthermore that the curvature κ satisfies the following constraint

$$|\kappa| \leq \kappa_{\max}. \tag{6}$$

To this end, we formulate the shortest path problem as a minimum-time control problem.

Problem 1: Given the system described by equations (1)-(4) and the cost functional

$$J(u) = \int_0^{T_f} 1 \, \mathrm{d}t = T_f, \tag{7}$$

where T_f is the free final time, determine the control input $u^* \in \mathcal{U}$, with $u^* : [0, T_f] \mapsto U$, such that

1) The trajectory $\mathsf{x}^* : [0, T_f] \mapsto \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$, with $\mathsf{x}^*(t) = (x^*(t), y^*(t), \theta^*(t), \kappa^*(t))$, generated by the control u^* satisfies
   a) The boundary conditions
   $$\mathsf{x}^*(0) = (x_0, y_0, \theta_0, \kappa_0), \tag{8}$$
   $$\mathsf{x}^*(T_f) = (x_f, y_f, \theta_f, \kappa_f). \tag{9}$$
   b) The global point-wise state constraint (6).
2) The control u^* minimizes the cost functional $J(u)$ given in (7).

III. Generation of G^2 Continuous Paths using Path Primitives

Boissonnat et al [9] and Sussmann [10] have examined Problem 1 when the state constraint (6) is not taken into account. In this special case, as shown in [9], [10], the problem is always feasible; however, when a line segment is part of the optimal path, then the corresponding optimal control u^* may switch infinitely fast (chattering). Therefore, the solution of Problem 1 is likely to be irregular as well. Chattering optimal controllers cannot be implemented easily in practice, only their approximations are possible [12].

If the bound on the derivative of the curvature is not necessarily finite, then as $\gamma_{\max} \to \infty$, Problem 1 reduces to the Dubins problem, the solution of which is characterized in [1]. For many applications, however, Dubins paths are not suited for path tracking since their discontinuous curvature profile induces an offset tracking error [7]. In this work, we use pieces of clothoid curves to allow smooth transitions between path arcs of different curvature. Clothoids are curves, whose curvature is an affine function of the arc length s, i.e., $\kappa(s) = \kappa_0 \pm \kappa_1 s$, where κ_0 and κ_1 are non-negative constants.

In this section we demonstrate a geometric approach to deal with Problem 1. The key idea of our approach is to associate the G^2 path generation problem with the standard Dubins problem by means of a family of geometric transformations. This approach allows us to characterize in closed form the paths that solve Problem 1 in a near-optimal fashion. To this end, we first present some of the basic properties of clothoids that we will use later on.

A. Clothoid Curves

The standard clothoid with scaling factor σ is expressed naturally in terms of the angle of its tangent ϑ. In particular, for $\vartheta > 0$ the coordinates of the standard clothoid curve are given by [14]

$$x(\vartheta) = \sigma \int_0^{\vartheta} \frac{\cos\tau}{\sqrt{\tau}} \mathrm{d}\tau, \quad y(\vartheta) = \sigma \int_0^{\vartheta} \frac{\sin\tau}{\sqrt{\tau}} \mathrm{d}\tau. \tag{10}$$

The curvature and the arc length of the clothoid as a function of the angle ϑ are given by

$$\kappa(\vartheta) = \frac{\sqrt{\vartheta}}{\sigma}, \quad \mathrm{d}s = \frac{\sigma}{\sqrt{\vartheta}} \, \mathrm{d}\vartheta. \tag{11}$$

It follows from (11) that $|\mathrm{d}\kappa/\mathrm{d}s| = 1/2\sigma^2$.

B. Family of Admissible Paths

In this section we examine the admissible paths of our scheme. First, we consider the problem of driving a vehicle whose kinematics are described by equations (1)-(4) from the initial configuration $\mathsf{x}_0 = (x_0, y_0, 0, 0)$ to the terminal configuration $\mathsf{x}_f = (x_f, y_f, \theta_f, 1/\rho_1)$ as shown in Fig. 1. This problem is equivalent to driving the vehicle traversing a line segment ϵ_2 to a circle $\mathcal{C}_1(\rho_1)$ of radius ρ_1 in finite time. The vehicle is initially located at $(x_0, y_0) \in \epsilon_2$ with orientation $\theta_0 = 0$ and the final position is $(x_f, y_f) \in \mathcal{C}_1(\rho_1)$ with orientation θ_f. The line segment ϵ_2 and the circle $\mathcal{C}_1(\rho_1)$ are connected with the clothoid $\mathcal{K}_1$. The line segment ϵ_2 is assumed to be parallel to a line segment ϵ_1 that is tangent to the circle $\mathcal{C}_1(\rho_1)$ at some point $\mathsf{A} \in \mathcal{C}_1(\rho_1)$. Let $\delta_s > 0$ be the distance between the lines ϵ_1 and ϵ_2 as shown in Fig. 1.

Given the radius of the circle ρ_1, the angle ϑ_1 of the tangent at the intersection point B of the clothoid $\mathcal{K}_1$ with $\mathcal{C}_1(\rho_1)$ and the scaling factor σ_1 of $\mathcal{K}_1$ are given by the following equations [14]

$$\sigma_1 - \rho_1 \sqrt{\vartheta_1} = 0, \tag{12}$$

$$\sqrt{\vartheta_1} \int_0^{\vartheta_1} \frac{\sin\tau}{\sqrt{\tau}} \mathrm{d}\tau + \cos\vartheta_1 - \left(1 + \frac{\delta_s}{\rho_1}\right) = 0. \tag{13}$$

In this work we confine ourselves to the case when $\vartheta_1 \in (0, \pi/2]$ for reasons that will become clear later on. Furthermore, we take δ_s to be a design parameter. In particular, the larger the value of δ_s the smaller the maximum value of $|\mathrm{d}\kappa/\mathrm{d}s|$ along the path from x_0 to x_f, hence the less stringent the condition on the gradient of the curvature. Once δ_s is given, σ_1 and ϑ_1 are derived uniquely from (12) and (13), as shown in the following proposition.

Proposition 1: Given $\rho_1 > 0$, the system of equations (12)-(13) has a unique solution $(\sigma, \vartheta_1) \in (0, \rho_1\sqrt{2\pi}/2] \times [0, \pi/2]$, provided that

$$\delta_s \in \mathcal{I}_\delta \triangleq \left(0, \rho_1 \left(\frac{\sqrt{2\pi}}{2}\int_0^{\pi/2} \frac{\sin\tau}{\sqrt{\tau}}\,\mathrm{d}\tau - 1\right)\right). \quad (14)$$

Proof: Consider the function $f : [0, \pi/2] \mapsto \mathbb{R}$, where

$$f(\vartheta) = \rho_1 \left(\sqrt{\vartheta}\int_0^{\vartheta} \frac{\sin\tau}{\sqrt{\tau}}\mathrm{d}\tau + \cos\vartheta - 1\right). \quad (15)$$

It follows from Bolzano's intermediate value theorem [15], that for $\vartheta \in [0, \pi/2]$ the function $f(\vartheta)$ takes all the values between $f(0) = 0$ and $f(\pi/2) = \rho_1 \left(\frac{\sqrt{2\pi}}{2}\int_0^{\pi/2} \frac{\sin\tau\mathrm{d}\tau}{\sqrt{\tau}} - 1\right)$. Thus, for $\delta_s \in \left(0, \rho_1 \left(\frac{\sqrt{2\pi}}{2}\int_0^{\pi/2} \frac{\sin\tau\mathrm{d}\tau}{\sqrt{\tau}} - 1\right)\right)$, the equation (13) has at least one solution in $[0, \pi/2]$. Furthermore, the function f is monotonically increasing since $f'(\vartheta) > 0$ for all $\vartheta \in (0, \pi/2)$. Thus, f is injective and the solution of the system of equations (12)-(13) is unique. ■

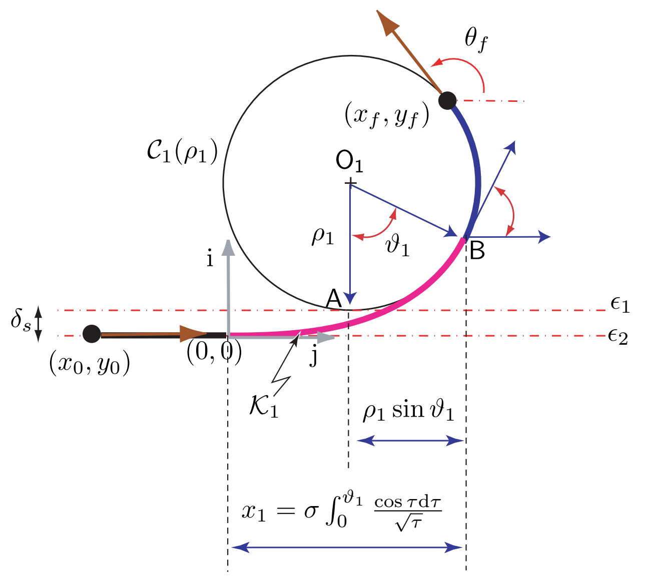

Fig. 1. Interconnecting a piece of clothoid with a line segment ($\mathsf{SC}^+(\rho_1, \delta_s)$ path). The path that corresponds to the near optimal solution to the steering Problem 1 is composed of a line segment, a piece of clothoid and a circular arc.

We write $\mathsf{SC}^+(\rho_1, \delta_s) \triangleq \epsilon_2 \circ \mathcal{K}_1 \circ \mathcal{C}_1$, where $\circ$ denotes concatenation of curves, to denote the G^2 continuous path shown in Fig. 1, and $\mathsf{SC}^+(\rho_1) \triangleq \epsilon_1 \circ \mathcal{C}_1$ to denote the corresponding Dubins path. This path constitutes a G^2 continuous *half turn* that plays the role of the archetype that we use to devise the rest of the admissible paths of our scheme. Throughout the sequel, the superscripts $+$ and $-$ denote arcs of positive and negative curvature respectively.

Let us now consider the path planning problem from the initial configuration $\mathsf{x}_0 = (x_0, y_0, \theta_0, -1/\rho_1)$ to the terminal configuration $\mathsf{x}_f = (x_f, y_f, \theta_f, 1/\rho_2)$ as shown in Fig. 2. The ensuing path is assumed to be the concatenation of two circles $\mathcal{C}_1(\rho_1)$ and $\mathcal{C}_2(\rho_2)$ and the interconnected composite curve, which is composed, in turn, of a piece of a clothoid curve $\mathcal{K}_1$, a piece of the line segment ϵ_2 and a piece of a second clothoid curve $\mathcal{K}_2$. Let $\epsilon_1 = \{(x, y)|\ A_1x + B_1y + C = 0\}$ be the common tangent line of $\mathcal{C}_1(\rho_1)$ and $\mathcal{C}_2(\rho_2)$ that is part of the Dubins path from x_0 to x_f as shown in Fig. 2. Let also $\epsilon_2 = \{(x, y)|\ A_2x+B_2y+C = 0\}$ be the line obtained after rotating ϵ_1 about the point P at an angle δ_r, where the pivot point P is the intersection of the line ϵ_1 and the line segment $\mathsf{O}_1\mathsf{O}_2$. The minimum distances $\delta_{s,1}$ and $\delta_{s,2}$ of the line ϵ_2 from the circles $\mathcal{C}_1(\rho_1)$ and $\mathcal{C}_2(\rho_2)$ are given by $\delta_{s,i} = |A_2X_i + B_2Y_i + C_2|/\sqrt{A_2^2 + B_2^2} - \rho_i,\ i = 1, 2,$ where (X_1, Y_1) and (X_2, Y_2) are the coordinates of the centers of $\mathcal{C}_1(\rho_1)$ and $\mathcal{C}_2(\rho_2)$ respectively. The composite curve that connects the two circles is the concatenation of a G^2 continuous SC^- and SC^+ path, as demonstrated in Fig. 1, with δ_s replaced by $\delta_{s,1}$ and $\delta_{s,2}$, respectively, where $\delta_{s,1}/\delta_{s,2} = \rho_1/\rho_2$. The scaling factors σ_1 and σ_2 of the clothoid curves $\mathcal{K}_1$ and $\mathcal{K}_2$ are specified by algebraic equations similar to those given in (12) and (13). We write $\mathsf{C}^-\mathsf{SC}^+(\rho_1, \rho_2, \delta_r) \triangleq \mathcal{C}_1 \circ \mathcal{K}_1 \circ \epsilon_2 \circ \mathcal{K}_2 \circ \mathcal{C}_2$, to denote this G^2 continuous path, and $\mathsf{C}^-\mathsf{SC}^+(\rho_1, \rho_2) = \mathcal{C}_1 \circ \epsilon_1 \circ \mathcal{C}_2$ to denote the corresponding Dubins path.

Let us now consider the path planning problem from the initial configuration $\mathsf{x}_0 = (x_0, y_0, \theta_0, -1/\rho_1)$ to the terminal configuration $\mathsf{x}_f = (x_f, y_f, \theta_f, -1/\rho_2)$ as depicted in Fig. 3. The line $\epsilon_1 = \{(x, y)|\ A_1x + B_1y + C_1 = 0\}$ is tangent to both $\mathcal{C}_1(\rho_1)$ and $\mathcal{C}_2(\rho_2)$ and the line $\epsilon_2 = \{(x, y)|\ A_2x + B_2y + C_2 = 0\}$ is parallel to ϵ_1 at a distance δ_s that is given by $\delta_s = |C_1 - C_2|/\sqrt{A_1^2 + B_1^2}$. The composite curve in this case is the concatenation of two G^2 continuous SC^- paths. We write $\mathsf{C}^-\mathsf{SC}^-(\rho_1, \rho_2, \delta_s) \triangleq \mathcal{C}_1 \circ \mathcal{K}_1 \circ \epsilon_2 \circ \mathcal{K}_2 \circ \mathcal{C}_2$, to denote the G^2 continuous path, and $\mathsf{C}^-\mathsf{SC}^-(\rho_1, \rho_2) = \mathcal{C}_1 \circ \epsilon_1 \circ \mathcal{C}_2$ to denote the corresponding Dubins path.

Finally, in Fig. 4 three circles, namely $\mathcal{C}_1(\rho_1), \mathcal{C}_2(\rho_2)$ and $\mathcal{C}_3(\rho_3)$, which are located at points $\mathsf{O}_1, \mathsf{O}_2$ and O_3 respectively, are interconnected by means of a $\mathsf{C}^-\mathsf{SC}^+$ or a $\mathsf{C}^+\mathsf{SC}^-$ path. The corresponding Dubins path is the concatenation of $\mathcal{C}_1(\rho_1)$ and $\mathcal{C}_3(\rho_1)$ with the common tangent circle $\mathcal{C}_2(\rho_2)$ located at O_2', where $\rho_2 = \min\{\rho_1, \rho_3\}$. By construction, the straight line passing through O_2' and O_2 bisects the line segment $\mathsf{O}_1\mathsf{O}_3$ at point A. Let δ_c denote the length of the segment $\mathsf{O}_2'\mathsf{O}_2$. For $\delta_c > 0$ the circles $\mathcal{C}_1(\rho_1)$ and $\mathcal{C}_2(\rho_2)$ can be interconnected by means of a $\mathsf{C}^-\mathsf{SC}^+$ path whereas the circles $\mathcal{C}_2(\rho_2)$ and $\mathcal{C}_3(\rho_3)$ can be interconnected by means of a $\mathsf{C}^+\mathsf{SC}^-$ path. We write $\mathsf{C}^-\mathsf{C}^+\mathsf{C}^-(\rho_1, \rho_2, \rho_3, \delta_\mathsf{C}) = \mathsf{C}^-\mathsf{SC}^+(\rho_1, \rho_2, \delta_{r,12}) \circ \mathsf{C}^+\mathsf{SC}^-(\rho_2, \rho_3, \delta_{r,23})$. Finally, we write $\mathsf{C}^-\mathsf{C}^+\mathsf{C}^-(\rho_1, \rho_2, \rho_3) = \mathcal{C}_1 \circ \mathcal{C}_2 \circ \mathcal{C}_3$ to denote the corresponding Dubins path.

IV. Path Admissibility and Length Minimality

The issue of admissibility of each element of the family of admissible paths introduced in Section III for different boundary configurations requires special attention. For simplicity, in this section we analyze in detail only the admis-

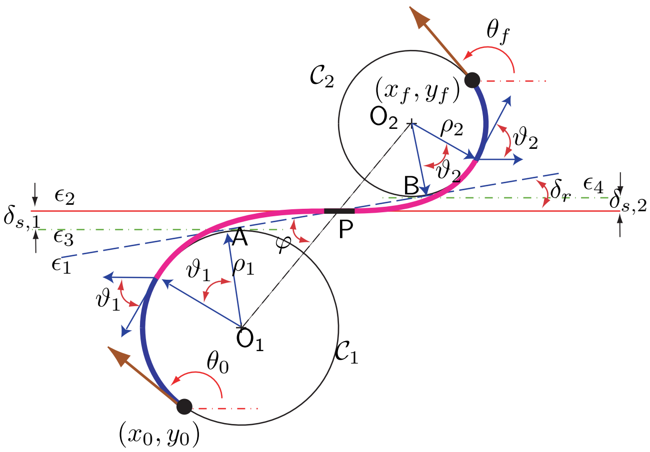

Fig. 2. Interconnecting two circles using clothoids traversed at opposite clock-wise orientation via a $\mathsf{C}^-\mathsf{S}\mathsf{C}^+(\rho_1, \rho_2, \delta_r)$ path.

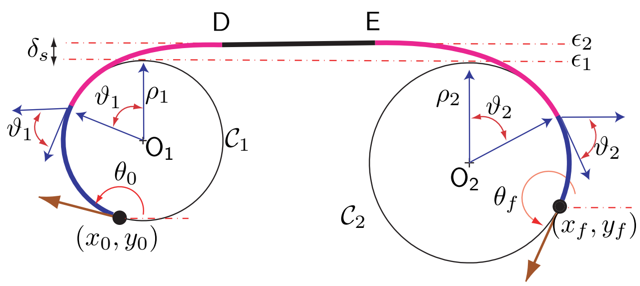

Fig. 3. Interconnecting two circles using clothoids traversed at the same clock-wise orientation via a $\mathsf{C}^-\mathsf{S}\mathsf{C}^-(\rho_1, \vartheta_1, \rho_2, \vartheta_2, \delta)$ path.

sibility of $\mathsf{C}^-\mathsf{S}\mathsf{C}^-$ paths. After the necessary modifications, similar results can be derived *mutatis mutandis* for the rest of the path primitives introduced in Section III.

The path admissibility is mainly related to the G^2 continuity requirement, which, in turn, reflects the smooth, forward-only motion requirement for the vehicle (paths with cusps or corners fail to satisfy the G^2 continuity requirement). In particular, we characterize any steering problem in which the two pieces of clothoids of the $\mathsf{C}^-\mathsf{S}\mathsf{C}^-$ path intersect as inadmissible. Note that if we do allow an intersection between the two clothoid curves to take place, then the vehicle would have to track either a curve with a corner, something that would result to a violation of the kinematic equations (1)-(3), or a path with an arc of backward motion along the line ϵ_2 and between the endpoints D and E of the two clothoid curves, where D is now aft of E compared to the situation depicted in Fig. 2. Before analyzing the connection between the G^2 continuity condition and the admissibility of the $\mathsf{C}^-\mathsf{S}\mathsf{C}^-$ path, we introduce the following lemma.

Lemma 1: For $\vartheta \in [0, \pi/2]$ we have that

$$\vartheta - \frac{\vartheta^3}{5} \leq \sqrt{\vartheta} \int_0^{\vartheta} \frac{\sin \tau}{\tau^{\frac{3}{2}}} \mathrm{d}\tau \leq \vartheta + \frac{\vartheta^3}{6}. \tag{16}$$

Proof: The result follows readily by integrating by parts $\int_0^{\vartheta} \frac{\sin \tau}{\tau^{\frac{3}{2}}} \mathrm{d}\tau$, and by virtue of the Taylor expansion theorem for the cosine and sine functions in $[0, \pi/2]$. ■

Proposition 2: The $\mathsf{C}^-\mathsf{S}\mathsf{C}^-$ path is admissible only if the distance L between the centers of $\mathcal{C}_1(\rho_1)$ and $\mathcal{C}_2(\rho_2)$ satisfy the following condition

$$L \geq \sqrt{(\rho_1 - \rho_2)^2 + \frac{1}{4} \left(\sum_{i=1}^{2} \rho_i \left(\vartheta_i - \frac{(\vartheta_i)^3}{5} \right) \right)^2}. \tag{17}$$

Proof: With the aid of Fig. 5 we observe that the two pieces of clothoids do not intersect only if $\Delta x_1 + \Delta x_2 \leq \sqrt{L^2 - (\rho_1 - \rho_2)^2}$, where $\Delta x_i = \sigma_i \int_0^{\vartheta_i} \frac{\cos \tau}{\sqrt{\tau}} \mathrm{d}\tau - \rho_i \sin \vartheta_i$, $(i = 1, 2)$. After integrating by parts the integral the result follows by virtue of Lemma 1. ■

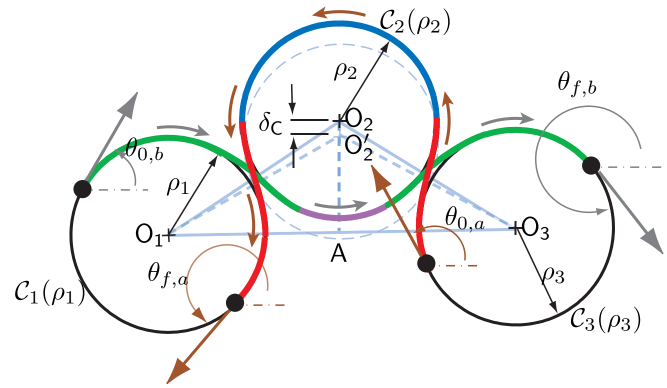

Fig. 4. Each $\mathsf{C}^-\mathsf{C}^+\mathsf{C}^-$ path is a concatenation of a $\mathsf{C}^-\mathsf{S}\mathsf{C}^+$ and a $\mathsf{C}^+\mathsf{S}\mathsf{C}^-$ path. The $\mathsf{C}^-\mathsf{C}^+\mathsf{C}^-$ path that solves the steering problem from $\theta_{0,a}$ to $\theta_{f,a}$ and $\theta_{0,b}$ and $\theta_{f,b}$ corresponds to an optimal and suboptimal Dubins path respectively.

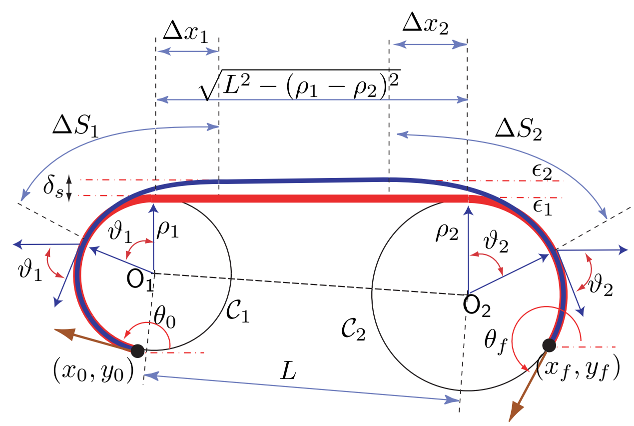

Fig. 5. Total length of a composite curve containing two pieces of clothoids.

It is important to examine how the use of clothoids affects the length minimality of a G^2 continuous $\mathsf{C}^-\mathsf{S}\mathsf{C}^-$ path compared with its corresponding Dubins path. In particular, equations (5) and (12) place a restriction over the minimum allowable value of the tangent angles ϑ_i as follows

$$\vartheta_i \geq \vartheta_{i,\min} \triangleq \frac{1}{2\gamma_{\max}\rho_i^2}, \quad i = 1, 2. \tag{18}$$

The constraint (18), in turn, restricts the set of initial configurations x_0 associated to $\mathcal{C}_1(\rho_1)$ and/or terminal configurations x_f associated to $\mathcal{C}_2(\rho_2)$ for which a nearly optimal, admissible $\mathsf{C}^-\mathsf{S}\mathsf{C}^-$ path exists. Before we address this problem in more detail we need the following definitions.

Definition 1: A path from x_0 to x_f is defined as a *weakly admissible* $\mathsf{C}^-\mathsf{SC}^-$ path if and only if the difference between the total length of this path and the corresponding Dubins path is bounded from below by a strictly positive quantity that does not depend on neither $\vartheta_{1,\min}$ nor $\vartheta_{2,\min}$. It is defined as *strongly admissible* otherwise.

In Fig. 6 we consider the steering problem from $\mathsf{x}_0 = (x_0, y_0, \theta_0, -1/\rho_1)$ to $\mathsf{x}_f = (x_f, y_f, \theta_f, -1/\rho_2)$. We observe that if $\theta_0 \in [0, \vartheta_{1,\min})$ and $\theta_f \in [0, 2\pi]$, or $\theta_0 \in [0, 2\pi]$ and $\theta_f \in (2\pi - \vartheta_{2,\min}, 2\pi]$, then x_0 and x_f can be connected by means of *weakly admissible* $\mathsf{C}^-\mathsf{SC}^-$ paths only. However, there always exists a *strongly admissible* $\mathsf{C}^-\mathsf{SC}^-$ path if $\theta_0 \in [\vartheta_{1,\min}, 2\pi)$ and $\theta_f \in (0, 2\pi - \vartheta_{2,\min}]$. It is possible that a set of boundary conditions that cannot be connected by a strongly admissible $\mathsf{C}^-\mathsf{SC}^-$ path may admit , for example, a strongly admissible $\mathsf{C}^-\mathsf{C}^+\mathsf{C}^-$ path.

The characterization of the type of admissible paths that solves a given steering problem in a nearly optimal fashion, if such a path exists, constitutes the synthesis problem. In this paper we do not address this problem.

The above observations reveal the intrinsic difficulties associated with the curvature-constrained, shortest path problem when one takes into account a constraint over the curvature derivative for a vehicle allowed to move only forward. Next, we compare the total length of a Dubins path and the corresponding G^2 continuous path in case the latter path is strongly admissible.

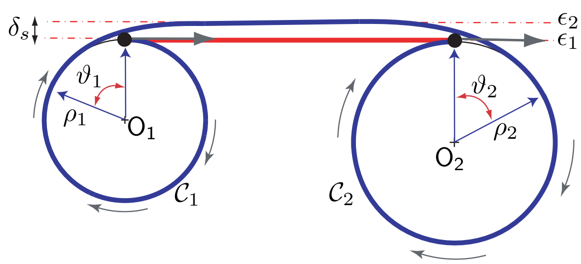

Fig. 6. A G^2 continuous path is not necessarily associated to a Dubins path in a near optimal fashion.

V. Comparison of Strongly Admissible G^2 Continuous Paths And The Corresponding Dubins Paths

A question that naturally rises from the previous analysis is how suboptimal, in terms of total length, is each of the G^2 continuous paths that we introduced in Section III, compared to the corresponding minimum-length Dubins path, in case the G^2 continuous path is strongly admissible. For the $\mathsf{C}^-\mathsf{SC}^-$ paths, and with the aid of Fig. 5, we can easily show that

$$S = \rho_1\theta_0 + \rho_2(2\pi - \theta_f) + \sqrt{L^2 - (\rho_1 - \rho_2)^2}, \tag{19}$$

$$S' = S + (\rho_1\vartheta_1 + \rho_2\vartheta_2) - \frac{1}{2}\sum_{i=1}^{2} \rho_i\sqrt{\vartheta_i}\int_0^{\vartheta_i} \frac{\sin\tau}{\tau^{\frac{3}{2}}}\mathrm{d}\tau, \tag{20}$$

where S and S' denotes the total length of the Dubins path and the corresponding composite G^2 continuous path respectively.

We next investigate whether a Dubins path $c^* = \mathsf{C}^-\mathsf{SC}^-(\rho, \rho)$ can be approximated as the limit of a sequence of strongly admissible $\mathsf{C}^-\mathsf{SC}^-(\rho, \rho, \delta_s)$ paths. The analysis for the other admissible paths follow similarly. Note that for $\rho_1 = \rho_2 = \rho$ and for given δ_s, equations (12)-(13) imply that $\sigma_1 = \sigma_2 = \sigma$ and $\vartheta_1 = \vartheta_2 = \vartheta$. Thus, the two pieces of the clothoid curves $\mathcal{K}_1$ and $\mathcal{K}_2$ differ only in terms of a plane isometry.

Proposition 2 implies that, given the distance L between $\mathcal{C}_1$ and $\mathcal{C}_2$, the G^2 continuity condition places a restriction over the maximum allowable value of ϑ, i.e., $\vartheta \leq \vartheta_{\max}$, where $\vartheta_{\max} \in (0, \pi/2]$. The previous inequality, in conjunction with inequality (18), characterize the set of admissible values of ϑ. This set, in turn, characterizes uniquely, via equation (13), the set of admissible values for the parameter δ_s of the path $\mathsf{C}^-\mathsf{SC}^-$, i.e., $\delta_s \in (\delta_{s,\min}, \delta_{s,\max}) \subset \mathcal{I}_\delta$, where $\delta_{s,j} = f(\vartheta_j)$, $j \in \{\min, \max\}$ and $\mathcal{I}_\delta$ and the function f are defined as in (14) and (15) respectively. To this end, let us consider the sequence $\{\delta_s^0, \delta_s^1, \ldots\} \in (\delta_{s,\min}, \delta_{s,\max})$ and the corresponding sequence $\{\vartheta_0, \vartheta_1, \ldots\} \in (\vartheta_{\min}, \vartheta_{\max})$, where, for each $n = 0, 1, \ldots$, the terms ϑ_n, δ_s^n of the two sequences are uniquely related by equation (13).

Proposition 3: The sequence $\{\delta_s^0, \delta_s^1, \ldots\}$ is non-increasing if and only if the corresponding sequence $\{\vartheta_0, \vartheta_1, \ldots\}$ is non-increasing.

Proof: The proof follows easily from the continuous function f being monotonically increasing. ∎

To investigate the relation between the G^2 continuous path $\mathsf{C}^-\mathsf{SC}^-(\rho, \rho, \delta_s)$ with the corresponding Dubins path c^*, we relax the constraint over the maximum allowable curvature gradient by letting $\delta_{s,\min} = \vartheta_{\min} = 0$. The following result is evident from the continuity of f along with the continuity of its inverse.

Proposition 4: Let $\{\delta_s^0, \delta_s^1, \ldots\}$ be a non-increasing sequence of positive real numbers. Then $\lim_{n\to\infty}\delta_s^n = 0$ if and only if $\lim_{n\to\infty}\vartheta_n = 0$.

Given $\rho > 0$, the sequence $\{\delta_s^0, \delta_s^1, \ldots\}$ induces through equations (12)-(13) a sequence of G^2 continuous, strongly admissible paths $\{c_0, c_1, \ldots\}$, where $c_n = \mathsf{C}^-\mathsf{SC}^-(\rho, \rho, \delta_s^n)$.

Proposition 5: Let $\{\delta_s^0, \delta_s^1, \ldots\}$ be a non-increasing sequence of positive real numbers, with $\lim_{n\to\infty}\delta_s^n = 0$. Then the sequence of G^2 continuous, strongly admissible paths $\{c_0, c_1, \ldots\}$ converges uniformly to the Dubins path $c^* = \mathsf{C}^-\mathsf{SC}^-(\rho, \rho)$.

Proof: Since $\theta_n \in (0, \pi/2)$ the path segments from $(x_0, y_0, \theta_0, -1/\rho)$ to $(x_0', y_0', \pi/2, -1/\rho)$ and from $(x_f', y_f', 3\pi/2, -1/\rho)$ to $(x_f, y_f, \theta_f, -1/\rho)$ for c^* and c_n coincide for all $n = 1, 2, \ldots$, as shown in Fig. 7. Thus, without loss of generality we consider the steering problem from $(x_0', y_0', \pi/2, -1/\rho)$ to $(x_f', y_f', 3\pi/2, -1/\rho)$. Let $d(z, y) = \sup_{\mathcal{I}}\big|\|z(t)\| - \|y(t)\|\big|$ be the metric function of the space $G^2(\mathcal{I})$ induced by the uniform norm. It follows that

$$\sup_{\mathcal{I}_x}\big|\|c_n(x)\| - \|c^*(x)\|\big| = \delta_s^n, \tag{21}$$

where $\mathcal{I}_x$ is the subinterval of the real line that corresponds to the projection of c^* on the x-axis. It follows that $d(c_n, c^*) = \delta_s^n$. Therefore, $\lim_{n\to\infty} d(c_n, c^*) = 0$. ∎

Thus, by constructing a non-increasing sequence $\{\delta_s^0, \delta_s^1, \dots\}$ we can approximate the Dubins curve $c^* = \mathsf{C}^-\mathsf{S}\mathsf{C}^-(\rho, \rho)$ by a sequence of G^2 continuous, strongly admissible paths $\{c_0, c_1, \dots\}$. Furthermore, equation (21) implies that the rate of convergence of $\{c_0, c_1, \dots\}$ to c^* is exactly the rate of convergence of the sequence $\{\delta_s^0, \delta_s^1, \dots\}$ to zero. The situation is depicted Fig. 7.

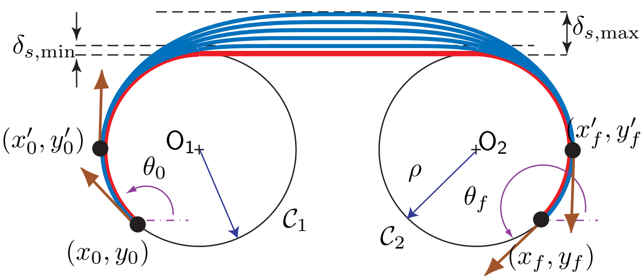

Fig. 7. Approximating a Dubins path by a sequence of G^2 continuous curves.

Equation (20) gives the total length of each curve in $\{c_0, c_1, \dots\}$. In particular,

$$S_n' - S = 2\rho\vartheta_n - \rho\sqrt{\vartheta_n}\int_0^{\vartheta_n} \frac{\sin\tau}{\tau^{\frac{3}{2}}}\,\mathrm{d}\tau, \quad n = 0, 1, \dots \quad (22)$$

Proposition 6: The sequence $\{S_0', S_1', \dots\}$ converges to S. Furthermore, $\rho m(\vartheta_N) \le S_N' - S \le \rho M(\vartheta_N)$, where

$$m(\vartheta_N) = \vartheta_N - \frac{\vartheta_N^3}{6}, \quad M(\vartheta_N) = \vartheta_N + \frac{\vartheta_N^3}{5}. \quad (23)$$

Proof: For any $N \in \{0, 1, \dots\}$ it follows by (22) and Proposition 1 that $\rho m(\vartheta_N) \le S_N' - S^* \le \rho M(\vartheta_N)$. Finally, as $N \to \infty$ both $m(\vartheta_N)$ and $M(\vartheta_N)$ go to zero. Thus, $S_N' \to S$ as $N \to \infty$. ■

Given $\rho, L > 0$, Proposition 6 implies that for large values of ϑ_N, or equivalently of δ_s^N, the relative error between the total length of c_N and the Dubins curve c^* increases rapidly.

VI. Conclusions

In this paper we present a path-planning scheme for the generation of smooth, continuous curvature, planar paths composed of line segments, pieces of clothoids, and circular arcs that have bounded curvature and bounded curvature gradient. Our analysis has revealed that the introduction of the constraint over the curvature gradient in conjunction with forward motion requirement increases the complexity of the shortest-path problem. The synthesis of nearly optimal, G^2 continuous paths for arbitrary boundary configurations, requires further and more thorough analysis, and shall be the focus of future research.

Acknowledgement: This work has been supported in part by NSF (award no. CMMI-0727768) and ARO (award no. W911NF-05-1-0331). The second author also acknowledges support from the A. Onassis Public Benefit Foundation.

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