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Stability Analysis of Positive Feedback Interconnections of Linear Negative Imaginary Systems

Junlin Xiong Ian R. Petersen Alexander Lanzon

Abstract— **The paper is concerned with the stability analysis of positive feedback interconnections of negative imaginary systems. Firstly, a previously established Negative Imaginary Lemma is shown to remain true even if the transfer function has poles on the imaginary axis. This is achieved by suitably extending the definitions of negative imaginary transfer functions. Secondly, a necessary and sufficient condition is established for the internal stability of the feedback interconnection of negative imaginary systems. Moreover, some properties of negative imaginary transfer functions are developed. Finally, a numerical example is presented to illustrate the theory.**

I. Introduction

Positive real systems have achieved great successes both in theory and in practice [2], [3]. Systems which dissipate energy often lead to positive real systems. The positive realness of a square transfer function matrix may be seen as a generalization of the positive definiteness of a matrix to the case of a dynamic system [3], where only the real part of the transfer function is considered. Positive real systems have many uses in practice. For instance, they can be realized with an electrical circuit using only resistors, inductors and capacitors [2]. For mechanical positive real systems, the use of velocity sensors and force actuators can be used to implement a control system with a guarantee of closed loop stability.

However, one major limitation of positive real systems is that their relative degree must be zero or one [3]. This limits the application of positive real theory. For example, a lightly damped flexible structure with collocated velocity sensors and force actuators can typically be modeled by a sum of second-order transfer functions as $G(s) = \sum_{i=0}^{\infty} \frac{\psi_i^2 s}{s^2+2\zeta_i\omega_i s+\omega_i^2}$, where ω_i is the mode frequency, and $\zeta_i > 0$ is the damping coefficient associated with the i-th mode, and ψ_i is determined by the boundary conditions on the partial differential equation. However, in some cases (for example, when using piezoelectric sensors), the sensor output is proportional to position rather than velocity. So the transfer function $G(s)$ given above is the transfer function from the actuator input to the derivative of the sensor output. In the case of a lightly damped flexible structure with collocated force actuators and position sensors, the transfer function will be of the form $G(s) = \sum_{i=0}^{\infty} \frac{\psi_i^2}{s^2+2\zeta_i\omega_i s+\omega_i^2}$. It can be seen that the relative degree of the system is more than unity. Hence, the standard positive real theory will not be helpful in establishing closed loop stability. However, such a transfer function would satisfy the following negative imaginary condition $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$. Such systems are called "systems with negative imaginary frequency response" in [4] or "negative imaginary systems" in this paper. For SISO systems, this is equivalent to the phase condition $\angle G(j\omega) \in [-\pi, 0]$.

The negative imaginariness of a square transfer function matrix may be seen as a generalization of the negative definiteness of a matrix to the case of a dynamic system, where the imaginary part of the transfer function matrix is considered. A complete state-space characterization of negative imaginary systems has been established in [4]. A necessary and sufficient condition has also been derived to guarantee the internal stability of a positive feedback interconnection of linear time-invariant multiple-input multiple-output negative imaginary systems.

However, all the results in [4] were built upon the requirement that the systems under consideration are asymptotically stable (that is, the poles of the systems lie in the open left half plane). Inspired by the Positive Real Lemma (see, e.g., Theorem 3.1 of [3] or Theorem 3 of [1]), which holds for dynamic systems that are Lyapunov stable (that is, the poles of the system are in the closed left half plane), we would like to extend the results in [4] to the case where the system poles may be on the imaginary axis.

The organization of the paper is as follows. Section II introduces the (strictly) negative imaginary concepts for square real-rational proper transfer function matrices, which extend the corresponding ones in [4]. The relationship between negative imaginariness and positive realness of transfer function matrices is also established in this section. In Section III, the Negative Imaginative Lemma (that is, Lemma 1 in [4]) is shown to remain true when the transfer functions have poles on the imaginary axis. In addition, a Strictly Negative Imaginary Lemma is established and some properties of negative imaginary systems are presented. Section IV studies the internal stability of a positive interconnection of two negative imaginary systems. A necessary and sufficient condition is proposed in terms of the DC loop gain (i.e., the loop gain at zero frequency) of the systems. This result extends the main result of [4] to allow for negative imaginary systems with purely imaginary eigenvalues. A numerical example is presented in Section V to illustrate the theory and Section VI concludes the paper.

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J. Xiong and I. R. Petersen are with the School of Information Technology and Electrical Engineering, University of New South Wales at the Australian Defence Force Academy, Canberra ACT 2600, Australia. junlin.xiong@gmail.com, i.r.petersen@gmail.com

A. Lanzon is with the Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Manchester M60 1QD, U. K. a.lanzon@ieee.org

Notation: $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$ denote the sets of $m \times n$ real and complex matrices, respectively. $\Re[s]$ denotes the real part of a complex number $s \in \mathbb{C}$. $\lambda_i(A)$ is the ith eigenvalue of a square complex matrix A, while $\lambda_{\max}(A)$ denotes the maximum eigenvalue for a square complex matrix A that has only real eigenvalues. A^T and A^* denotes the transpose and the complex conjugate transpose of a complex matrix A. $R^{\sim}(s)$ presents the adjoint of transfer function matrix $R(s)$ given by $R^T(-s)$. When $s = j\omega$, we have $R^{\sim}(j\omega) = R^T(-j\omega) = R^*(j\omega)$.

II. Negative Imaginary Transfer Functions

In this section, we propose new definitions of negative imaginary transfer functions. These new definitions generalize the corresponding ones in [4] to the case where the transfer functions are allowed to have poles on the imaginary axis. The transfer functions being considered are assumed to be square real-rational proper transfer function matrices.

Definition 1: The real-rational strictly proper transfer function matrix $\hat{R}(s) \in \mathbb{C}^{m\times m}$ is termed negative imaginary if

1) $\hat{R}(s)$ has no poles at the origin and in $\Re[s] > 0$;
2) $j[\hat{R}(j\omega) - \hat{R}^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $\hat{R}(s)$;
3) If $j\omega_0$ is a pole of $\hat{R}(s)$, it is at most a simple pole, and the residue matrix $K_0 \triangleq \lim_{s\to j\omega_0}(s - j\omega_0)s\hat{R}(s)$ is positive-semidefinite Hermitian.

Remark 1: In the above definition, the requirement of $\hat{R}(s)$ having no poles at $s = 0$ avoids the possible pole-zero cancellation at the origin in $s\hat{R}(s)$ (see [6] for more details). Also, it ensures the DC gain of the transfer function exists. This plays an essential role in the internal stability analysis of a positive feedback interconnection of negative imaginary systems.

Remark 2: When $\hat{R}(s)$ is asymptotically stable, Definition 1 coincides with the one given in [4]. Hence, the definitions here can be considered as a generalization of those given in [4].

Remark 3: It can be seen that if a scalar transfer function $\hat{R}(s)$ is negative imaginary, then $\hat{R}(s)$ will have non-positive imaginary part when $s = j\omega$, $\omega \in (0, \infty)$. However, the inverse of the statement is not necessarily true. For example, $\hat{R}(s) = \frac{1}{s+a}$ has negative imaginary part for all $a \in \mathbb{R}$ when $s = j\omega$, $\omega \in (0, \infty)$. However, it is negative imaginary according to Definition 1 only if $a > 0$.

Based upon Definition 1, we introduce the following additional definitions.

Definition 2: The real-rational strictly proper transfer function matrix $\hat{R}(s) \in \mathbb{C}^{m\times m}$ is termed strictly negative imaginary if

1) $\hat{R}(s)$ has no poles in $\Re[s] \geq 0$;
2) $j[\hat{R}(j\omega) - \hat{R}^*(j\omega)] > 0$ for $\omega \in (0, \infty)$.

Definition 3: The real-rational proper transfer function matrix $R(s) \in \mathbb{C}^{m\times m}$ is termed negative imaginary if

1) $R(\infty) = R^T(\infty)$;
2) $\hat{R}(s) \triangleq R(s) - R(\infty)$ is negative imaginary according to Definition 1.

Remark 4: The above definition is reasonable since

1) For Condition 1, $R(\infty) = R^T(\infty)$ has been shown to be a necessary condition for $R(s)$ to be negative imaginary (see Lemma 1 in [4]);
2) When Condition 1 holds, we will have $j[R(j\omega) - R^*(j\omega)] = j[\hat{R}(j\omega) - \hat{R}^*(j\omega)]$. So the negative imaginariness of $R(s)$ can be characterized by $\hat{R}(s)$.

Definition 4: The real-rational proper transfer function matrix $R(s) \in \mathbb{C}^{m\times m}$ is termed strictly negative imaginary if

1) $R(\infty) = R^T(\infty)$;
2) $\hat{R}(s) \triangleq R(s) - R(\infty)$ is strictly negative imaginary according to Definition 2.

The concepts of negative imaginariness are closely related to the concepts of positive realness of transfer functions. This point can be seen from the following lemma, which provides a necessary and sufficient condition for the positive realness of real-rational transfer function matrices.

Lemma 1 (Theorem 2.7.2 of [2]): Let $F(s)$ be a real-rational matrix of functions of s. Then $F(s)$ is positive real if and only if

1) No element of $F(s)$ has a pole in $\Re[s] > 0$;
2) $F(j\omega) + F^*(j\omega) \geq 0$ for all real ω except values of ω where $j\omega$ is a pole of $F(s)$;
3) If $j\omega_0$ is a pole of any element of $F(s)$, it is at most a simple pole, and the residue matrix, $K_0 = \lim_{s\to j\omega_0}(s - j\omega_0)F(s)$ in case ω_0 is finite, and $K_\infty = \lim_{\omega\to\infty} \frac{F(j\omega)}{j\omega}$ in case ω_0 is infinite, is non-negative definite Hermitian.

Another useful lemma is as follows.

Lemma 2: If $A = A^* \geq 0$, then $\bar{A} = (\bar{A})^* \geq 0$.

Proof: The lemma follows immediately from $x^*\bar{A}x = (\bar{\bar{x}})^*\bar{A}(\bar{\bar{x}}) = \overline{\bar{x}^*A\bar{x}} = \bar{x}^*A\bar{x} \geq 0$. ∎

Now, we are ready to give a description of the relationship between negative imaginary transfer functions and positive real transfer functions.

Lemma 3: Given a real-rational strictly proper transfer function matrix $\hat{R}(s) \in \mathbb{C}^{m\times m}$. Then $\hat{R}(s)$ is negative imaginary if and only if

1) $\hat{R}(s)$ has no poles at the origin;
2) $F(s) \triangleq s\hat{R}(s)$ is positive real.

Proof: The proof is based on Definition 1 and Lemma 1. Note that $F(s)$ and $\hat{R}(s)$ have the same set of poles.

(Necessity) Suppose $\hat{R}(s)$ is negative imaginary. Condition 1 in Definition 1 implies that $\hat{R}(s)$ has no poles at the origin and $F(s)$ has no poles in $\Re[s] > 0$.

When $j\omega$, $\omega > 0$, is not a pole of $F(s)$, Condition 2 in Definition 1 implies that $F(j\omega) + F^*(j\omega) = j\omega[\hat{R}(j\omega) - \hat{R}^*(j\omega)] \geq 0$. Since $s = 0$ is not a pole of $\hat{R}(s)$, we have that $F(0) + F^*(0) = 0$. So $F(j\omega) + F^*(j\omega) \geq 0$ for $\omega \in [0, \infty)$ with $j\omega$ not a pole of $F(s)$. In view of Lemma 2, we have $\overline{F(j\omega)} + \overline{F^*(j\omega)} \geq 0$ for $\omega \geq 0$. That is, $F(-j\omega) + F^*(-j\omega) \geq 0$ for $\omega \geq 0$. So $F(j\omega) + F^*(j\omega) \geq 0$ for $\omega \leq 0$. Finally, we have that $F(j\omega) + F^*(j\omega) \geq 0$ for any $\omega \in (-\infty, \infty)$ with $j\omega$ not a pole.

If $j\omega_0$ is a pole of $F(s)$, then Condition 3 in Definition 1 implies that $j\omega_0$ is at most a simple pole, and the residue matrix $K_0 = K_0^* \geq 0$. Moreover, $F(s)$ has no infinite poles since $\hat{R}(s)$ is strictly proper. According to Lemma 1, $F(s)$ is positive real.

(Sufficiency) Suppose $F(s)$ is positive real and has no poles at the origin. According to Lemma 1, Condition 1 and Condition 3 of Definition 1 hold and $j\omega[\hat{R}(j\omega) - \hat{R}^*(j\omega)] \geq 0$ for $\omega \in (-\infty, \infty)$ with $j\omega$ not a pole of $\hat{R}(s)$. So $j[\hat{R}(j\omega) - \hat{R}^*(j\omega)] \geq 0$ for $\omega \in (0, \infty)$ with $j\omega$ not a pole of $\hat{R}(s)$. Therefore, $\hat{R}(s)$ is negative imaginary. ■

Example 1: As an application of Lemma 3, we can say that $\hat{R}(s) = \frac{1}{s^2+1}$ is negative imaginary if and only if $F(s) = \frac{s}{s^2+1}$ is positive real. This can be actually verified by directly using Definition 1 and Lemma 1.

Remark 5: Suppose $R(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a negative imaginary transfer function. Then $\hat{R}(s) = R(s) - R(\infty) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$, and $F(s) = s\hat{R}(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$.

The following lemma gives a property of these state-space realizations.

Lemma 4: Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, $m \leq n$, and $\det(A) \neq 0$. Then the following statements are equivalent:

1) (A, B, C, D) is a minimal realization;
2) $(A, B, C, 0)$ is a minimal realization;
3) (A, B, CA, CB) is a minimal realization.

Proof: The equivalence between Statement 1 and Statement 2 is obvious. The equivalence between Statement 2 and Statement 3 follows from the fact that A is invertible. ■

The following lemma characterizes the property of the sum of negative imaginary transfer functions.

Lemma 5: Given two negative imaginary transfer functions $R(s)$, $\Delta(s)$, and a strictly negative imaginary transfer function $R_s(s)$. Then

1) $R(s) + \Delta(s)$ is negative imaginary;
2) If $\Delta(s)$ has no poles on the imaginary axis, then $R_s(s) + \Delta(s)$ is strictly negative imaginary.

Proof: To prove Part 1, let $H(s) \triangleq R(s) + \Delta(s)$, and $\hat{H}(s) \triangleq H(s) - H(\infty)$. Then $H(\infty) = R(\infty) + \Delta(\infty) = R^T(\infty) + \Delta^T(\infty) = H^T(\infty)$, and the set of poles of $\hat{H}(s)$ is the union of the sets of poles of $\hat{R}(s)$ and $\hat{\Delta}(s)$ where $\hat{R}(s) = R(s) - R(\infty)$ and $\hat{\Delta}(s) = \Delta(s) - \Delta(\infty)$. So $\hat{H}(s)$ has no poles at the origin and in $\Re[s] > 0$. For any given $\omega \in (0, \infty)$, if $j\omega$ is not a pole of $\hat{H}(s)$, then $j[\hat{H}(j\omega) - \hat{H}^*(j\omega)] = j[\hat{R}(j\omega) - \hat{R}^*(j\omega)] + j[\hat{\Delta}(j\omega) - \hat{\Delta}^*(j\omega)] \geq 0$. If $j\omega$ is a pole of $\hat{H}(s)$, we have three cases:

1) $j\omega$ is a pole of $\hat{R}(s)$ but not a pole of $\hat{\Delta}(s)$. Then $K_H \triangleq \lim_{s\to j\omega}(s - j\omega)s\hat{H}(s) = \lim_{s\to j\omega}(s - j\omega)s\hat{R}(s) + \lim_{s\to j\omega}(s - j\omega)s\hat{\Delta}(s) = K_R + 0 \geq 0$, where $K_R = \lim_{s\to j\omega}(s - j\omega)s\hat{R}(s)$;
2) $j\omega$ is not a pole of $\hat{R}(s)$ but a pole of $\hat{\Delta}(s)$, Then $K_H = K_\Delta \geq 0$, where $K_\Delta = \lim_{s\to j\omega}(s - j\omega)s\hat{\Delta}(s)$;
3) $j\omega$ is a pole of both $\hat{R}(s)$ and $\hat{\Delta}(s)$, Then $K_H = K_R + K_\Delta \geq 0$.

Therefore, we have $0 \leq K_H < \infty$. Also $j\omega$ must be a simple pole (otherwise $K_H = \infty$). This proves that $\hat{H}(s)$ is negative imaginary and so is $H(s)$.

The proof for Part 2 is trivial since both $R_s(s)$ and $\Delta(s)$ have no poles on the imaginary axis in this case. ■

III. NEGATIVE IMAGINARY LEMMA

The Negative Imaginary Lemma proposed in this section extends the Negative Imaginary Lemma in [4] to the case where the transfer functions may have poles on the imaginary axis. Also a Strictly Negative Imaginary Lemma is established for strictly negative imaginary transfer functions.

The following lemma is analogous to the Positive Real Lemma (e.g., see Lemma 3.1 of [3] or Theorem 3 of [1]), where the systems may have purely imaginary poles.

Lemma 6 (Negative Imaginary Lemma): Let (A, B, C, D) be a minimal state-space realization of a real-rational proper transfer function $R(s) \in \mathbb{C}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, $m \leq n$. Then $R(s)$ is negative imaginary if and only if

1) $\det(A) \neq 0$, $D = D^T$, and
2) there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that

$$AY + YA^T \leq 0, \quad \text{and} \quad B + AYC^T = 0.$$

Proof: The equivalence follows from the following sequence of equivalent reformulations.

$R(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is negative imaginary.

$\Leftrightarrow$ $D = D^T$, and $\hat{R}(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ is negative imaginary (see Definition 3).

$\Leftrightarrow$ $\det(A) \neq 0$, $D = D^T$, and $F(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$ is positive real (according to Lemma 3 and Lemma 4).

$\Leftrightarrow$ $\det(A) \neq 0$, $D = D^T$, and there exists $X = X^T > 0$, Q, W such that

$$XA + A^TX = -Q^TQ,$$
$$B^TX + W^TQ = CA,$$
$$CB + B^TC^T = W^TW.$$

This equivalence is via the Positive Real Lemma (see e.g. Lemma 3.1 of [3]). The rest of the proof follows along the lines of the proof of Lemma 1 in [4]. ■

Remark 6: It follows from the equation $B + AYC^T = 0$ that $\text{rank}(B) = \text{rank}(C) \leq m$ since both A and Y are invertible.

A useful property of negative imaginary systems is stated in the following corollary.

Corollary 1: If $R(s)$ is negative imaginary and has the minimal state-space realization (A, B, C, D), then there exists a real-rational strictly proper transfer function matrix $M(s) \sim \left[\begin{array}{c|c} A & B \\ \hline LY^{-1}A^{-1} & 0 \end{array}\right]$ such that

$$j[R(j\omega) - R^*(j\omega)] = \omega M^*(j\omega)M(j\omega)$$

when $j\omega$ is not a pole of $R(s)$. Here, $Y = Y^T > 0$ and L are the solutions of $L^T L = -AY - YA^T$ and $B + AYC^T = 0$.

Proof: Define

$$W(s) \triangleq sM(s) = \left[\begin{array}{c|c} A & B \\ \hline LY^{-1} & LY^{-1}A^{-1}B \end{array}\right],$$

$$\hat{R}(s) \triangleq R(s) - R(\infty) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right],$$

$$F(s) \triangleq s\hat{R}(s) = \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right].$$

Then

$$\begin{aligned}
&W^{\sim}(s)W(s) \\
&= \left[\begin{array}{c|c} -A^T & -Y^{-1}L^T \\ \hline B^T & B^T A^{-T} Y^{-1} L^T \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline LY^{-1} & LY^{-1}A^{-1}B \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & 0 & B \\ -Y^{-1}L^T LY^{-1} & -A^T & -Y^{-1}L^T LY^{-1}A^{-1}B \\ \hline B^T A^{-T} Y^{-1} L^T LY^{-1} & B^T & B^T A^{-T} Y^{-1} L^T LY^{-1} A^{-1} B \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & 0 & B \\ Y^{-1}(AY + YA^T)Y^{-1} & -A^T & Y^{-1}(AY + YA^T)Y^{-1}A^{-1}B \\ \hline B^T A^{-T} Y^{-1}(-AY - YA^T)Y^{-1} & B^T & B^T A^{-T} Y^{-1}(-AY - YA^T)Y^{-1}A^{-1}B \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & 0 & B \\ Y^{-1}A + A^T Y^{-1} & -A^T & Y^{-1}B - A^T C^T \\ \hline CA - B^T Y^{-1} & B^T & CB + B^T C^T \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & 0 & B \\ 0 & -A^T & -A^T C^T \\ \hline CA & B^T & CB + B^T C^T \end{array}\right] \\
&= \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right] + \left[\begin{array}{c|c} -A^T & -A^T C^T \\ \hline B^T & B^T C^T \end{array}\right] \\
&= F(s) + F^{\sim}(s) \\
&= s\hat{R}(s) - s\hat{R}^{\sim}(s) \\
&= s[\hat{R}(s) - \hat{R}^{\sim}(s)] \\
&= s[R(s) - R^{\sim}(s)].
\end{aligned}$$

That is,

$$s[R(s) - R^{\sim}(s)] = -s^2 M^{\sim}(s)M(s).$$

When $s \neq 0$, we have

$$R(s) - R^{\sim}(s) = -sM^{\sim}(s)M(s).$$

When $s = 0$, we have

$$R(0) - R^{\sim}(0) = R(0) - R^T(0) = 0$$

since

$$R(0) = -CA^{-1}B = CYC^T = R^T(0).$$

So

$$R(s) - R^{\sim}(s) = -sM^{\sim}(s)M(s)$$

holds for any s with s not a pole of $R(s)$. Let $s = j\omega$ with $j\omega$ not a pole of $R(s)$. Then, we have

$$R(j\omega) - R^*(j\omega) = -j\omega M^*(j\omega)M(j\omega).$$

That is,

$$j[R(j\omega) - R^*(j\omega)] = \omega M^*(j\omega)M(j\omega).$$

This completes the proof. ■

Remark 7: It follows from the above proof that $R(s) - R^{\sim}(s) = -sM^{\sim}(s)M(s)$ for all $s \in \mathbb{C}$ with s not a pole of $R(s)$.

For strictly negative imaginary transfer functions, a strict version of negative imaginary lemma is also derived. The result is analogous to the Weakly Strictly Positive Real Lemma (see e.g., Lemma 3.18 of [3] or Theorem 1 of [5]).

Lemma 7 (Strictly Negative Imaginary Lemma): Let (A, B, C, D) be a minimal state-space realization of a real-rational proper transfer function $R(s) \in \mathbb{C}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, $m \leq n$. Then $R(s)$ is strictly negative imaginary if and only if

1) A is Hurwitz, $D = D^T$, $\operatorname{rank}(B) = \operatorname{rank}(C) = m$;
2) There exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that
$$AY + YA^T \leq 0, \quad \text{and} \quad B + AYC^T = 0;$$
3) The transfer function $M(s) \sim \left[\begin{array}{c|c} A & B \\ \hline LY^{-1}A^{-1} & 0 \end{array}\right]$ has full column rank at $s = j\omega$ for any $\omega \in (0, \infty)$. Here $L^T L = -AY - YA^T$. That is, $\operatorname{rank}(M(j\omega)) = m$ for any $\omega \in (0, \infty)$.

Proof: (Necessity) Suppose $R(s)$ is strictly negative imaginary. Then $R(s)$ has no poles in $\Re[s] \geq 0$, which implies that A is Hurwitz. Also $R(s)$ is negative imaginary. Hence, according to the Negative Imaginary Lemma, we have $D = D^T$ and there exists a matrix $Y = Y^T > 0$ such that $AY + YA^T \leq 0$ and $B + AYC^T = 0$.

To prove $\operatorname{rank}(B) = m$, suppose that, on the contrary, $\operatorname{rank}(B) < m$. Then there exists a nonzero vector $x \in \mathbb{R}^m$ such that $Bx = 0$. Therefore,

$$\begin{aligned}
&x^*\{j[R(j\omega) - R^*(j\omega)]\}x \\
&= jx^T C(j\omega I - A)^{-1}Bx - jx^T B^T(-j\omega I - A^T)^{-1}C^T x \\
&= 0
\end{aligned}$$

for any $\omega \in (0, \infty)$. This contradicts $j[R(j\omega) - R^*(j\omega)] > 0$ for all $\omega \in (0, \infty)$. Thus, we conclude that $\operatorname{rank}(B) = m$. That is, B has full column rank. Similarly, we have $\operatorname{rank}(C) = m$.

Next, we prove Condition 3. It follows from the proof of Corollary 1 that

$$\omega M^*(j\omega)M(j\omega) = j[R(j\omega) - R^*(j\omega)] > 0$$

for any $\omega \in (0, \infty)$. This implies that $M(j\omega)x \neq 0$ for any $\omega \in (0, \infty)$ and any nonzero $x \in \mathbb{C}^m$. Therefore, $\text{rank}(M(j\omega)) = m$ for any $\omega \in (0, \infty)$.

(Sufficiency) Condition 1 and Condition 2 imply that $R(s)$ is negative imaginary. According to Corollary 1, we have

$$j[R(j\omega) - R^*(j\omega)] = \omega M^*(j\omega)M(j\omega)$$

for any $\omega \in (0, \infty)$. Condition 3 implies that $M(j\omega)x \neq 0$ for any nonzero vector $x \in \mathbb{C}^m$. Therefore,

$$jx^*[R(j\omega) - R^*(j\omega)]x = \omega x^* M^*(j\omega)M(j\omega)x > 0$$

for any $\omega \in (0, \infty)$ and any nonzero vector $x \in \mathbb{C}^m$. That is,

$$j[R(j\omega) - R^*(j\omega)] > 0$$

for $\omega \in (0, \infty)$. This completes the proof. ■

Remark 8: Condition 3 in Lemma 7 can be equivalently replaced by the condition that $\begin{bmatrix} A - j\omega I & B \\ LY^{-1}A^{-1} & 0 \end{bmatrix}$ has full column rank for all $\omega \in (0, \infty)$. The equivalence follows from Lemma 3.33 of [6].

Similar to Corollary 1 for negative imaginary transfer functions, we have the following corollary for strictly negative imaginary transfer functions.

Corollary 2: A real-rational proper transfer function $R(s)$ is strictly negative imaginary if and only if there exists a stable real-rational strictly proper transfer function $M(s)$ with full column rank at $j\omega$ for any $\omega \in (0, \infty)$ such that

$$j[R(j\omega) - R^*(j\omega)] = \omega M^*(j\omega)M(j\omega)$$

for all $\omega \in (0, \infty)$.

Proof: The result follows from the proofs of Corollary 1 and Lemma 7. ■

At frequencies of zero and infinity, negative imaginary transfer functions have the following properties.

Corollary 3:
1) Given a negative imaginary transfer function $R(s)$, then $R(0) - R(\infty) \geq 0$.
2) Given a strictly negative imaginary transfer function $R(s)$, then $R(0) - R(\infty) > 0$.

Proof: The proof is the same as the proof of Lemma 2 of [4]. ■

IV. Interconnection of Negative Imaginary Systems

In this section, we consider the internal stability of a positive feedback interconnection of two negative imaginary systems as shown in Figure 1.

A necessary and sufficient condition is provided for the stability of the system given in Figure 1 in terms of the DC loop gain (i.e., the loop gain at zero frequency).

Theorem 1: Given a negative imaginary transfer function $R(s)$ and a strictly negative imaginary transfer function $R_s(s)$ that also satisfy $R(\infty)R_s(\infty) = 0$ and $R_s(\infty) \geq 0$. Then the positive feedback interconnection $[R(s), R_s(s)]$ is internally stable if and only if $\lambda_{\max}(R(0)R_s(0)) < 1$.

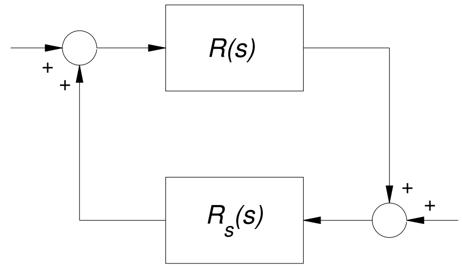

Fig. 1. Positive feedback interconnection

Proof: The proof follows along the same lines as the proof of Theorem 5 of [4] except that Theorem 5.7 of [6] is used (instead of Corollary 5.6 of [6]). ■

The following corollaries are a restatement of the above theorem, written in the same form as the small gain theorem (see Theorem 9.1 of [6]).

Corollary 4: Given $\gamma > 0$ and a strictly negative imaginary transfer function $R(s)$ with $R(\infty) \geq 0$. Then, the positive feedback interconnection $[\Delta(s), R(s)]$ is internally stable for all negative imaginary transfer functions $\Delta(s)$ satisfying $\Delta(\infty)R(\infty) = 0$ and $\lambda_{\max}(\Delta(0)) < \gamma$ (respectively $\leq \gamma$) if and only if $\lambda_{\max}(R(0)) \leq \frac{1}{\gamma}$ (respectively $< \frac{1}{\gamma}$).

Proof: The proof is the same as the proof of Corollary 6 of [4]. ■

Corollary 5: Given $\gamma > 0$ and a negative imaginary transfer function $R(s)$. Then the positive feedback interconnection $[\Delta(s), R(s)]$ is internally stable for all strictly negative imaginary transfer functions $\Delta(s)$ satisfying $\Delta(\infty)R(\infty) = 0$, $\Delta(\infty) \geq 0$ and $\lambda_{\max}(\Delta(0)) < \gamma$ (respectively $\leq \gamma$) if and only if $\lambda_{\max}(R(0)) \leq \frac{1}{\gamma}$ (respectively $< \frac{1}{\gamma}$).

Proof: The proof is the same as the proof of Corollary 6 of [4]. ■

Remark 9: The results in this section are simply "restatements" of the results in [4] with the new definitions in this paper. However, the results here allow one of the interconnected systems to have purely imaginary poles.

V. Illustrative Example

To illustrate the main results of this paper, we have modified the example in [4] so that the system uncertainty has poles on the imaginary axis. The example is a two-input two-output fourth-order linear system. Let

$$p(s) \triangleq \frac{1}{s^2 + s + 1},$$
$$\delta(s) \triangleq \frac{1}{s^2 + (2k + 1)}.$$

The transfer function of the uncertain plant is given by

$$P_\Delta(s) = p(s)\delta(s) \begin{bmatrix} s^2 + k + 1 & k \\ k & s^2 + k + 1 \end{bmatrix}$$

where $k > 0$ is unknown and presents the uncertainty in the system.

For the purpose of control system design, we now choose to split the uncertain plant $P_\Delta(s)$ as

$$P_\Delta(s) = P(s) + \Delta(s)$$

where $P(s)$ is the nominal completely known plant model and $\Delta(s)$ is the uncertain reminder. Using a partial fraction expansion, we see that

$$P(s) = 0.5p(s)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},$$
$$\Delta(s) = 0.5\delta(s)\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}.$$

It is then a simple computation to check that $\Delta(s)$ is negative imaginary. Note that $\Delta(s)$ has poles on the imaginary axis while $P(s) \in \mathcal{RH}_\infty$.

Now, let us consider the controlled closed-loop system given in Figure 2, and let $C(s)$ be chosen as

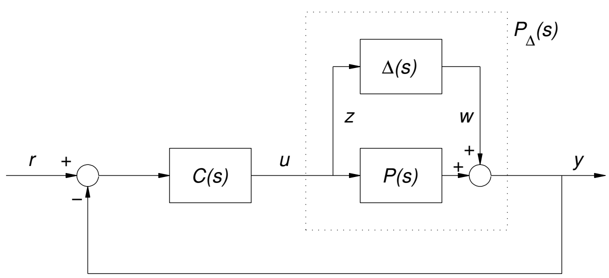

Fig. 2. Closed-loop system

$$C(s) = \frac{-1}{[(s^2+s+1)(s+1)+1.5](s+1)} \times \begin{bmatrix}(s^2+s+1)(s+1) & -1\\ -1 & 2(s^2+s+1)(s+1)+1\end{bmatrix}.$$

Then, define

$$R(s) \triangleq -C(s)\left(I + P(s)C(s)\right)^{-1}$$

to be the transfer function matrix mapping w to z so that the closed-loop system in Figure 2 can be rearranged into the form shown in Figure 3 for robustness analysis.

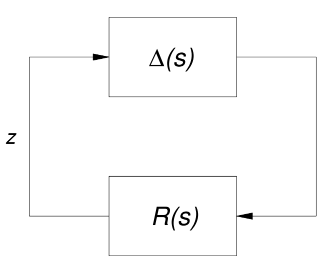

Fig. 3. Rearranged closed-loop system

For our particular choice of $C(s)$, it is easy to see that

$$R(s) = \frac{1}{s+1}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},$$

which clearly is strictly negative imaginary. Also

$$R(\infty) = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \geq 0, \quad R(0) = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},$$
$$\lambda_{\max}(R(0)) = 2.$$

On the other hand, we know that

$$\Delta(s) = \frac{0.5}{s^2+(2k+1)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}.$$

So

$$\Delta(\infty) = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},$$
$$\Delta(0) = \frac{0.5}{2k+1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix},$$
$$\lambda_{\max}(\Delta(0)) = \frac{1}{2k+1}.$$

Also, we have

$$\Delta(0)R(0) = \frac{0.5}{2k+1}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix},$$
$$\lambda_{\max}(\Delta(0)R(0)) = \frac{1.5}{2k+1}.$$

Consequently, Corollary 4 states that the feedback interconnection system given in Figure 2 is robustly stable for all negative imaginary uncertainties $\Delta(s)$ (not just those of the form $\Delta(s) = \frac{0.5}{s^2+(2k+1)}\left[\begin{smallmatrix}1 & -1\\ -1 & 1\end{smallmatrix}\right]$) satisfying $\lambda_{\max}(\Delta(0)) < \frac{1}{2}$. In our example, this implies that $k > 0.5$. Additionally, Theorem 1 states that the uncertain plant is robustly stabilized by the controller $C(s)$ defined earlier if and only if $k > 0.25$ (obtained through the condition $\lambda_{\max}(\Delta(0)R(0)) < 1$).

VI. Conclusions

This paper has studied the negative imaginary properties of square real-rational proper transfer functions which may have poles on the imaginary axis. Dynamic systems with such negative imaginary transfer functions have applications in position feedback control of undamped flexible structures. The Negative Imaginary Lemma was derived for transfer functions that may have poles on the imaginary axis. Moreover, a necessary and sufficient condition was established for the internal stability analysis of positive feedback interconnections of negative imaginary systems. These results extend corresponding results of a previous paper which no poles on the imaginary axis were allowed. Finally, the theory in the paper was illustrated by a numerical example.

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