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Probabilistically-robust performance optimization for controlled linear stochastic systems

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Abstract—**This study discusses a robust controller synthesis methodology for linear time invariant systems characterized by probabilistic parameter uncertainty. The optimization of the robust performance is considered. The extension of pre-existing, synthesis approaches, such as multi-objective $\mathcal{H}_2$ design, to account for probabilistic uncertainty is investigated. A design based on the concept of the reliability of the system response output is also considered. Analysis and synthesis methodologies based on stochastic simulation techniques are discussed. The design approach is applied in a structural control example. The results illustrate the differences between the various probabilistic performance objectives and the importance of adopting a probabilistic characterization for model uncertainty when compared to nominal design or to the design using a worst-case scenario approach.**

I. Introduction

THE existence of model uncertainty is important for modern control applications, as one of the main objectives is to establish optimum robustness over all possible operational conditions. Standard tools for robust control design, such as $\mathcal{H}_\infty$, μ-synthesis [1] and the many offshoots of these, consider only compact set of possible models for the system. Information implying that some model parameters are more probable than others is not explicitly treated. However in most real engineering applications, there is considerable knowledge about the relative plausibility of the different model parameter values. A probability logic approach provides a rational framework for quantifying this knowledge [2] by characterizing the relative plausibility of the system properties by appropriate, based on the available information, probability models.

This observation has motivated researchers to investigate the stability and performance of linear and nonlinear controlled systems under probabilistic parameter uncertainty. This is established by optimizing *statistics* of the objective function (*probabilistic performance*) under plant uncertainty, rather than the objective function resulting from the nominal model (*nominal performance*). The design process incorporating such measures is called *robust stochastic design* [3, 4], where the term robustness pertains to the stochastic, i.e. probabilistic, model description.

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A number of studies have been developed exploring such ideas. The methods proposed in [4, 5] characterize the robust performance of a controller in terms of the probabilities that the closed loop system will have unacceptable response in terms of either its stability or performance. The design objective was expressed as a weighted sum of these probabilities, and evolutionary algorithms were proposed for performing the optimization for the controller parameters. Field et al. [6] focused on the probability of instability for controlled systems and used first order reliability calculations to approximate it. Boers et al. [7, 8] discussed the expected performance related to the $\mathcal{H}_2$ norm of the closed loop system. They proved that the design problem is well-posed but restricted their attention to a simple subclass of parametric model uncertainty characterizations. The concept of the reliability for the system output has also been used as a controller design objective, an approach initially introduced in [9]. The optimal *reliability-based* controller is selected by minimizing the probability of first-passage of the output over a region that defines acceptable performance. Theoretical issues related to this approach were investigated in detail in [10].

The present paper discusses the robust-performance optimization of linear time invariant dynamical systems with probabilistically-described parametric model uncertainties and focuses on systems including a stochastic disturbance input. It discusses efficient analysis and synthesis methodologies, based on recently-developed stochastic simulation techniques. It also sheds light on appropriate characterization of the probabilistic performance and discussed differences for reliability-based and multi-objective $\mathcal{H}_2$ control. Finally, it draws comparisons between probabilistically-robust control design, and nominal system design or the "worst-case" interpretation of robustness.

II. Problem Formulation and Nominal Design

We assume a linear dynamical system under stochastic excitation with a state space model that depends on a set of parameters $\boldsymbol{\theta}$; i.e.,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(\boldsymbol{\theta})\mathbf{x}(t) + \mathbf{B}(\boldsymbol{\theta})\mathbf{u}(t) + \mathbf{E}(\boldsymbol{\theta})\mathbf{w}(t) \\ \mathbf{z}(t) &= \mathbf{C}(\boldsymbol{\theta})\mathbf{x}(t) + \mathbf{D}(\boldsymbol{\theta})\mathbf{u}(t)\ , \quad \mathbf{y}(t) = \mathbf{L}(\boldsymbol{\theta})\mathbf{x}(t)\end{aligned} \tag{1}$$

where $\mathbf{x}(t) \in \Re^{n_x}$ is the system state vector, consisting of the structural states together with any ancillary states used to model sensor and actuator dynamics, spectral characteristics of the external excitation and dynamic states of the controller. Vector $\mathbf{u}(t)\in\Re^{n_u}$ consists of control forces that are assumed to be formulated based on a feedback vector $\mathbf{y}(t)\in\Re^{n_y}$, which is linearly related to $\mathbf{x}(t)$. The performance of the controlled system is assessed through $\mathbf{z}(t)\in\Re^{n_z}$. Disturbance input $\mathbf{w}(t)$ is a zero-mean Gaussian white-noise vector process. Control input $\mathbf{u}(t)$ is assumed to be a feedback function of the response measurements, $\mathbf{u}(t)=\mathbf{K}(\boldsymbol{\varphi})\mathbf{y}(t)$ where $\mathbf{K}\in\Re^{n_u}\text{x}\Re^{n_y}$ is the feedback gain matrix and $\boldsymbol{\varphi}\in\Phi$ denotes the free parameters of this matrix, which constitute the design variables of the problem. $\Phi\in\Re^{n_\varphi}$ corresponds to the admissible design space. The image of Φ under $\mathbf{K}$ is denoted $\mathcal{K}$.

The nominal performance is conditioned on an assumed system model, and thus is always parameterized by a presumed $\boldsymbol{\theta}$ vector. The measure assessing the nominal performance of the system corresponding to vector $\boldsymbol{\theta}$ will be denoted by $J(\mathbf{K}|\boldsymbol{\theta})$. Probabilistic performance measures, on the other hand, presume a distribution for $\boldsymbol{\theta}$ and evaluate some statistical representation of the nominal performance measure over this distribution. Note that in the present work the nominal performance is evaluated considering a stochastic influence, the one of the disturbance input $\mathbf{w}$. In principle though, the methodologies described can extend many standard optimal control methods predicated on completely deterministic system models. The design adopting the probabilistic performance as objective function is defined as *robust stochastic design*, with the term stochastic referring here to the probabilistic (i.e. stochastic) description for the system model [3, 4] and not to the disturbance input $\mathbf{w}$. In this context, the design adopting the nominal performance as objective function is going to be referenced as *nominal design*.

Considering first the nominal performance, note that in stationarity the closed loop linear controlled system yields a zero-mean Gaussian distribution for the output $\mathbf{z}$ with zero mean and covariance matrices that can be readily obtained (see, for example, [10]). Thus, the uncertainty stemming from the stochastic disturbance can be analytically propagated to the system response output, in terms of its stationary statistics. Based on this observation, two different instances for the nominal performance characterization will be considered here. The first one is the multi-objective $\mathcal{H}_2$ performance (referenced herein $m\mathcal{H}_2$) defined as:

$$J_{m\mathcal{H}_2}(\mathbf{K}\,|\,\boldsymbol{\theta}) = E\{\max_{1\le i\le n_z}(\lim_{t\to\infty} 1/T\int_0^\infty z(t)_i^2\,dt)\} = E\{\max_{1\le i\le n_z}(\sigma_{z_i}^2)\} \quad (2)$$

where $E\{.\}$ denotes expectation with respect to the stochastic input $\mathbf{w}$, given $\boldsymbol{\theta}$ and σ_z^2 denotes the stationary variance for response variable z. The associated controller design problem has received a lot of attention and though it corresponds to a non-convex optimization, convexifying techniques have been developed for its solution [11].

The second, and less familiar, quantification is motivated by the concept of system-reliability, defined here as the probability that the response output will not exceed acceptable bounds. These bounds define a hyper-cubic safe region $D_s=\{|z_i(t)<1|, i=1,\ldots,n_z\}$. The optimal reliability-based controller, introduced initially in [9], is then the one that minimizes the probability of unacceptable performance over some time duration $t\in[0,T]$ usually chosen to correspond to the duration of the event causing the dynamic excitation of the system . This probability associated with controller $\mathbf{K}$, $P_F(\mathbf{K}|T,\boldsymbol{\theta})=P(\mathbf{z}(t)\notin D_s$ for some $t\in[0,T]|\mathbf{K},\boldsymbol{\theta})$, is commonly referred to as probability of failure and in stationarity, may be expressed as the probability of first passage across S_D, the boundary of the safe region D_s,

$$J_R(\mathbf{K}\,|\,T,\boldsymbol{\theta}) \triangleq P_F(\mathbf{K}\,|\,T,\boldsymbol{\theta}) = 1-\exp(-\nu_z(\mathbf{K}\,|\,\boldsymbol{\theta})T)\,, \quad (3)$$

where $\nu_z(\mathbf{K}|\boldsymbol{\theta})$ is the mean out-crossing rate of the boundary S_D, conditioned on no previous out-crossing having occurred [10]. This rate can be approximated as a sum of the out-crossing rates corresponding to each failure mode i and can be analytically evaluated based on the stationary statistics for $\mathbf{z}$ [10]. The reliability optimal controller is finally equivalent to the minimization of the stationary out-crossing rate, and the dependence on the time duration T vanishes

$$\mathbf{K}_R^* = \arg\min_{\mathbf{K}\in\mathcal{K}} \nu_z(\mathbf{K}\,|\,\boldsymbol{\theta})\,. \quad (4)$$

Numerical details pertaining to this optimization and theoretical comparisons to other controller synthesis methods (including $m\mathcal{H}_2$) are presented in detail in [10]. Also, Field and Bergman [12] have discussed incorporation of similar reliability constraints in covariance control.

III. Robust Stochastic Design

Let Θ denote the set of possible model parameter values for the model in (1). Some of these values may be more probable than others. This relative plausibility of different model parameter vectors constitutes prior knowledge about the system and can be quantified by assigning a probability distribution function PDF $p(\boldsymbol{\theta})$ to the model parameters. Non-parametric modeling uncertainty may be addressed by introducing a model prediction error, i.e. an error between the response of the actual system and the response of the model adopted for it. This error may be probabilistically characterized and augmented into $\boldsymbol{\theta}$.

A. Robust performance: general case

The robust performance quantification requires the extension of the nominal performance $J(\mathbf{K}|\boldsymbol{\theta})$ to a probabilistic one $H(\mathbf{K})$. This can established in various ways but in general will be expressed by an integral of the following form, which we will refer to as *stochastic* integral:

$$H(\mathbf{K}) \triangleq E_\theta[j(\mathbf{K}\,|\,\boldsymbol{\theta})] = \int_\Theta j(\mathbf{K}\,|\,\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}\,, \quad (5)$$

where $E_\theta[.]$ denotes expectation over the uncertain parameter space and the utility function $j(\mathbf{K}|\boldsymbol{\theta})$ is related to the deterministic performance measure $J(\mathbf{K}|\boldsymbol{\theta})$ by some mapping. By appropriate definition of $j(\mathbf{K}|\boldsymbol{\theta})$ different probabilistic measures $H(\mathbf{K})$ can be quantified. The *stochastically-robust controller* is then given by:

$$\mathbf{K}^* = \arg\min_{\mathbf{K}\in\mathcal{K}} \int_{\boldsymbol{\theta}} j(\mathbf{K}\mid\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}. \tag{6}$$

In the present work we will focus on two basic choices for $H(\mathbf{K})$ the (a) average value of $J(\mathbf{K}|\boldsymbol{\theta})$, or (b) the probability that $J(\mathbf{K}|\boldsymbol{\theta})$ will exceed some acceptable threshold. The corresponding performance measures (and designs) will be characterized as *average robustness* (*AR*) and *reliability robustness* (*RB*) and are given, respectively, by:

$$H_{AR}(\mathbf{K}) \triangleq E_\theta[J(\mathbf{K}\mid\boldsymbol{\theta})] = \int_\Theta J(\mathbf{K}\mid\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta} \tag{7}$$

$$H_{RB}(\mathbf{K}) \triangleq P(J(\mathbf{K}\mid\boldsymbol{\theta}) > b\mid\mathbf{K}) = \int_\Theta I_b(\mathbf{K}\mid\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta} \tag{8}$$

where the indicator function $I_b(.)=1$ if the system behavior is unacceptable, i.e., $J(\mathbf{K}|\boldsymbol{\theta})>b$, and $I_b=0$ if not. Which of the two probabilistic performance quantifications, *AR* or *RB*, is more appropriate for control design is directly related to the nature of the metric $J(\mathbf{K}|\boldsymbol{\theta})$ and the criteria adopted in the design, i.e., which objective is more important, regulation of the average performance or the performance that exceeds acceptable bounds? With respect to the multi-objective $\mathcal{H}_2$ nominal performance characterization, both probabilistic quantifications could be appropriate depending on the application, but *RB* seems in general to be a better choice.

B. Robust performance: reliability-based design

In reliability-based design the robust performance definition follows from the basic principals of probability logic; the failure probability may be simply expressed by the total probability theorem:

$$\begin{aligned} P_F\left(\mathbf{K}\mid T\right) &= \int_\Theta P_F\left(\mathbf{K}\mid T,\boldsymbol{\theta}\right)p(\boldsymbol{\theta})d\boldsymbol{\theta} \\ &= 1-\int_\Theta p(\boldsymbol{\theta})\exp\left(-\nu_z\left(\mathbf{K}\mid\boldsymbol{\theta}\right)T\right)d\boldsymbol{\theta}. \end{aligned} \tag{9}$$

Contrary to the certain parameter case (4), the choice of the time duration, T, influences the design optimization. To further characterize this influence, a Taylor series expansion is implemented for (9), leading to the optimal controller

$$\mathbf{K}^*_{RR_c} = \arg\min_{\mathbf{K}\in\mathcal{K}} \{-\sum_{j=1}^{\infty}\left(-T\right)^j / j!\,E_\theta[\left(\nu_z\left(\mathbf{K}\mid\boldsymbol{\theta}\right)\right)^j]\}. \tag{10}$$

Thus, robust reliability-based design weighs the mean value of ν_z (obtained for j=1 in the last infinite sum) against its higher-order moments over the uncertain parameter space. Time duration T enters the problem as a sensitivity parameter which defines the relative importance of the higher-order statistics. For small time durations $T\to 0$, the optimal controller is the one that minimizes the expected value of the out-crossing rate, evaluated over the uncertain parameter space, without considering the higher order statistics. As T increases these statistics become important.

This discussion shows that choice of T must therefore be made with some care. A logical assumption is to take this duration T as the duration of the dynamic excitation, (depending on the purpose of the control system), which suggests that it should be also treated as an uncertain parameter. A reasonable probability distribution for T is the exponential distribution; i.e., $p(T)=1/T_m\exp(-T/T_m)$ if $T>0$; 0 else, where T_m corresponds to the mean value. The probability of failure may be then calculated as:

$$\begin{aligned} P_F\left(\mathbf{K}\mid T_m\right) &= \int_\Theta\int_0^\infty P_F\left(\mathbf{K}\mid T,\boldsymbol{\theta}\right)p(\boldsymbol{\theta})p(T)dTd\boldsymbol{\theta} \\ &= 1-\int_\Theta \frac{p(\boldsymbol{\theta})}{\nu_z\left(\mathbf{K}\mid\boldsymbol{\theta}\right)T_m+1}d\boldsymbol{\theta} \end{aligned}, \tag{11}$$

leading to a reliability-based optimal controller:

$$\mathbf{K}^*_{RR} = \arg\min_{\mathbf{K}\in\mathcal{K}}\{1-\frac{p(\boldsymbol{\theta})}{\nu_z(\mathbf{K}\mid\boldsymbol{\theta})T_m+1}\}. \tag{12}$$

Employing a Taylor series expansion, as before, leads to interesting limiting cases for the controller optimization. For small time durations we have

$$\mathbf{K}^*_{RR}\big|_{T_m\to 0} = \arg\min_{\mathbf{K}\in\mathcal{K}}\{E_\theta[\nu_z(\mathbf{K}\mid\boldsymbol{\theta})]\}. \tag{13}$$

This optimization is identical to the one for deterministic short-time durations. This is not surprising, because T is being treated as probabilistic with an arbitrarily-narrow distribution, thus converging to the case of deterministic $T\to 0$. For the infinite time duration we have:

$$\mathbf{K}^*_{RR}\Big|_{T_m\to\infty} = \arg\max_{\mathbf{K}\in\mathcal{K}}\{E_\theta[\nu_z(\mathbf{K}\mid\boldsymbol{\theta})^{-1}]\}. \tag{14}$$

This expression has a very intuitive interpretation. It is straight-forward to show that, for a given $\boldsymbol{\theta}\in\Theta$, the quantity $\nu_z(\mathbf{K}|\boldsymbol{\theta})^{-1}$ is the expected (i.e., average) time duration *between* out-crossings in stationary response. Thus the optimal robust reliability-based controller for the infinite time horizon case is the one that maximizes the expected time between out-crossings. Note that a similar result does not hold for (10), the case with deterministic time duration.

IV. STOCHASTIC ANALYSIS AND OPTIMIZATION

A. Stochastic analysis and optimization

The general form of the stochastic integrals encountered in the robust performance quantification, expressed in terms of the design variables for the controller $\boldsymbol{\varphi}$, is

$$H(\mathbf{K}) = H(\boldsymbol{\varphi}) \triangleq E_{\boldsymbol{\theta}}[h(\boldsymbol{\varphi},\boldsymbol{\theta})] = \int_\Theta h(\boldsymbol{\varphi},\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}, \tag{15}$$

for some appropriate selection of the utility function $h(\boldsymbol{\varphi},\boldsymbol{\theta})$. If the dimension of $\boldsymbol{\theta}$ is large the integral in (15) can rarely be numerically evaluated. An efficient alternative approach is to estimate the integral by stochastic simulation [13]. Using a finite number, N, of samples of $\boldsymbol{\theta}$ drawn from some importance sampling density $p_{is}(\boldsymbol{\theta})$, an estimate for (15) is given by the *stochastic analysis*:

$$\hat{H}(\boldsymbol{\varphi},\boldsymbol{\Omega}_N) \triangleq 1/N\sum_{i=1}^{N} h(\boldsymbol{\varphi},\boldsymbol{\theta}_i)p(\boldsymbol{\theta}_i)/p_{is}(\boldsymbol{\theta}_i)\,, \tag{16}$$

where $\boldsymbol{\Omega}_N=[\boldsymbol{\theta}_1 \ldots \boldsymbol{\theta}_N]$ is defined as the sample set, and vector $\boldsymbol{\theta}_i$ denotes the sample of the uncertain parameters used in the i^{th} simulation. As $N \to \infty$, then $\hat{H}\to H$ but even for finite, large enough N (16) gives a good approximation for (15). The importance sampling density $p_{is}(\boldsymbol{\theta})$ may be used to improve the efficiency of this estimation. This is established by focusing on regions of the Θ space that contribute more to the integrand of the stochastic integral in (15) [13].

The optimal design choice is finally given by the *stochastic optimization*:

$$\boldsymbol{\varphi}^* = \arg\min_{\boldsymbol{\varphi}\in\Phi} \hat{H}(\boldsymbol{\varphi},\boldsymbol{\Omega}_N)\,. \tag{17}$$

The estimate for the objective function for this optimization involves an unavoidable estimation error and significant computational cost (since N evaluations of the model response are needed for each stochastic analysis), which make the optimization problem challenging. References [14, 15] provide a review of appropriate algorithms. These algorithms require though a large number of iterations, thus a large number of stochastic analyses. An efficient, two-stage alternative framework is discussed next.

B. SSO and two-stage framework

In the first stage of the framework a novel method called Stochastic Subset Optimization (SSO) [3] is applied for efficiently exploring the sensitivity of $E_{\boldsymbol{\theta}}[h(\boldsymbol{\varphi},\boldsymbol{\theta})]$ to both $\boldsymbol{\varphi}$ and $\boldsymbol{\theta}$. The basic idea in SSO is the formulation of an augmented stochastic problem where the design variables are artificially considered as uncertain with uniform distribution $p(\boldsymbol{\varphi})$ over the design space Φ. An auxiliary PDF $\pi(\boldsymbol{\varphi},\boldsymbol{\theta})$ is defined then as:

$$\pi(\boldsymbol{\varphi},\boldsymbol{\theta}) = h(\boldsymbol{\varphi},\boldsymbol{\theta})p(\boldsymbol{\varphi},\boldsymbol{\theta})/E_{\boldsymbol{\varphi},\boldsymbol{\theta}}[h(\boldsymbol{\varphi},\boldsymbol{\theta})] \tag{18}$$

with $p(\boldsymbol{\varphi},\boldsymbol{\theta})=p(\boldsymbol{\varphi})p(\boldsymbol{\theta})$. The integral in the denominator of $\pi(\boldsymbol{\varphi},\boldsymbol{\theta})$ is simply a normalization constant; it corresponds to the expected value for $h(\boldsymbol{\varphi},\boldsymbol{\theta})$ in the augmented uncertain space and is not required in the analysis. In the context of the augmented stochastic problem, the objective function $E_{\boldsymbol{\theta}}[h(\boldsymbol{\varphi},\boldsymbol{\theta})]$ is proportional to the marginal PDF $\pi(\boldsymbol{\varphi})$:

$$E_{\boldsymbol{\theta}}[h(\boldsymbol{\varphi},\boldsymbol{\theta})] = \pi(\boldsymbol{\varphi})E_{\boldsymbol{\varphi},\boldsymbol{\theta}}[h(\boldsymbol{\varphi},\boldsymbol{\theta})]/p(\boldsymbol{\varphi}) \propto \pi(\boldsymbol{\varphi}) \tag{19}$$

$$\pi(\boldsymbol{\varphi}) = \int_{\Theta} \pi(\boldsymbol{\varphi},\boldsymbol{\theta})d\boldsymbol{\theta}\,. \tag{20}$$

Since $E_{\boldsymbol{\varphi},\boldsymbol{\theta}}[h(\boldsymbol{\varphi},\boldsymbol{\theta})]$ and $p(\boldsymbol{\varphi})$ are constants, the marginal PDF $\pi(\boldsymbol{\varphi})$ expresses the sensitivity of the objective function $E_{\boldsymbol{\theta}}[h(\boldsymbol{\varphi},\boldsymbol{\theta})]$ with respect to the design variables. Samples of this PDF can be obtained through stochastic simulation techniques [3], for example using direct Monte Carlo or Markov Chain Monte Carlo sampling. These algorithms will give sample pairs $[\boldsymbol{\varphi},\boldsymbol{\theta}]$ that are distributed according to the joint distribution $\pi(\boldsymbol{\varphi},\boldsymbol{\theta})$, that is, according to $h(\boldsymbol{\varphi},\boldsymbol{\theta})p(\boldsymbol{\varphi},\boldsymbol{\theta})$. Their $\boldsymbol{\varphi}$ component corresponds to samples from the marginal distribution $\pi(\boldsymbol{\varphi})$.

A sensitivity analysis can be efficiently performed by exploiting the information in these samples; this is established in SSO by identifying the subset, within some class of admissible subsets A, that has the smallest estimated average value of $E_{\boldsymbol{\theta}}[h(\boldsymbol{\varphi},\boldsymbol{\theta})]$. This is equivalent to identifying the subset, within A, that has the smallest density of samples of $\pi(\boldsymbol{\varphi})$. At iteration k of the algorithm, additional samples that are distributed according to $\pi(\boldsymbol{\varphi})$ are obtained in the subset that was identified in the previous step I_{k-1}. A new region I_k for the optimal design parameters is then identified. Fig. 1 illustrates some of these concepts for a two dimensional design application. Note that for each iteration of the SSO algorithm, the computational burden for obtaining the samples is similar to the one for the analysis in (16); thus the computational cost is comparable to the cost required for a single evaluation of the objective function in (16). More details about SSO, including discussion on appropriate selection of the admissible subsets and stopping criteria for the iterative process may be found in [3].

When SSO has converged, all designs in the identified subset, I_{SSO}, give nearly the same value of $H(\boldsymbol{\varphi})$ and so can be considered near-optimal [3]. The center of the set I_{SSO}, $\boldsymbol{\varphi}_{SSO}$, can be taken as an approximation for $\boldsymbol{\varphi}^*$. If higher accuracy is required, a second optimization stage can be performed for pinpointing more precisely the optimal solution. Problem (17) is solved in this second stage exploiting *all information* available from SSO about the sensitivity with respect to *both* the design variables as well as the uncertain model parameters. The second type of

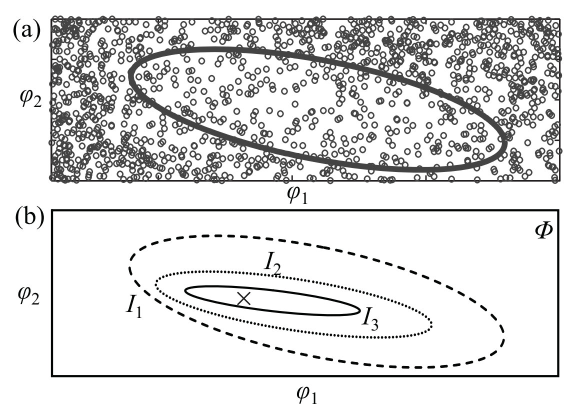

Fig. 1. Illustration of SSO process: (a) samples of design variables and first optimal subset (ellipse) and (b) sequence of identified subsets over 3 iterations. Rectangle corresponds to Φ and x to $\boldsymbol{\varphi}^*$.

information can be used to develop importance sampling densities $p_{is}(\boldsymbol{\theta})$ since the $\boldsymbol{\theta}$ component of the available samples is distributed proportional to the integrand in (15). The information for the sensitivity of the objective function to the design variables (corresponding to the orientation and relative size of the ellipses in Fig. 1) can be used to tune the characteristics of the algorithms used in the second stage. The search may be restricted only within I_{SSO} and efficient normalization of the design space or selection of the starting point for iterative algorithms may be established. Reference [3] provides details on how SSO can be efficiently combined with the Simultaneous Perturbation Stochastic Approximation (SPSA) algorithm which has been proven efficient for controller optimization problems [15].

V. Illustrative Example

The design concepts discussed in this paper are illustrated through a control example that considers the protection of a 3 story-building against dynamic earthquake excitation. The structure, shown in Fig. 2, is modeled as a 3-story shear building. An ideal active actuator between the ground and first floor implements a critically-damped positive position feedback control law with transfer function, as illustrated in Fig. 2, where K and ω_c are the free parameters (design variables) of the controller, corresponding to the controller gain and bandwidth parameter respectively, and $2\zeta=(2)^{1/2}$.

The excitation model chosen is the stationary response of a modified Kanai-Tajimi filter with transfer function:

$$H_w(s)=\sigma_o(2\zeta_g\omega_g s+\omega_g^2)/[(s^2+2\zeta_g\omega_g s+\omega_g^2)(s+\omega_v)] \quad (21)$$

where ω_g is the bandwidth, ζ_g=0.5 is the damping ratio and ω_v=15 Hz is a high-frequency pole. The gain σ_o is selected so that the RMS intensity of the earthquake input is a_{RMS}.

The performance variables $\mathbf{z}(t)$ are taken as the vector $\mathbf{d}(t)$ of inter-story drifts, the vector $\mathbf{a}(t)$ of absolute story accelerations, and the actuator force $u(t)$, normalized by appropriate thresholds (quantifying their relative importance) $\mathbf{z}(t)=[\mathbf{d}^T(t)/0.01\text{m}\ \mathbf{a}^T(t)/0.9g\ u(t)/500\text{kN}]^T$.

The model parameters assumed to be uncertain are the bandwidth and RMS intensity of the earthquake excitation and the inter-story stiffnesses k_i. Each is parameterized by

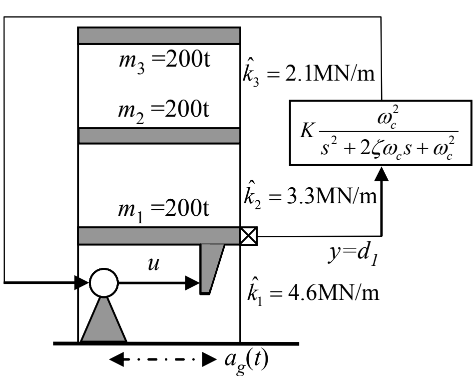

Fig. 2. Structural model.

$\omega_g=\hat{\omega}_g\theta_g$, $a_{RMS}=\hat{a}_{RMS}\theta_R$ and $k_i=\hat{k}_i\theta_{s,i}$, i=1,2,3, where $\hat{\omega}_g$, $\hat{a}_{RMS}$, $\hat{k}_i$ denote the most probable values for these parameter, selected as 2 Hz, 0.09g and the parameters given in Fig. 2, respectively. These model parameter values define the nominal system model. The parameters θ_g and θ_R are modeled to be independent uncertain variables while the $\{\theta_{s,i}\}$ are assumed to be to correlated with correlation matrix $\mathbf{R}$ with elements $R_{ij}=\{\exp[-(j-i)^2/2^2]\}$. The linear transformation $\boldsymbol{\theta}_{us}=\mathbf{S}^{-1}\boldsymbol{\theta}_s$ where $\mathbf{S}$ is the upper triangular Cholesky decomposition matrix for $\mathbf{R}$, is then introduced to obtain uncorrelated parameters $\{\theta_{us,i}\}$. Finally the set of model parameters for the problem is chosen as the vector with independent components $\boldsymbol{\theta}=[\theta_R, \theta_g, \theta_{us,i}\ i=1,2,3]$. Each of these parameters is assumed to take values in range [0.875 1.125] which creates a compact feasible set Θ_c for the model parameters. The set of admissible controllers is defined as the set that guarantee closed loop stable system for all $\boldsymbol{\theta}\in\Theta_c$. The probability models considered for $\boldsymbol{\theta}$ correspond to independent to Gaussian distributions with mean value 1 and standard deviation 0.05, truncated within Θ_c. The optimization is performed using the two stage framework discussed before with characteristics same as the ones reported in [3]. SPSA is used for the second stage.

A worst-case scenario design (denoted *WC*), a notion closer to the classical interpretation of robust feedback control, is considered for the $m\mathcal{H}_2$ case, described by:

$$\mathbf{K}^*_{WC}=\arg\min_{\mathbf{K}\in\mathcal{K}}(\max_{\boldsymbol{\theta}\in\Theta_c} J_{m\mathcal{H}_2}(\mathbf{K}\,|\,\boldsymbol{\theta})). \quad (22)$$

This problem is solved by means of a nonlinear min-max optimization using the powerful TOMLAB toolbox [16].

For better comparison when presenting the results the controller gain is normalized with respect to the optimal nominal reliability gain given by (4) and the bandwidth parameter with respect to the fundamental natural frequency of the uncontrolled structure 18.54 rad/sec.

Initially the $m\mathcal{H}_2$ performance is discussed. We consider all designs instances discussed in this study, that is, nominal (*Nom*), *WC*, *AR* and *RB*. For the *RB* design the threshold b is considered as a scaling of the optimal nominal performance, $b=0.0622\gamma$, where 0.0622 is the optimal $m\mathcal{H}_2$ performance for the nominal system and γ is the scaling factor that defines the acceptable performance bound relative to that

TABLE I
Results for multi-objective $\mathcal{H}_2$ design

Design Case	$\mathbf{K}^*$		Performance				
	K	ω_c	AR (10^{-2})	RB_1	$RB_{1.4}$ (10^{-1})	RB_2 (10^{-3})	WC
Nom	0.89	1.56	6.91	0.78	0.44	0.68	0.21
AR	0.98	1.52	**6.72**	0.76	0.41	0.60	0.22
RB_1	0.99	1.60	6.76	**0.75**	0.57	0.84	0.27
$RB_{1.4}$	0.95	1.44	6.76	0.81	**0.31**	0.32	0.19
RB_2	0.91	1.31	7.21	0.93	0.69	**0.28**	0.16
WC	0.93	1.17	7.51	0.96	1.14	0.31	**0.15**

optimal performance. Three values are considered for γ, 1, 1.4 and 2, and the associated design and performance are denoted by RB_γ. The results are reported in Table 1. For each performance quantification, corresponding to the columns of the table, the associated optimal design is denoted by bold characters. The *AR* and *Nom* designs are close with respect to both optimal controllers as well as associated *AR* performance. Also the sensitivity of the *AR* performance around the optimal design configuration is small. The *RB* optimal design configuration, now, is close to the aforementioned two for γ=1. As the threshold for acceptable performance increases though, i.e., as we focus on rare events for quantifying failure, that design moves further away from the *Nom* and the *AR* designs and it gets closer to a worst-case scenario design approach. Also the sensitivity of the performance objective around that optimal configuration becomes larger. This leads to an important implication: for designs problems for which the focus is on rare events, i.e., larger thresholds that determine acceptable system performance, the benefits from using an explicit reliability-based design approach are greater, compared to the designs that consider the nominal or the average performance. These remarks illustrate that important differences may exist between the two probabilistic objectives, *AR* and *RB*, especially for rare events. Thus, the designer needs to exercise some level of caution when defining performance in the stochastic design framework.

For the reliability-based design we discuss the nominal and the robust design, denoted respectively by *R* and *RR*. For the latter we consider the cases with fixed time duration (RR_c) and uncertain time duration (*RR*). Fig. 3 shows the *RR* and RR_c optimal controllers for various selections of T_m and *T* respectively, along with the nominal reliability controller. For small time durations (T_m or *T*), *RR* and RR_c designs are practically identical and converge to the controller that minimizes the expected value of the out-crossing rate (13). Only for time durations above a certain threshold differences become apparent. Comparing the nominal and robust reliability-based designs now, it is evident that again differences exist, and these differences become greater for large time horizons. A similar remark holds for the comparison between optimal reliability-based and optimal $m\mathcal{H}_2$ controllers, which shows that these two objectives represent different priorities for controller synthesis.

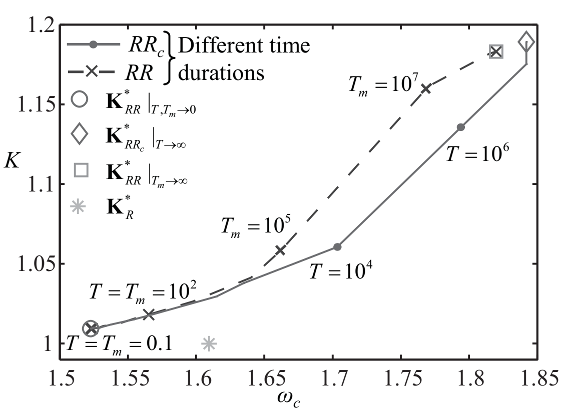

Fig. 3. Optimal controllers for reliability-based design.

VI. Conclusion

The robust stochastic performance optimization of linear dynamical systems with probabilistically described parametric model uncertainties was discussed in this paper. Both (i) reliability-based design and (ii) extension of pre-existing control methodologies to account for probabilistic information were considered. The design approach was illustrated in a structural control application. Probabilistically-robust controllers were demonstrated to yield considerable different designs compared to controllers optimized using only a nominal model, or the notion of worst-case scenario design. Also, significant differences were shown in the design characteristics between the concepts of average robustness and reliability robustness for quantifying the probabilistic performance.

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