

On the Solution of the Path Following Problem for the PVTOL Aircraft

L. Consolini, M. Maggiore, M. Tosques, C. Nielsen

Abstract—This article proposes a solution to the path following problem for the planar vertical take-off and landing aircraft (PVTOL) applicable to a class of smooth Jordan curves. Our solution relies on the stabilization of two nested embedded submanifolds of the state space that are defined based on the path one wishes to follow. The stabilization of these sets is performed using the ideas of transverse feedback linearization and finite-time stabilization. Our path following controller enjoys the two crucial properties of output invariance of the path (i.e., if the PVTOL's centre of mass is initialized on the path and its initial velocity is tangent to the path, then the PVTOL remains on the path at all future time) and boundedness of the roll dynamics. Further, our controller guarantees that, after a finite time, the time average of the roll angle is zero, and the PVTOL does not perform multiple revolutions about its longitudinal axis.

I. INTRODUCTION

In this paper we investigate the model of a V/STOL aircraft in planar vertical take-off and landing (PVTOL) mode, introduced by Hauser and co-workers in [1]

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= -u_1 \sin x_5 + \epsilon u_2 \cos x_5 \\
 \dot{x}_3 &= x_4 \\
 \dot{x}_4 &= -g + u_1 \cos x_5 + \epsilon u_2 \sin x_5 \\
 \dot{x}_5 &= x_6 \\
 \dot{x}_6 &= \mu u_2 \\
 y &= h(x) = \text{col}(x_1, x_3),
 \end{aligned} \tag{1}$$

where (x_1, x_3) are the coordinates of the centre of mass of the aircraft in the vertical plane, x_5 is the roll angle, and (x_2, x_4, x_6) are the corresponding velocities. The constants ϵ and μ are positive, and g denotes the acceleration due to gravity. The state space of the PVTOL is $M = \mathbb{R}^4 \times (\mathbb{R} \bmod 2\pi) \times \mathbb{R}$. The output of the system is the position of the aircraft's centre of mass. In various computations we will

denote by f , g_1 , and g_2 the drift and control vector fields in (1), so that $\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2$.

This paper continues the line of research initiated in [2], aimed at designing explicit path following controllers (as opposed to path following control *algorithms* relying on numerical computations) for the PVTOL aircraft. We refer the reader to the introduction of [2] for a literature overview on the subject. In [2], we designed a path following controller to make the PVTOL follow a unit circle. Two main problems remained open. First, it wasn't clear whether the approach in [2] could be extended to general curves. Second, the analysis of the dynamics on the so-called roll dynamics manifold in Section V.C of [2] relies on non rigorous considerations based on numerical plots. The roll dynamics manifold is a two-dimensional manifold on which the PVTOL is constrained to lie on the path and its roll angle is constrained to have a value that depends only on the displacement of the PVTOL on the path.

In this paper, we overcome both problems above by developing an explicit path following controller for the PVTOL, applicable to a large class of closed curves. The analysis of the dynamics on the roll dynamics manifold is now entirely justified thanks to the realization that the dynamics on this manifold are Hamiltonian.

Throughout the paper, we use the following notation. If N is a positive real number, $[\cdot]_N : \mathbb{R} \rightarrow \mathbb{R} \bmod N$ is the function mapping real numbers to their value modulo N . Given vectors $x, y \in \mathbb{R}^n$, we will denote by $\langle x, y \rangle$ the Euclidean inner product and by $\|x\|$ the associated Euclidean norm. Given a set $A \subset \mathbb{R}^n$, and a point $x \in \mathbb{R}^n$, we let $\|x\|_A := \inf_{a \in A} \|x - a\|$. We let $\text{col}(x_1, \dots, x_k) = [x_1 \ \dots \ x_k]^\top$. Given a function $\sigma : A \rightarrow B$, we let $\text{Im}(\sigma)$ denote its image. If $f(x_1, \dots, x_n)$ is a differentiable function, we denote by $\partial_{x_i} f$ its partial derivative with respect to x_i .

If M and N are two smooth manifolds and $F : M \rightarrow N$ is a map, we denote by dF_p the differential of F at $p \in M$. If M and N are open subsets of Euclidean spaces \mathbb{R}^m and \mathbb{R}^n , respectively, then dF_p is the familiar derivative of F at p , whose matrix representation is the $n \times m$ Jacobian of F . In this case, we will not distinguish, notationally, between the map dF_p and its matrix representation. In particular, if $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ is a real-valued function then, depending on the context, $d\lambda_x$ may represent the differential map $\mathbb{R}^n \rightarrow \mathbb{R}$ or the row vector $[\partial_{x_1} \lambda \ \dots \ \partial_{x_n} \lambda]$. On the other hand, we will denote by $\nabla_x \lambda$ the column vector $d\lambda_x^\top$. Given a vector field f , the directional derivative of λ along f , denoted by $L_f \lambda$, is given by $L_f \lambda(x) = \langle d\lambda_x, f(x) \rangle$. If $\phi : \mathbb{D} \rightarrow M$ is a smooth map between manifolds, with either $\mathbb{D} = \mathbb{R}$ or

L. Consolini is with the Dipartimento di Ingegneria dell'Informazione, Parco Area delle Scienze 181/a, 43100 Parma, Italy, E-mail: luca.consolini@polirone.mn.it

M. Maggiore is with the Dept. of Electrical and Computer Engineering, University of Toronto, 10 King's College Rd., Toronto, ON, M5S 3G4, Canada, E-mail: maggiore@control.utoronto.ca

M. Tosques is with the Dipartimento di Ingegneria Civile, Parco Area delle Scienze 181/a, 43100 Parma, Italy, E-mail: mario.tosques@unipr.it

C. Nielsen is with the Dept. of Electrical and Computer Engineering, University of Toronto, 10 King's College Rd., Toronto, ON, M5S 3G4, Canada, E-mail: nielsen@control.utoronto.ca

L. Consolini and M. Tosques were partially supported by MIUR scientific funds in the framework of a PRIN project.

M. Maggiore and C. Nielsen were supported by the National Science and Engineering Research Council (NSERC) of Canada.

$\mathbb{D} = S^1$, and $d/d\theta$ is the tangent vector to \mathbb{D} at θ , we will denote by $\phi'(\theta) := d\phi_\theta(d/d\theta)$ the tangent vector at $\phi(\theta)$.

II. PATH FOLLOWING PROBLEM

Consider a regular Jordan curve \mathcal{C} of length L in the y plane with smooth parameterization $\tilde{\sigma}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^2$, $\text{Im}(\tilde{\sigma}) = \mathcal{C}$. Assume, without loss of generality, that $\tilde{\sigma}$ is a unit speed parameterization, i.e., $\|\tilde{\sigma}'(\cdot)\| \equiv 1$. With this assumption, the map $\tilde{\sigma}$ is L -periodic. We will also assume that \mathcal{C} can be expressed in implicit form as

$$\mathcal{C} = \{y \in U : \gamma(y) = 0\},$$

where $\gamma : W \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is a smooth function such that $d\gamma_y \neq 0$ on W , and W is an open set containing \mathcal{C} . Without loss of generality, we will assume that $\|d\gamma_y\| = 1$ for all $y \in \mathcal{C}$ (for, if that isn't the case, we may replace $\gamma(y)$ by $\gamma(y)/\|d\gamma_y\|$, whose differential has unit norm on \mathcal{C}).

Being Jordan and smooth, the curve \mathcal{C} is diffeomorphic to the set $\mathbb{R} \bmod L$, and the diffeomorphism between the two sets is produced as follows. Since $\tilde{\sigma}$ is L -periodic, any two points t and $t + L$ in the domain of $\tilde{\sigma}$ can be identified. Henceforth, we denote by S^1 the set $\mathbb{R} \bmod L$. We define a map $\sigma : S^1 \rightarrow \mathbb{R}^2$ through the identity $\sigma([t]_L) = \tilde{\sigma}(t)$ for all $t \in \mathbb{R}$. Now σ maps S^1 diffeomorphically onto \mathcal{C} , and it has the same properties of $\tilde{\sigma}$: $\text{Im}(\sigma) = \mathcal{C}$, $\|\sigma'(\cdot)\| \equiv 1$. Let $\varphi(\theta) : S^1 \rightarrow \mathbb{R} \bmod 2\pi$ be the map associating to each θ the angle of the tangent vector $\sigma'(\theta)$ to \mathcal{C} at $\sigma(\theta)$. Then, the derivative φ' is the signed curvature of \mathcal{C} . Throughout this paper, we restrict the geometry of \mathcal{C} by means of the next assumption.

Assumption 1 (Curve geometry): (i) There exists $\theta_0 \in S^1$ such that $\varphi(\theta_0) = 0$ and

$$\varphi'(\theta_0 + \theta) = \varphi'(\theta_0 - \theta), \text{ for all } \theta \in S^1. \quad (2)$$

(ii) The curvature satisfies the inequality

$$|\varphi'(\theta)| < \sqrt{\left(\frac{2\pi}{L}\right)^2 + \left(\frac{\mu}{\epsilon}\right)^2} \text{ for all } \theta \in S^1. \quad (3)$$

Remark 2.1: Part (i) of the assumption implies that

$$\varphi(\theta_0 + \theta) = -\varphi(\theta_0 - \theta), \quad (4)$$

(note that the identity holds modulo 2π) and, using this fact, it easy to show that

$$\begin{aligned} \sigma_1(\theta_0 + \theta) - \sigma_1(\theta_0) &= -(\sigma_1(\theta_0 - \theta) - \sigma_1(\theta_0)) \\ \sigma_2(\theta_0 + \theta) - \sigma_2(\theta_0) &= \sigma_1(\theta_0 - \theta) - \sigma_1(\theta_0), \end{aligned}$$

where σ_1 and σ_2 are the components of the map σ . Therefore, part (i) requires \mathcal{C} to have a vertical symmetry axis passing through the point $\sigma(\theta_0)$. In particular, then, $\varphi(\theta_0 + L/2) = [\pi]_{2\pi}$. In the special case when \mathcal{C} is a circle of length L , radius $L/(2\pi)$, and unit speed parameterization

$$\sigma(\theta) = \frac{L}{2\pi} \text{col} \left(\cos \left(\frac{2\pi}{L} \theta \right), \sin \left(\frac{2\pi}{L} \theta \right) \right),$$

the assumption is satisfied with $\theta_0 = [-L/4]_L$.

The curvature of a circle of length L is $2\pi/L$. Thus, part (ii) of the assumption requires that the maximum curvature

of \mathcal{C} be not too much higher than that of a circle of the same length. In intuitive terms, the assumption limits the amount of deformation that one has to apply to a circle in order to obtain \mathcal{C} . The higher the ratio μ/ϵ is, the more the maximum curvature of \mathcal{C} can deviate from that of a circle of the same length, and hence the more one may deform the circle to obtain \mathcal{C} .

Path Following Problem (PFP): Given a Jordan curve $\sigma : S^1 \rightarrow \mathbb{R}^2$, with $\text{Im}(\sigma) = \mathcal{C}$, satisfying Assumption 1, find a continuous feedback $u(x) = \text{col}(u_1(x), u_2(x)) : M \rightarrow \mathbb{R}^2$ and an open set of initial conditions $U \subset M$ such that $\mathcal{C} \subset h(U)$, and the closed-loop system meets the following goals:

- G1** For each initial condition in U , at least one solution $x(t)$ to (1) exists for all $t \geq 0$ and all solutions are such that $y(t) := h(x(t)) \rightarrow \mathcal{C}$ in finite time.
- G2** The set \mathcal{C} is *output invariant* for the closed-loop system. In other words, if the centre of mass (x_1, x_3) is initialized on \mathcal{C} , and if the velocity vector (x_2, x_4) is initialized tangent to \mathcal{C} , then all solutions of the closed-loop system give output signals $y(t) \in \mathcal{C}$ for all $t \geq 0$.
- G3** For each initial condition in U , there exists a time $T_1 > 0$ such that after time T_1 , all output signals $y(t)$ trace the entire curve \mathcal{C} , i.e., $\text{Im}(y([T_1, +\infty))) = \mathcal{C}$, in a desired direction.
- G4** For each initial condition in U , there exists a time $T_2 > T_1$ after which the roll angle oscillates around its zero value, and its time average is zero. In other words, the aircraft does not undergo multiple revolutions about its longitudinal axis.

The reason for allowing continuous feedback, and therefore non-unique solutions, is that, in solving PFP, we will utilize the finite-time stabilization theory of [3], [4]. As mentioned in the introduction, goal **G2** is a crucial feature that distinguishes our control strategy from other path following control approaches in the literature and has considerable practical value. To illustrate its importance, suppose that a sudden disturbance slows down the PVTOL or even stops its motion without making its centre of mass abandon \mathcal{C} . If goal **G2** is met, then as soon as the disturbance vanishes, the PVTOL resumes its normal operation without leaving \mathcal{C} . On the other hand, a controller not meeting goal **G2** may have the undesirable property of making the PVTOL leave \mathcal{C} .

Our approach to solving PFP is summarized in the following steps.

- 1) We find the four-dimensional *path following submanifold* Γ_1^* associated with \mathcal{C} (see [5]), i.e., the maximal controlled invariant subset of $h^{-1}(\mathcal{C})$.
- 2) We use transverse feedback linearization (see [6]) to decompose system (1) into subsystems *tangential* and *transversal* to Γ_1^* , with the property that the transversal subsystem is linear time invariant (LTI). The tangential and transversal subsystems are driven by tangential and transversal control inputs, v^\parallel and v^\perp , respectively.
- 3) Using the theory of [4], we design the transversal controller v^\perp to finite-time stabilize the origin of the transversal subsystem. This controller meets goals **G1**

and **G2**.

- 4) We find a two-dimensional controlled invariant submanifold $\Gamma_2^* \subset \Gamma_1^*$, henceforth called the *roll dynamics submanifold*, on which the roll dynamics (subsystem with state (x_5, x_6)) meet goal **G4**. More precisely, for all initial conditions on Γ_2^* , the resulting roll angle, $x_5(t)$, is a periodic function with zero mean.
- 5) We design the tangential controller v^\flat to finite-time stabilize the roll dynamics submanifold Γ_2^* .
- 6) We show that the two-dimensional dynamics on Γ_2^* are Hamiltonian with energy $\mathcal{H} = T + V$ given by kinetic plus potential energy. Using this fact, we are able to completely characterize the motion on Γ_2^* . In particular, we show that there exist two open subsets of Γ_2^* corresponding to clockwise and counterclockwise motion of the PVTOL on the curve \mathcal{C} , thus satisfying goal **G3**.

III. SOLUTION OF PFP

In this section we carry out in detail each point of the program outline above.

A. Finding the path following manifold

In general, the path following manifold Γ_1^* associated with the curve \mathcal{C} (see [5]) is defined to be the maximal controlled invariant submanifold (if it exists) contained in $h^{-1}(\mathcal{C}) = \{x \in M : \gamma(\text{col}(x_1, x_3)) = 0\}$. As such, Γ_1^* is the collection of all possible motions generated by the control system with the property that their associated outputs lie in \mathcal{C} at all times. Equivalently, Γ_1^* is the zero dynamics manifold of (1) with output function $\tilde{\gamma}(x) := \gamma(\text{col}(x_1, x_3))$. In order to characterize Γ_1^* for the problem at hand, it suffices to notice that $\tilde{\gamma}$ yields a well-defined relative degree 2 everywhere on $h^{-1}(\mathcal{C})$, since $L_{g_i}\tilde{\gamma}(x) = 0$, for $i = 1, 2$ and for all $x \in M$, and

$$[L_{g_1}L_f\tilde{\gamma} \quad L_{g_2}L_f\tilde{\gamma}] = d\gamma_{(\text{col}(x_1, x_3))} \begin{bmatrix} -\sin x_5 & \epsilon \cos x_5 \\ \cos x_5 & \epsilon \sin x_5 \end{bmatrix}$$

has full rank 1 for all x in $h^{-1}(W)$. Therefore, the path following submanifold of (1) associated with \mathcal{C} is the four-dimensional submanifold

$$\begin{aligned} \Gamma_1^* &= \{x \in M : \tilde{\gamma}(x) = 0, L_f\tilde{\gamma} = 0\} \\ &= \{x \in M : \gamma(\text{col}(x_1, x_3)) = 0, \\ &\quad (\partial_{x_1}\gamma)x_2 + (\partial_{x_3}\gamma)x_4 = 0\}. \end{aligned} \quad (5)$$

B. Transverse feedback linearization

We now seek to transverse feedback linearize system (1) with respect to the set Γ_1^* (see [7], [8], [6]). Since Γ_1^* is the zero dynamics manifold associated with an output (the function $\tilde{\gamma}$) yielding a well-defined relative degree, transverse feedback linearization with respect to Γ_1^* amounts to standard input-output linearization (see Theorem 3.1 in [6]) from the output $\tilde{\gamma}$.

By the tubular neighborhood theorem (see [9]), there exists a sufficiently small constant $\epsilon > 0$ such that, letting $\mathcal{C}^\epsilon :=$

$\{y \in \mathbb{R}^2 : \|y\|_C < \epsilon\}$, the relation $\pi : \mathcal{C}^\epsilon \rightarrow S^1$ defined by

$$\pi(y) = \arg \min_{\theta \in S^1} \|y - \sigma(\theta)\|$$

is a well-defined smooth map and, for all $y \in \mathcal{C}^\epsilon$, $d\pi_y \neq 0$. For instance, if \mathcal{C} were a circle centred at the origin, then we would have $\pi(y) = \frac{L}{2\pi} \arg(y_1 + iy_2)$. This function is smooth everywhere except at the origin.

Lemma 3.1: For each $y \in \mathcal{C}$, the matrix

$$D(y) := \begin{bmatrix} d\pi_y \\ d\gamma_y \end{bmatrix} = \begin{bmatrix} \partial_{y_1}\pi & \partial_{y_2}\pi \\ \partial_{y_1}\gamma & \partial_{y_2}\gamma \end{bmatrix} \quad (6)$$

is orthogonal.

Unfortunately, due to space constraints, the proof of Lemma 3.1 and all subsequent proofs have been omitted. We invite the interested reader to contact the authors for an extended version of this article containing these proofs.

Lemma 3.2: The coordinate transformation

$$T : x \mapsto (\eta_1, \eta_2, \eta_3, \eta_4, \xi_1, \xi_2) \in (\mathbb{R} \bmod 2\pi) \times \mathbb{R} \times S^1 \times \mathbb{R}^3$$

defined as

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} x_5 \\ x_6 \\ \pi(\text{col}(x_1, x_3)) \\ d\pi_{\text{col}(x_1, x_3)}(\text{col}(x_2, x_4)) \\ \gamma(\text{col}(x_1, x_3)) \\ d\gamma_{\text{col}(x_1, x_3)}(\text{col}(x_2, x_4)) \end{bmatrix}, \quad (7)$$

is a diffeomorphism of a neighborhood V of Γ_1^* onto its image.

The system in new coordinates reads as

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \mu u_2 \\ \dot{\eta}_3 &= \eta_4 \\ \dot{\eta}_4 &= d\pi \text{col}(x_2, x_4) - g \partial_{x_3}\pi \\ &\quad + d\pi_{\text{col}(x_1, x_3)} \begin{bmatrix} -\sin x_5 & \epsilon \cos x_5 \\ \cos x_5 & \epsilon \sin x_5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= d\gamma \text{col}(x_2, x_4) - g \partial_{x_3}\gamma \\ &\quad + d\gamma_{\text{col}(x_1, x_3)} \begin{bmatrix} -\sin x_5 & \epsilon \cos x_5 \\ \cos x_5 & \epsilon \sin x_5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \end{aligned}$$

where $\dot{d}\pi$ and $\dot{d}\gamma$ are the time derivatives of the row vectors $d\pi_{\text{col}(x_1, x_3)}$ and $d\gamma_{\text{col}(x_1, x_3)}$ along the vector field f .

Since $d\pi_{\text{col}(x_1, x_3)}$ and $d\gamma_{\text{col}(x_1, x_3)}$ are linearly independent on Γ_1^* , they remain so in a neighborhood of Γ_1^* , without loss of generality on V . Consider the regular feedback transformation on V ,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin x_5 & \epsilon \cos x_5 \\ \cos x_5 & \epsilon \sin x_5 \end{bmatrix}^{-1} D(\text{col}(x_1, x_3))^{-1} \cdot \left(- \begin{bmatrix} \dot{d}\pi \\ \dot{d}\gamma \end{bmatrix} \text{col}(x_2, x_4) + g \begin{bmatrix} \partial_{x_3}\pi \\ \partial_{x_3}\gamma \end{bmatrix} + \begin{bmatrix} v^\parallel \\ v^\flat \end{bmatrix} \right), \quad (8)$$

where v^{\parallel} and $v^{\hat{\eta}}$ are new control inputs. The PVTOL after coordinate and feedback transformation reads as

$$\begin{aligned}\dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \mu u_2(\eta_1, \eta_3, \xi_1, v^{\parallel}, v^{\hat{\eta}}) \\ \dot{\eta}_3 &= \eta_4 \\ \dot{\eta}_4 &= v^{\parallel} \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v^{\hat{\eta}}.\end{aligned}\quad (9)$$

In (η, ξ) coordinates, we have $T(\Gamma_1^*) = \{\xi = 0\}$. Therefore, the ξ subsystem describes the dynamics of the PVTOL transversal to Γ_1^* , and for this reason it is called the *transversal subsystem*, driven by the *transversal input* $v^{\hat{\eta}}$. On the other hand, the restriction of the η subsystem to $T(\Gamma_1^*)$ when $v^{\hat{\eta}} = 0$ represents the dynamics of the PVTOL on Γ_1^* , and is therefore referred to as the *tangential subsystem*, driven by the *tangential input* v^{\parallel} . We will explore the structure of the tangential subsystem in more detail in Lemma 3.3.

C. Transversal control design

This part is identical to that of our previous work in [2] and so the presentation is succinct. Since the transversal subsystem in (9) is a double integrator, following the work in [4], one defines a controller $v^{\hat{\eta}}(\xi)$ guaranteeing that trajectories of the ξ subsystem are stable and converge to zero in a finite time which is uniform over compact sets of initial conditions.

D. Finding the roll dynamics submanifold

In this section we consider the motion of the PVTOL on the path following manifold Γ_1^* .

Lemma 3.3: The tangential subsystem on Γ_1^* , obtained from (9) by setting $\xi = 0$ and $v^{\hat{\eta}} = 0$, is given by

$$\begin{aligned}\dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \frac{\mu}{\epsilon} \left(g \sin \eta_1 + \sin(\eta_1 - \varphi(\eta_3)) \varphi'(\eta_3) \eta_4^2 \right. \\ &\quad \left. + \cos(\eta_1 - \varphi(\eta_3)) v^{\parallel} \right) \\ \dot{\eta}_3 &= \eta_4 \\ \dot{\eta}_4 &= v^{\parallel}.\end{aligned}\quad (10)$$

Next, in order meet goal **G4**, we will impose a ‘‘virtual constraint’’ for the roll angle η_1 , $\eta_1 = f(\eta_3)$, where f is a smooth function to be determined so that

- (i) f is a well-defined function on S^1 . In other words, there exists an L -periodic smooth function $\tilde{f}(x)$ on \mathbb{R} such that $f([x]_L) = \tilde{f}(x)$ for all $x \in \mathbb{R}$, and
- (ii) f has zero mean: $\int_0^L f(\eta_3) d\eta_3 = 0$.

The constraint in question will identify a submanifold $\Gamma_2^* \subset \Gamma_1^*$. In Corollary 3.6 we will show that property (ii) of f implies that, on Γ_2^* , the signal $\eta_1(t) = f(\eta_3(t))$ has time average zero, and thus goal **G4** is met. In order for $\eta_1 = f(\eta_3)$ to be a feasible constraint for (10), we need

$$\begin{aligned}\dot{\eta}_1 &= \eta_2 = f'(\eta_3) \eta_4 \\ \dot{\eta}_1 &= \dot{\eta}_2 = f''(\eta_3) \eta_4^2 + f'(\eta_3) v^{\parallel}.\end{aligned}$$

Thus, if

$$\begin{aligned}\frac{\mu}{\epsilon} \left(g \sin \eta_1 + \sin(\eta_1 - \varphi) \varphi' \eta_4^2 + \cos(\eta_1 - \varphi) v^{\parallel} \right) \Big|_{\eta_1=f(\eta_3)} \\ = f''(\eta_3) \eta_4^2 + f'(\eta_3) v^{\parallel},\end{aligned}$$

then the two dimensional submanifold

$$T(\Gamma_2^*) := \{(\eta, \xi) : \eta_1 = f(\eta_3), \eta_2 = f'(\eta_3) \eta_4, \xi = 0\} \quad (11)$$

is controlled invariant. Solving the above equation for v^{\parallel} we obtain

$$v^{\parallel} = \frac{\mu g \sin f(\eta_3) + [\sin(f(\eta_3) - \varphi(\eta_3)) \varphi'(\eta_3) - \frac{\epsilon}{\mu} f''(\eta_3)] \eta_4^2}{\epsilon f'(\eta_3) - \frac{\mu}{\epsilon} \cos(f(\eta_3) - \varphi(\eta_3))} \quad (12)$$

The feedback v^{\parallel} is smooth if its denominator is bounded away from zero. If there exists a function f satisfying properties (i) and (ii) above, and such that, for all $\eta_3 \in S^1$, $f'(\eta_3) - \frac{\mu}{\epsilon} \cos(f(\eta_3) - \varphi(\eta_3)) \neq 0$, then on the corresponding controlled invariant set Γ_2^* defined by (11), the system meets goal **G4**. Note that there may be many choices of f yielding the desired result. We find one such function by imposing that

$$f'(\eta_3) = \frac{\mu}{\epsilon} \cos(f(\eta_3) - \varphi(\eta_3)) + \varphi'(\eta_3) - \delta_0, \quad (13)$$

where δ_0 is a positive constant yet to be specified such that $|\varphi'(\eta_3) - \delta_0| > 0$ for all $\eta_3 \in S^1$. Letting $\lambda := f - \varphi$, the above equation becomes

$$\lambda' = \frac{\mu}{\epsilon} \cos \lambda - \delta_0,$$

where the prime indicates differentiation with respect to η_3 . The above first-order ODE can be explicitly integrated on \mathbb{R} ,

$$\lambda(x) = \left[-\frac{\pi}{2} + 2 \arctan \left(\frac{1}{\delta_0} \left(\alpha \tan \left(\frac{\alpha}{2} (K - x) \right) + \frac{\mu}{\epsilon} \right) \right) \right]_{2\pi}$$

where $\alpha = \sqrt{\delta_0^2 - \left(\frac{\mu}{\epsilon}\right)^2}$ and K is the integration constant. The function $\arctan(\cdot)$ has 2π jumps whenever the argument of $\tan(\cdot)$ is $\pm\pi/2$, but λ is smooth because its codomain is $\mathbb{R} \bmod 2\pi$. Next, we impose that $\alpha/2 = \pi/L$, or

$$\delta_0 = \sqrt{\left(\frac{2\pi}{L}\right)^2 + \left(\frac{\mu}{\epsilon}\right)^2}, \quad (14)$$

so that the function $\tilde{f}(x) := [\lambda(x) + \varphi([x]_L)]_{2\pi}$ is smooth and L -periodic. This latter property allows us to define $f(\eta_3)$ by replacing x with η_3 in \tilde{f} :

$$\begin{aligned}f(\eta_3) &= \left[\varphi(\eta_3) - \frac{\pi}{2} \right. \\ &\quad \left. + 2 \arctan \left(\frac{1}{\delta_0} \left(\frac{2\pi}{L} \tan \left(\frac{\pi}{L} (K - \eta_3) \right) + \frac{\mu}{\epsilon} \right) \right) \right]_{2\pi}.\end{aligned}\quad (15)$$

The function $f(\eta_3)$ is smooth, well-defined on S^1 , and satisfies (13). Moreover, by inequality (3) in Assumption 1, the choice of δ_0 above yields $|\varphi'(\eta_3) - \delta_0| > 0$ for all $\eta_3 \in S^1$, and so the feedback v^{\parallel} in (12) is smooth. Next, we choose the integration constant K in $\lambda(x)$ to ensure that f has zero mean. We'll do that by imposing that f is odd with

respect to θ_0 , which occurs when f' is even with respect to θ_0 and $f(\theta_0) = 0$. Letting

$$\bar{\eta}_3(K) := K - \frac{L}{\pi} \arctan\left(\frac{L\delta_0}{2\pi} - \frac{\mu L}{2\pi\epsilon}\right),$$

we have $\lambda(\bar{\eta}_3(K)) = 0$ and it is not hard to see that $\cos \lambda(\eta_3)$ is an even function with respect to $\bar{\eta}_3$, i.e., $\cos[\lambda(\bar{\eta}_3(K) + \eta_3)] = \cos[\lambda(\bar{\eta}_3(K) - \eta_3)]$.

By (2) in Assumption 1, φ' is an even function with respect to θ_0 . We choose K so that $\bar{\eta}_3(K) = \theta_0$, i.e.,

$$K = \frac{L}{\pi} \arctan\left(\frac{L\delta_0}{2\pi} - \frac{\mu L}{2\pi\epsilon}\right) + \theta_0, \quad (16)$$

so that $\cos \lambda(\eta_3)$ is now even with respect to θ_0 . With this choice, from (13) we conclude that $f'(\eta_3)$ is even with respect to θ_0 . Since

$f(\theta_0) = [\lambda(\theta_0) + \varphi(\theta_0)]_{2\pi} = [\lambda(\bar{\eta}_3(K)) + \varphi(\theta_0)]_{2\pi} = [0]_{2\pi}$ and f' is even, we have that $f(\eta_3)$ is odd with respect to θ_0 , i.e.,

$$f(\theta_0 + \eta_3) = -f(\theta_0 - \eta_3). \quad (17)$$

This fact ensures that f has zero mean, as required.

To summarize, picking K as in (16), the function $f : S^1 \rightarrow \mathbb{R} \bmod 2\pi$ given by (15) is smooth, L -periodic, and has zero mean. Moreover, the feedback v^\parallel in (12) is smooth and renders the submanifold $\Gamma_2^* \subset \Gamma_1^*$ defined in (11) invariant. We call this submanifold the *roll dynamics manifold*. From a physical point of view, when the state of the PVTOL is on Γ_2^* , the roll angle x_5 (equal to η_1) is completely determined by the position $\pi(y)$ (equal to η_3) of the aircraft on \mathcal{C} by means of the virtual constraint $\eta_1 = f(\eta_3)$. Hence, no matter what are the dynamics of the aircraft's centre of mass on \mathcal{C} , the roll angle does not perform multiple revolutions about its longitudinal axis.

E. Tangential control design

Having identified a submanifold $\Gamma_2^* \subset \Gamma_1^*$ on which the roll angle of the PVTOL exhibits desired properties, we use the tangential input v^\parallel in (10) to stabilize Γ_2^* . The development in this section is similar to that in our previous work [2] and so the presentation is concise. Define error variables $e_1 = \eta_1 - f(\eta_3)$, $e_2 = \eta_2 - f'(\eta_3)\eta_4$. On Γ_1^* , stabilizing Γ_2^* is equivalent to stabilizing the origin of the error dynamics

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= \beta(\eta_1, \eta_3, \eta_4) - \left(f'(\eta_3) - \frac{\mu}{\epsilon} \cos(\eta_1 - \varphi(\eta_3))\right) v^\parallel, \end{aligned}$$

where $\beta = \frac{\mu}{\epsilon} \left(g \sin \eta_1 + [\sin(\eta_1 - \varphi(\eta_3))\varphi'(\eta_3) - \frac{\epsilon}{\mu} f''(\eta_3)]\eta_4^2\right)$. We pick the tangential controller on Γ_1^* to be

$$v^\parallel(\eta) = \frac{\beta(\eta_1, \eta_3, \eta_4) - w(e)}{f'(\eta_3) - \frac{\mu}{\epsilon} \cos(\eta_1 - \varphi(\eta_3))}, \quad (18)$$

where $w(e)$ is a continuous function chosen to finite-time stabilize the rotational (since e_1 is a variable in $\mathbb{R} \bmod 2\pi$) double integrator $\dot{e}_1 = e_2$, $\dot{e}_2 = w(e)$. The expression for $w(e)$ is found in equation (15) of [2] and is a slight

modification of the control law originally introduced in [4]. We omit it for brevity.

The tangential feedback v^\parallel in (18) is well-defined in the region of Γ_1^* where the denominator is bounded away from zero. On $\Gamma_2^* \subset \Gamma_1^*$, the denominator in question takes the form

$$f'(\eta_3) - \frac{\mu}{\epsilon} \cos(f(\eta_3) - \varphi(\eta_3))$$

which, by our choice of f , is equal to $\varphi'(\eta_3) - \delta_0$ and is bounded away from zero for all η_3 . Therefore, the denominator of (18) is bounded away from zero in a neighborhood of Γ_2^* in Γ_1^* . An estimate of this neighborhood is found by noting that

$$\begin{aligned} f'(\eta_3) - \frac{\mu}{\epsilon} \cos(\eta_1 - \varphi(\eta_3)) &= \varphi'(\eta_3) - \delta_0 \\ &+ \frac{\mu}{\epsilon} [\cos(\eta_1 - \varphi)(\cos e_1 - 1) + \sin(\eta_1 - \varphi) \sin e_1], \end{aligned}$$

and thus the left-hand side is bounded away from zero for all those values of e_1 such that $(\mu/\epsilon)(|1 - \cos e_1| + |\sin e_1|) < \min_{\eta_3 \in S^1} (\varphi'(\eta_3) - \delta_0)$. By (3), the right-hand side of this inequality is > 0 .

F. Motion on the roll dynamics manifold

So far we picked the transversal controller $v^\perp(\xi)$ to locally finite-time stabilize the path following manifold Γ_1^* defined in (5), and the tangential controller $v^\parallel(\eta)$ to locally finite-time stabilize Γ_2^* from within Γ_1^* . What is left is to do is to investigate the dynamics of the closed-loop system on the two dimensional submanifold Γ_2^* . On Γ_2^* , v^\parallel is given by (12). Substituting that expression in (10) and setting $\eta_1 = f(\eta_3)$, $\eta_2 = f'(\eta_3)\eta_4$ we obtain the reduced second-order system describing the motion on the roll dynamics manifold Γ_2^* ,

$$\begin{aligned} \dot{\eta}_3 &= \eta_4 \\ \dot{\eta}_4 &= \phi_1(\eta_3) + \phi_2(\eta_3)\eta_4^2 \end{aligned} \quad (19)$$

where,

$$\begin{aligned} \phi_1(\eta_3) &= \frac{(\mu/\epsilon)g \sin f(\eta_3)}{\varphi'(\eta_3) - \delta_0} \\ \phi_2(\eta_3) &= \frac{(\mu/\epsilon) \sin(f(\eta_3) - \varphi(\eta_3))\varphi'(\eta_3) - f''(\eta_3)}{\varphi'(\eta_3) - \delta_0}. \end{aligned}$$

Physically, the dynamics in (19) describe the evolution of the position and velocity of the PVTOL's centre of mass along the curve \mathcal{C} . In order to meet goal **G3**, it is desirable for η_3 to traverse the entire S^1 (which corresponds to the PVTOL traversing the entire curve \mathcal{C}) in a desired direction while η_4 remains bounded. We will show that almost all phase curves of \mathcal{C} can be of three types: equilibria, closed curves corresponding to oscillations (i.e., the PVTOL oscillates back and forth on a segment of \mathcal{C} without traversing the entire \mathcal{C}), and closed curves corresponding to complete rotations around S^1 . The latter is the type we are interested in, and we'll precisely characterize the domain of initial conditions of interest. The crucial realization to understanding the dynamics in (19) is that the system is Hamiltonian with energy function

$$\mathcal{H}(\eta_3, \eta_4) = \frac{1}{2}M(\eta_3)\eta_4^2 + V(\eta_3),$$

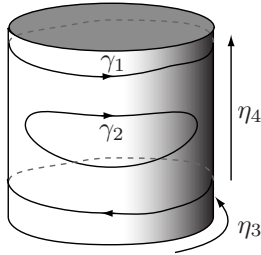


Fig. 1. Two phase curves illustrating Lemma 3.5. Curve γ_1 is homeomorphic to a circle $\{\eta_4 = \text{constant}\}$, while curve γ_2 is homeomorphic to a circle $\{(\eta_3 - \bar{\eta}_3)^2 + \eta_4^2 = \text{constant}\}$.

and canonical coordinates $(q, p) = (\eta_3, M(\eta_3)\eta_4)$. The functions M and V are given by

$$M(\eta_3) = \exp\left(-2 \int_0^{\eta_3} \phi_2(\tau) d\tau\right)$$

$$V(\eta_3) = - \int_0^{\eta_3} \phi_1(\mu) M(\mu) d\mu.$$

Note that $M'(\eta_3) = -2M(\eta_3)\phi_2(\eta_3)$ and $V'(\eta_3) = -M(\eta_3)\phi_1(\eta_3)$. Using these identities, it immediately follows that \mathcal{H} is a first integral of (19). Moreover, it is easy to see that in (q, p) coordinates (19) takes on the canonical form of Hamilton's equations.

Lemma 3.4: Consider the dynamics in (19) of the PVTOL on the roll manifold. The equilibria are pairs $(\eta_3^*, 0)$ given by values of η_3^* such that either $f(\eta_3^*) = [0]_{2\pi}$ or $f(\eta_3^*) = [\pi]_{2\pi}$. There are at least two equilibria corresponding to $\eta_3^* = \theta_0, \theta_0 + L/2$. The equilibrium at $(\theta_0 + L/2, 0)$ is always unstable. If there are only two equilibria, then the equilibrium at $(\theta_0, 0)$ is stable.

In the case that \mathcal{C} is a circle of length L , there are two equilibria at $(\theta_0, 0) = ([-L/4]_L, 0)$ and $(\theta_0 + L/2, 0) = ([L/4]_L, 0)$. They correspond to the south and north poles of the circle. The south pole is stable, while the north pole is unstable, so if the PVTOL is initialized on the roll manifold near the south pole of the circle, the aircraft oscillates back and forth around this point while maintaining a bounded roll angle.

Lemma 3.5: Let $\underline{\mathcal{H}} := \min_{\eta_3 \in S^1} V(\eta_3)$ and $\bar{\mathcal{H}} := \max_{\eta_3 \in S^1} V(\eta_3)$. Then, all phase curves of (19) in the set $\{(\eta_3, \eta_4) \in S^1 \times \mathbb{R} : \mathcal{H}(\eta_3, \eta_4) > \bar{\mathcal{H}}\}$ are homeomorphic to circles $\{(\eta_3, \eta_4) \in S^1 \times \mathbb{R} : \eta_4 = \text{constant}\}$. On the other hand, almost all phase curves in the set $\{(\eta_3, \eta_4) \in S^1 \times \mathbb{R} : \underline{\mathcal{H}} < \mathcal{H}(\eta_3, \eta_4) < \bar{\mathcal{H}}\}$ are homeomorphic to circles $\{(\eta_3, \eta_4) : (\eta_3 - \bar{\eta}_3)^2 + \eta_4^2 = \text{constant}\}$.

In words, almost all phase curves are closed; high-energy phase curves correspond to complete rotations of η_3 around S^1 in either direction, while low-energy ones correspond to oscillations around a point in S^1 , see Figure 1.

To better illustrate Lemma 3.5, consider the periodic cubic spline represented in Figure 2. We set $\mu = \epsilon = 1$, $g = 9.8$, the length of γ is numerically computed as $L = 85.63$. For this curve, Assumption 1 is satisfied because the curve has a vertical symmetry axis and it can be verified that $|\varphi'(\theta)| \leq$

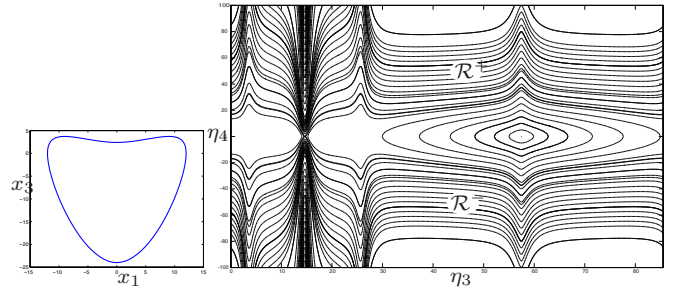


Fig. 2. A cubic spline satisfying Assumption 1 and the associated level sets of the energy \mathcal{H} .

$0.49, \forall \theta \in \mathbb{R}$ and $\sqrt{(\frac{2\pi}{L})^2 + (\frac{\mu}{\epsilon})^2} = 1.0027$. The level sets of $\mathcal{H}(\eta_3, \eta_4)$ and regions \mathcal{R}^+ and \mathcal{R}^- are shown in Figure 2.

Corollary 3.6: Let $\mathcal{R}^+ = \{(\eta_3, \eta_4) \in S^1 \times \mathbb{R} : \mathcal{H}(\eta_3, \eta_4) > \bar{\mathcal{H}}, \eta_4 > 0\}$ and $\mathcal{R}^- = \{(\eta_3, \eta_4) \in S^1 \times \mathbb{R} : \mathcal{H}(\eta_3, \eta_4) > \bar{\mathcal{H}}, \eta_4 < 0\}$. Then, for any initial condition in \mathcal{R}^+ (resp., \mathcal{R}^-), the PVTOL traverses \mathcal{C} in the positive (resp., negative) direction with bounded speed. Moreover, its roll angle $\eta_1(t) = f(\eta_3(t))$ is a periodic function with zero mean.

IV. CONCLUSION

Theorem 4.1: The smooth feedback given by the coordinate transformation $x \mapsto (\eta, \xi)$ in (7), feedback transformation (8), transversal feedback $v^{\text{th}}(\xi)$, tangential feedback $v^{\text{ll}}(\eta)$ in (18) (with $f(\eta_3)$ as in (15), δ_0 as in (14), and K as in (16)), solves PFP.

REFERENCES

- [1] J. Hauser, S. Sastry, and G. Meyer, "Nonlinear control design for slightly non-minimum phase systems: Applications to VSTOL aircraft," *Automatica*, vol. 28, no. 4, pp. 665–679, 1992.
- [2] C. Nielsen, L. Consolini, M. Maggiore, and M. Tosques, "Path following for the pvtol: A set stabilization approach," in *IEEE Conference on Decision and Control*, Cancun, Mexico, December 2008.
- [3] S. Bhat and D. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal of Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.
- [4] —, "Continuous finite-time stabilization of the translational and rotational double integrators," *IEEE Trans. on Automatic Control*, vol. 43, no. 5, pp. 678–682, 1998.
- [5] C. Nielsen, "Set stabilization using transverse feedback linearization," Ph.D. dissertation, Dept. of Electrical and Computer Engineering, Univ. of Toronto, 2008.
- [6] C. Nielsen and M. Maggiore, "On local transverse feedback linearization," *SIAM J. Control and Optimization*, vol. 47, no. 5, pp. 2227–2250, 2008.
- [7] A. Banaszuk and J. Hauser, "Feedback linearization of transverse dynamics for periodic orbits," *Systems and Control Letters*, vol. 26, no. 2, pp. 95–105, Sept. 1995.
- [8] C. Nielsen and M. Maggiore, "Output stabilization and maneuver regulation: A geometric approach," *Systems Control Letters*, vol. 55, pp. 418–427, 2006.
- [9] V. Guillemin and A. Pollack, *Differential Topology*. New Jersey: Prentice Hall, 1974.