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Uniform Semistability for Time-Varying Dynamical Systems and Network Consensus with Time-Dependent Communication Links

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Abstract— **This paper focuses on uniform semistability and uniform finite-time semistability for time-varying dynamical systems. Semistability is the property whereby the solutions of a dynamical system converge to Lyapunov stable equilibrium points determined by the system initial conditions. Using these results we develop a framework for designing semistable protocols in dynamical networks with time-dependent communication links. Specifically, we present distributed nonlinear time-varying control architectures for multiagent network consensus with dynamic communication links.**

I. INTRODUCTION

Modern complex dynamical systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks. Distributed decision-making for coordination of networks of dynamic agents involving information flow can be naturally captured by graph-theoretic notions. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles (UAV's) [1], autonomous underwater vehicles (AUV's) [2], distributed sensor networks [3], air and ground transportation systems [4], swarms of air and space vehicle formations [5], and congestion control in communication networks [6], to cite but a few examples. Hence, it is not surprising that a considerable research effort has been devoted to control of networks and control over networks in recent years.

Since communication links among multiagent systems are often unreliable due to multipath effects and exogenous disturbances, the information exchange topologies in network systems are often dynamic. In particular, communication links between different agents may be time-varying, which results in a time-dependent communication topology. In this case, the vector field defining the dynamical system is a time-varying function, and hence, system stability should be analyzed using Lyapunov theory for nonautonomous systems involving concepts such as weak and strong invariance notions, differential inclusions, and generalized gradients of locally Lipschitz functions [7].

In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop information consensus protocols for networks of dynamic agents wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or *consensus*. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. For such systems possessing a continuum of equilibria, *semistability* [8], [9], and not asymptotic stability, is the relevant notion of stability. Semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable.

In this paper we develop uniform semistability and uniform finite-time semistability theory for nonautonomous dynamical systems. Using these results, we develop distributed time-varying control algorithms for addressing consensus problems for nonlinear multiagent dynamical systems with time-dependent communication links. Unlike the results in [10], which deal with discrete-time dynamical systems, here we focus on continuous-time dynamical systems. The proposed controller architectures are predicated on the recently developed notion of system thermodynamics [11] resulting in time-varying controller architectures involving the exchange of information between agents that guarantee that the closed-loop dynamical network is consistent with basic thermodynamic principles.

II. LYAPUNOV-BASED SEMISTABILITY THEORY FOR NONAUTONOMOUS DYNAMICAL SYSTEMS

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $(\cdot)^{\mathrm{T}}$ denotes transpose. For $A \in \mathbb{R}^{n\times m}$ we write $\operatorname{rank} A$ to denote the rank of A and $\overline{\mathcal{S}}$ to denote the closure of the subset $\mathcal{S} \subset \mathbb{R}^n$. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball *centered* at α with *radius* ε, $\operatorname{dist}(p, \mathcal{M})$ for the distance from a point p to the set $\mathcal{M}$, that is, $\operatorname{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p - x\|$, and $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the

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set $\mathcal{M}$, that is, for each $\varepsilon > 0$ there exists $T > 0$ such that $\mathrm{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$.

Consider the time-varying dynamical system given by

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{1}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^q$, and $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ is a *Carathéodory function* [12], [13], that is, $f(\cdot, x)$ is measurable for all $x \in \mathbb{R}^q$, $f(t, \cdot)$ is continuous for all $t \in \mathbb{R}$, and f is locally integrably bounded on compact sets; that is, for every compact set $\mathcal{C} \subset \mathbb{R}^q$, there exists a Lebesgue measurable function $m(\cdot)$ such that the function $t \mapsto \|m(t)\|$ is locally Lebesgue integrable and $\|f(t,x)\| \leq m(t)$ for all $t \in \mathbb{R}$ and all $x \in \mathcal{C}$. A locally absolutely continuous function $x : [t_0, \tau) \to \mathbb{R}^q$ is said to be a *(Carathéodory) solution* of (1) on the interval $[t_0, \tau)$ with initial condition $x(t_0) = x_0$ if $x(t)$ satisfies (1) for almost all $t \in [t_0, \tau)$, and every solution can be maximally extended to a maximal interval $[t_0, \tau)$. Moreover, if x is maximal and $\tau < \infty$, then x is unbounded. A maximal solution x is called *global* if $\tau = \infty$. Let $\omega(x)$ denote the *positive limit set* (possibly empty) of a global solution x and recall that, if x is a bounded maximal solution, then $\tau = \infty$ and $\omega(x)$ is a nonempty, compact, connected set and is approached by x [7].

Let $\mathcal{U}$ denote the class of *set-valued maps* $\mathcal{F}(x) : \mathbb{R}^q \to \mathcal{B}(\mathbb{R}^q)$ that are upper semicontinuous at every $x \in \mathbb{R}^q$ and take nonempty convex compact values, where $\mathcal{B}(\mathbb{R}^q)$ denotes the collection of all subsets of $\mathbb{R}^q$. We assume that $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ is *weakly asymptotically autonomous* in the sense that there exists $\mathcal{F} \in \mathcal{U}$ such that for all compact $\mathcal{C} \subset \mathbb{R}^q$ and all $\varepsilon > 0$, there exists $T \geq 0$ such that for all $x \in \mathcal{C}$,

$$\operatorname*{ess\,sup}_{t \geq T} \mathrm{dist}(f(t,x), \mathcal{F}(x)) < \varepsilon. \tag{2}$$

If, in addition, $\mathcal{F}$ is singleton-valued, that is, $\mathcal{F} : x \mapsto \{g(x)\}$ for some continuous function $g : \mathbb{R}^q \to \mathbb{R}^q$, then we say f is *asymptotically autonomous* [7].

The *Filippov solution* [13], [14] of the *differential inclusion* [15] given by

$$\dot{x}(t) \in \mathcal{F}(x(t)), \quad \mathrm{a.\,a.} \in [0, \tau], \tag{3}$$

is defined by an absolutely continuous function $x : [0, \tau] \to \mathbb{R}^q$ such that (3) holds almost everywhere for all $t \in [0, \tau]$. Since the set-valued map given by $\mathcal{F}$ is upper semicontinuous with nonempty, convex, and compact values, and is also locally bounded, it follows that Filippov solutions to (3) exist [13]. We say that a set $\mathcal{M} \subset \mathbb{R}^q$ is *weakly invariant* (resp., *strongly invariant*) with respect to (3) if for every $x_0 \in \mathcal{M}$, $\mathcal{M}$ contains a maximal solution (resp., all maximal solutions) of (3) [16], [17].

An equilibrium point of (3) is a point $x_{\mathrm{e}} \in \mathbb{R}^q$ such that $0 \in \mathcal{F}(x_{\mathrm{e}})$. It is easy to see that x_{e} is an equilibrium point of (3) if and only if the constant function $x(\cdot) = x_{\mathrm{e}}$ is a Filippov solution of (3). We denote the set of equilibrium points of (3) by $\mathcal{E}$. Since the set-valued map $\mathcal{F}$ is upper semicontinuous, it follows that $\mathcal{E}$ is closed. To develop Lyapunov-based stability theory for nonautonomous dynamical systems of the form given by (1), we need to introduce the notion of generalized derivatives and gradients. Here we focus on Clarke generalized derivatives and gradients [18].

Definition 2.1 ([18]): Let $V : \mathbb{R}^q \to \mathbb{R}$ be a locally Lipschitz continuous function. The *Clarke upper generalized derivative* of $V(x)$ at x in the direction of v is defined by

$$V^o(x, v) \triangleq \limsup_{y \to x, h \to 0^+} \frac{V(y + hv) - V(y)}{h}. \tag{4}$$

The *Clarke generalized gradient* $\partial V : \mathbb{R}^q \to \mathcal{B}(\mathbb{R}^q)$ of $V(x)$ at x is the set

$$\partial V(x) \triangleq \mathrm{co}\left\{ \lim_{i \to \infty} \nabla V(x_i) : x_i \to x, \, x_i \notin \mathcal{N} \cup \mathcal{S} \right\}, \tag{5}$$

where "co" denotes the convex hull, ∇ denotes the nabla operator, $\mathcal{N}$ is the set of measure zero points where ∇V does not exist, and $\mathcal{S}$ is an arbitrary set of measure zero in $\mathbb{R}^q$.

The next definition introduces the notion of semistability for Filippov dynamical systems. For this definition, Lyapunov stability for the solution $x(t) \equiv z$ to (3) can be found in [13] and [17].

Definition 2.2 ([19]): Let $\mathcal{D} \subseteq \mathbb{R}^q$ be an open strongly invariant set with respect to the differential inclusion (3). An equilibrium point $z \in \mathcal{D}$ of (3) is *semistable* with respect to $\mathcal{D}$ if it is Lyapunov stable and there exists an open subset $\mathcal{D}_0$ of $\mathcal{D}$ containing z such that for all initial conditions in $\mathcal{D}_0$, the Filippov solutions of (3) converge to a Lyapunov stable equilibrium point. The system (3) is *semistable* with respect to $\mathcal{D}$ if every equilibrium point in $\mathcal{E}$ is semistable with respect to $\mathcal{D}$. Finally, (3) is said to be globally semistable if (3) is semistable and $\mathcal{D} = \mathbb{R}^q$.

Next, we introduce the definition of finite-time semistability of (3).

Definition 2.3 ([19]): Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to the differential inclusion (3). An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (3) is said to be *finite-time-semistable* if there exist an open neighborhood $\mathcal{U} \subseteq \mathcal{D}$ of x_{e} and a function $T : \mathcal{U} \backslash \mathcal{E} \to (0, \infty)$, called the *settling-time function*, such that the following statements hold:

$i)$ For every $x \in \mathcal{U} \backslash \mathcal{E}$ and any Filippov solution $\psi(t)$ of (3) with $\psi(0) = x$, $\psi(t) \in \mathcal{U} \backslash \mathcal{E}$ for all $t \in [0, T(x))$, and $\lim_{t \to T(x)} \psi(t)$ exists and is contained in $\mathcal{U} \cap \mathcal{E}$.

$ii)$ x_{e} is semistable.

An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (3) is said to be *globally finite-time-semistable* if it is finite-time-semistable with $\mathcal{D} = \mathcal{U} = \mathbb{R}^q$. The system (3) is said to be *finite-time-semistable* if every equilibrium point in $\mathcal{E}$ is finite-time-semistable. Finally, (3) is said to be *globally finite-time-semistable* if every equilibrium point in $\mathcal{E}$ is globally finite-time-semistable.

The following result asserts that if f is weakly asymptotically autonomous and x is a bounded global solution of (1), then the positive limit set $\omega(x)$ of x is weakly invariant with respect to the associated autonomous differential inclusion (3).

Lemma 2.1 ([7]): Let $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ be a Carathéodory function and weakly asymptotically autonomous with some $\mathcal{F} \in \mathcal{U}$. Let $x_0 \in \mathbb{R}^q$ and assume that x is a global solution of (1). If x is bounded, then the positive limit set $\omega(x)$ of x is nonempty, compact, connected, is approached by x, and is weakly invariant with respect to (3).

Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to the differential inclusion (3). For $x \in \mathcal{D}$, let $\psi(\cdot)$ denote the Filippov solution to (3) with $\psi(0) = x$ and let $\Omega(\psi)$ be the positive limit set of ψ.

Lemma 2.2: Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to (3) and let $x \in \mathcal{D}$. If $z \in \Omega(\psi) \cap \mathcal{D}$ is a Lyapunov stable equilibrium point with respect to $\mathcal{D}$, then $z = \lim_{t\to\infty} \psi(t)$ with $\psi(0) = x$ and $\Omega(\psi) = \{z\}$.

Combining Lemmas 2.1 and 2.2 yields the following result.

Lemma 2.3: Let $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ be a Carathéodory function and weakly asymptotically autonomous with $\mathcal{F} \in \mathcal{U}$. Let $x_0 \in \mathbb{R}^q$ and assume that x is a global solution of (1). If x is bounded and $z \in \omega(x)$ is a Lyapunov stable equilibrium point, then $z = \lim_{t\to\infty} x(t)$ and $\omega(x) = \{z\}$.

Recall that an equilibrium point of (1) is a point $x_{\mathrm{e}} \in \mathbb{R}^q$ such that $f(t, x_{\mathrm{e}}) = 0$ for all $t \in \mathbb{R}$. We denote the set of equilibrium points of (1) by $\mathcal{E}_{\mathrm{n}}$.

Lemma 2.4: Let $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ be a Carathéodory function and weakly asymptotically autonomous with $\mathcal{F} \in \mathcal{U}$. Then $\mathcal{E} = \mathcal{E}_{\mathrm{n}}$.

Lemma 2.5: Let $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ be a Carathéodory function and weakly asymptotically autonomous with some $\mathcal{F} \in \mathcal{U}$. Then (1) is uniformly (Lyapunov) stable if and only if (3) is Lyapunov stable.

The following definition gives the notion of uniform semistability for the nonautonomous dynamical system (1).

Definition 2.4: An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}_{\mathrm{n}}$ of (1) is *uniformly semistable* if x_{e} is uniformly Lyapunov stable and there exists $\delta > 0$ such that, for every $x_0 \in \mathbb{R}^q$ satisfying $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, there exists a uniformly Lyapunov stable equilibrium point $z_{x_0} \in \mathcal{E}_{\mathrm{n}}$ such that every solution $x(t)$, $t \geq t_0$, with the initial condition $x(t_0) = x_0$ satisfies $\lim_{t\to\infty} x(t) = z_{x_0}$ uniformly in $t_0 \in \mathbb{R}$, that is, for every $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that $\|x(t) - z_{x_0}\| < \varepsilon$ for every $t \geq t_0 + T(\varepsilon)$. The system (1) is *uniformly semistable* if all the equilibrium points of (1) are uniformly semistable.

The relationship between uniform semistability of (1) and semistability of (3) is given by the following result.

Proposition 2.1: Let $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ be a Carathéodory function and weakly asymptotically autonomous with some $\mathcal{F} \in \mathcal{U}$. Then (1) is uniformly semistable if and only if (3) is semistable.

Next, the notion of uniform finite-time semistability can be defined by using uniform semistability and Definition 2.3.

Definition 2.5: Let $\mathcal{D} \subseteq \mathbb{R}^q$ be nonempty and open. Assume that all the global solutions $x(t)$ satisfy $x(t) \in \mathcal{D}$ for all $t \geq t_0$ and all $x_0 \in \mathcal{D}$. An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}_{\mathrm{n}}$ of (1) is said to be *uniformly finite-time-semistable* if there exist an open neighborhood $\mathcal{U} \subseteq \mathcal{D}$ of x_{e} and a function $T : \mathcal{U}\backslash\mathcal{E}_{\mathrm{n}} \to (0, \infty)$, called the *uniform settling-time function*, such that the following statements hold:

$i)$ For every $x \in \mathcal{U}\backslash\mathcal{E}_{\mathrm{n}}$ and any global solution $\psi(t)$ of (1) with $\psi(t_0) = x$, $\psi(t) \in \mathcal{U}\backslash\mathcal{E}_{\mathrm{n}}$ for all $t - t_0 \in [0, T(x))$, and $\lim_{t-t_0\to T(x)} \psi(t - t_0) < \infty$ and is contained in $\mathcal{U} \cap \mathcal{E}_{\mathrm{n}}$.

$ii)$ x_{e} is uniformly semistable.

An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}_{\mathrm{n}}$ of (1) is said to be *globally uniformly finite-time-semistable* if it is finite-time-semistable with $\mathcal{D} = \mathcal{U} = \mathbb{R}^q$. The system (1) is said to be *uniformly finite-time-semistable* if every equilibrium point in $\mathcal{E}_{\mathrm{n}}$ is finite-time-semistable. Finally, (1) is said to be *globally uniformly finite-time-semistable* if every equilibrium point in $\mathcal{E}_{\mathrm{n}}$ is globally uniformly finite-time-semistable.

The following result gives a relationship between uniform semistability of (1) and semistability of (3).

Proposition 2.2: Let $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ be a Carathéodory function and weakly asymptotically autonomous with some $\mathcal{F} \in \mathcal{U}$. Then (1) is uniformly finite-time-semistable if and only if (3) is finite-time-semistable.

Now, we present a sufficient condition to guarantee uniform semistability of (1).

Theorem 2.1: Let $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ be a Carathéodory function and weakly asymptotically autonomous with some $\mathcal{F} \in \mathcal{U}$. Assume that all the global solutions $x(t)$ of (1) are bounded. Furthermore, assume there exist a locally Lipschitz function $V : \mathbb{R}^q \to \mathbb{R}$, an upper semicontinuous function $g : \mathbb{R}^q \to \mathbb{R}$, and a Lebesgue measurable function $\gamma(\cdot)$ with Lebesgue integrable map $t \mapsto \|\gamma(t)\|$, such that

$$V^o(z, f(t, z)) - \gamma(t) \leq g(z) \leq 0, \quad t \geq t_0, \quad z \in \mathbb{R}^q. \quad (6)$$

If every point in the largest weakly invariant subset of $g^{-1}(0)$ with respect to (3) is a Lyapunov stable equilibrium point of (3), then (1) is uniformly semistable.

Corollary 2.1: Let $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ be a Carathéodory function and weakly asymptotically autonomous with some $\mathcal{F} \in \mathcal{U}$. Assume that all the global solutions $x(t)$ of (1) are bounded. Furthermore, assume there exist a locally Lipschitz function $V : \mathbb{R}^q \to \mathbb{R}$ and an upper semicontinuous function $g : \mathbb{R}^q \to \mathbb{R}$ such that

$$V^o(z, f(t, z)) \leq g(z) \leq 0, \quad t \geq t_0, \quad z \in \mathbb{R}^q. \quad (7)$$

If (3) is Lyapunov stable, then (1) is uniformly semistable.

The following result gives a sufficient condition for uniform finite-time semistability. For this result, we need the notion of homogeneity with respect to semi-Euler vector fields, which can be found in [20].

Proposition 2.3: Let $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ be a Carathéodory function and weakly asymptotically autonomous with some $\mathcal{F} \in \mathcal{U}$. Assume that all the global solutions $x(t)$ of (1) are bounded. Furthermore, assume that $\mathcal{F}(x) = \{g(x)\}$, where $g : \mathbb{R}^q \to \mathbb{R}^q$ is continuous and homogeneous of degree $k < 0$ with respect to a semi-Euler vector field $\nu(x)$. If (3) is Lyapunov stable, then (1) is uniformly finite-time-semistable.

Example 2.1: Consider the nonlinear time-varying dy-

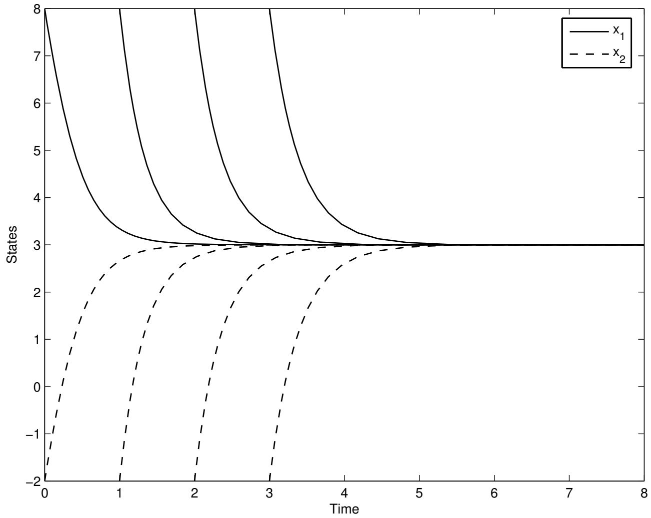

Fig. 1. Solutions for Example 2.1

namical system given by

$$\dot{x}_1(t) = [c+h(t)][f(x_2(t)) - g(x_1(t))], \quad x_1(0) = x_{10}, \quad t \geq 0, \quad (8)$$

$$\dot{x}_2(t) = [c+h(t)][g(x_1(t)) - f(x_2(t))], \quad x_2(0) = x_{20}, \quad (9)$$

where $x_1, x_2 \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous, $h : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, $|h(t)|$ is locally Lebesgue integrable, $h(t) > -c$ for all $t \in \mathbb{R}$, $\mathrm{dist}(h(t), I) \to 0$ as $t \to \infty$, $I \subset \mathbb{R}$ is a compact interval, $c > 0$, $f(x_2) - g(x_1) = 0$ if and only if $x_1 = x_2$, and $(x_1 - x_2)(f(x_2) - g(x_1)) \leq 0$, $x_1, x_2 \in \mathbb{R}$. Note that $\mathcal{E}_{\mathrm{n}} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$.

Let

$$\tilde{f}(x_1, x_2) \triangleq \begin{bmatrix} f(x_2) - g(x_1) \\ g(x_1) - f(x_2) \end{bmatrix}. \quad (10)$$

Clearly the function $(t, z_1, z_2) \mapsto [c + h(t)]\tilde{f}(z_1, z_2)$ is a Carathéodory function and weakly asymptotically autonomous with $\mathcal{F} \in \mathcal{U}$ given by

$$\mathcal{F}(z_1, z_2) \triangleq \overline{\mathrm{co}}\{(c+v)\tilde{f}(z_1, z_2) : v \in I\}. \quad (11)$$

To show that (8) and (9) is uniformly semistable, consider the nonnegative function $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. Now, it follows that the derivative of V along the trajectory of (8) and (9) is given by

$$\begin{aligned} &\dot{V}(x_1(t), x_2(t)) \\ &\quad = x_1(t)[c+h(t)][f(x_2(t)) - g(x_1(t))] \\ &\qquad + x_2(t)[c+h(t)][g(x_1(t)) - f(x_2(t))] \\ &\quad = (c+h(t))[x_1(t) - x_2(t)][f(x_2(t)) - g(x_1(t))] \\ &\quad \leq m[x_1(t) - x_2(t)][f(x_2(t)) - g(x_1(t))] \\ &\quad \leq 0, \quad (x_1(t), x_2(t)) \in \mathbb{R} \times \mathbb{R}, \quad t \geq 0, \end{aligned} \quad (12)$$

where $0 < m \leq c + \inf_{t \geq 0} h(t)$, which implies that $(x_1(\cdot), x_2(\cdot))$ is bounded.

Next, we consider the differential inclusion (3) where $x \triangleq [x_1, x_2]^{\mathrm{T}} \in \mathbb{R}^2$. Let v_x be an arbitrary element of $\mathcal{F}(x)$ and recall that the Clarke upper generalized derivative of $V(x - \alpha\mathbf{e})$, $\mathbf{e} \triangleq [1, 1]^{\mathrm{T}}$, along a vector $v_x \in \mathcal{F}$ is given by

$$V^o(x - \alpha\mathbf{e}, v_x) = (x - \alpha\mathbf{e})^{\mathrm{T}} v_x = x^{\mathrm{T}} v_x. \quad (13)$$

Now, consider $\max V^o(x - \alpha\mathbf{e}, v_x) \triangleq \max_{v_x \in \mathcal{F}}\{x^{\mathrm{T}} v_x\}$. It follows from Theorem 1 of [21] and the definition of a differential inclusion that $\max V^o(x - \alpha\mathbf{e}, v_x) = \max \overline{\mathrm{co}}\{(c+v)(x_1 - x_2)(f(x_2) - g(x_1)) : v \in I\}$. Note that $(x_1 - x_2)(f(x_2) - g(x_1)) \leq 0$, $x \in \mathbb{R}^2$, it follows that $\max V^o(x - \alpha\mathbf{e}, v_x)$ cannot be positive, and hence, the largest value $\max V^o(x - \alpha\mathbf{e}, v_x)$ can achieve is zero. Thus, $V^o(x - \alpha\mathbf{e}, v_x) \leq \max V^o(x - \alpha\mathbf{e}, v_x) \leq 0$ for all $x \in \mathbb{R}^2$ and all $v_x \in \mathcal{F}$. Hence, it follows that $x_1 = x_2 = \alpha$ is Lyapunov stable with respect to (3) for all $\alpha \in \mathbb{R}$.

Finally, note that it follows from (12) that

$$\begin{aligned} V^o(z, \tilde{f}(t, z)) &\leq m(z_1 - z_2)[f(z_2) - g(z_1)] \\ &\leq 0, \quad z \in \mathbb{R}^2, \quad t \geq 0, \end{aligned} \quad (14)$$

where $z \triangleq [z_1, z_2]^{\mathrm{T}}$ and $\tilde{f}(t, z) \triangleq (c + h(t))\tilde{f}(z_1, z_2)$. Now, it follows from Corollary 2.1 that (8) and (9) is uniformly semistable. Figure 1 shows the solutions of (8) and (9) for $f(x) = x$, $g(x) = x$, $h(t) = \frac{t}{1+t^2}$, $c = 1$, $I = \{0\}$, $x_{10} = 8$, $x_{20} = -2$, and $t_0 = 0, 1, 2, 3$. △

Example 2.2: Consider the nonautonomous dynamical system given by

$$\dot{x}_1(t) = [c+h(t)]f(x_2(t) - x_1(t)), \quad x_1(0) = x_{10}, \quad t \geq 0, \quad (15)$$

$$\dot{x}_2(t) = [c+h(t)]f(x_1(t) - x_2(t)), \quad x_2(0) = x_{20}, \quad (16)$$

where $x_1, x_2 \in \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, $|h(t)|$ is locally Lebesgue integrable, $h(t) > -c$ for all $t \in \mathbb{R}$, $\lim_{t\to\infty} h(t) = 0$, $c > 0$, $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(\lambda z) = \lambda^{1+r} f(z)$ for all $z \in \mathbb{R}$ and all $\lambda > 0$ with $r < 0$, $f(z) = -f(-z)$ for all $z \in \mathbb{R}$, and $f(z)z > 0$ for all $z \neq 0$, $z \in \mathbb{R}$. Note that $\mathcal{E}_{\mathrm{n}} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$.

Let $g(x_1, x_2) \triangleq [cf(x_2 - x_1), cf(x_1 - x_2)]^{\mathrm{T}}$. Clearly the function $(t, x_1, x_2) \mapsto [1 + h(t)/c]g(x_1, x_2)$ is a Carathéodory function and weakly asymptotically autonomous with $\mathcal{F} \in \mathcal{U}$ given by $\mathcal{F}(x) = \{g(x)\}$, where $x \triangleq [x_1, x_2]^{\mathrm{T}}$. Next, it follows from [22] that g is homogeneous of degree $r < 0$ with respect to the semi-Euler vector field $\nu(x) = (x_1 - x_2)\frac{\partial}{\partial x_1} + (x_2 - x_1)\frac{\partial}{\partial x_2}$. Consider $V(x_1, x_2) = \frac{1}{2}(x_1 - \alpha)^2 + \frac{1}{2}(x_2 - \alpha)^2$, where $\alpha \in \mathbb{R}$. Now, it follows that the Lyapunov derivative of V along the trajectories of (3) is given by

$$\begin{aligned} \dot{V}(x_1, x_2) &= (x_1 - \alpha)cf(x_2 - x_1) \\ &\quad + (x_2 - \alpha)cf(x_1 - x_2) \\ &= x_1 cf(x_2 - x_1) + x_2 cf(x_1 - x_2) \\ &= c(x_1 - x_2)f(x_2 - x_1) \\ &\leq 0, \quad (x_1, x_2) \in \mathbb{R} \times \mathbb{R}, \end{aligned} \quad (17)$$

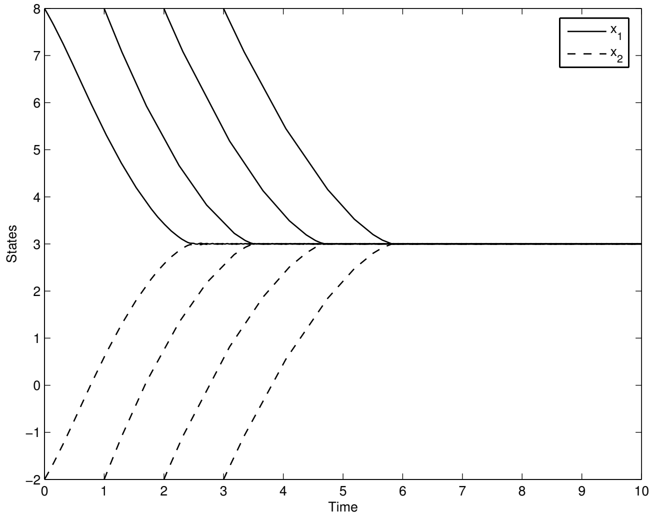

Fig. 2. Solutions for Example 2.2

which implies that $x_1 = x_2 = \alpha$ is Lyapunov stable with respect to (3). Now, it follows from Proposition 2.3 that (15) and (16) is uniformly finite-time-semistable. Figure 2 shows the solutions of (15) and (16) for $f(x) = x^{\frac{1}{3}}$, $h(t) = \frac{t}{1+t^2}$, $c = 1$, $x_{10} = 8$, $x_{20} = -2$, and $t_0 = 0, 1, 2, 3$. △

III. Consensus Problems in Dynamical Networks

In this section, we develop a thermodynamically motivated information consensus framework for multiagent nonlinear systems that achieve semistability and state equipartition. Specifically, consider q continuous-time integrator agents with dynamics

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \tag{18}$$

where for each $i \in \{1, \ldots, q\}$, $x_i(t) \in \mathbb{R}$ denotes the information state and $u_i(t) \in \mathbb{R}$ denotes the information control input for all $t \geq 0$. The general consensus protocol is given by

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \phi_{ij}(x_i(t), x_j(t)), \tag{19}$$

where $\phi_{ij}(\cdot,\cdot)$, $i, j = 1, \ldots, q$, are locally Lipschitz continuous. Note that (18) and (19) describe an interconnected network with a graph topology $\mathfrak{G}$ where information states are updated using a distributed nonlinear controller involving neighbor-to-neighbor interaction between agents. The following assumptions are needed for the main results of the paper.

Assumption 1: For the *connectivity matrix*[1] $\mathcal{C} \in \mathbb{R}^{q\times q}$ associated with the multiagent dynamical system $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)} \triangleq \begin{cases} 0, & \text{if } \phi_{ij}(x_i, x_j) \equiv 0, \\ 1, & \text{otherwise}, \end{cases} \quad i \neq j, \quad i, j = 1, \ldots, q, \tag{20}$$

[1]The negative of the connectivity matrix, that is, $-\mathcal{C}$, is known as the Laplacian of the directed graph $\mathfrak{G}$ in the literature.

and

$$\mathcal{C}_{(i,i)} \triangleq -\sum_{k=1,\, k\neq i}^{q} \mathcal{C}_{(i,k)}, \quad i = 1, \ldots, q, \tag{21}$$

$\operatorname{rank}\mathcal{C} = q-1$, and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\phi_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$.

Assumption 2: For $i, j = 1, \ldots, q$, $(x_i - x_j)\phi_{ij}(x_i, x_j) \leq 0$, $x_i, x_j \in \mathbb{R}$.

The fact that $\phi_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$, $i \neq j$, implies that agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are *connected*, and hence can share information; alternatively, $\phi_{ij}(x_i, x_j) \equiv 0$ implies that agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are *disconnected* and hence cannot share information. Assumption 1 implies that if the energies or information in the connected agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are equal, then energy or information exchange between these agents is not possible. This statement is reminiscent of the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. Furthermore, if $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ and $\operatorname{rank}\mathcal{C} = q-1$, then it follows that the connectivity matrix $\mathcal{C}$ and the adjacency matrix $\mathcal{A}$ are irreducible, which implies that for any pair of agents $\mathcal{G}_i$ and $\mathcal{G}_j$, $i \neq j$, of $\mathcal{G}$ there exists a sequence of information connectors (information arcs) of $\mathcal{G}$ that connect $\mathcal{G}_i$ and $\mathcal{G}_j$. Assumption 2 implies that energy or information flows from more energetic or information rich agents to less energetic or information poor agents and is reminiscent of the *second law of thermodynamics*, which states that heat (energy) must flow in the direction of lower temperatures. For further details, see [11].

Proposition 3.1 ([20]): Consider the multiagent dynamical system (18) and (19) and assume that Assumptions 1 and 2 hold. Then $f_i(x) = 0$ for all $i = 1, \ldots, q$ if and only if $x_1 = \cdots = x_q$. Furthermore, $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, is an equilibrium state of (18) and (19).

IV. Applications to Network Consensus with Time-Dependent Communication Links

Communication links among multiagent systems are often time-varying due to multipath effects and exogenous disturbances leading to dynamic information exchange topologies. In this section, we develop a time-varying consensus protocol to achieve agreement over a network with time-dependent communication links. First, we design a time-varying nonlinear consensus protocol for (18). Specifically, consider q mobile agents with the dynamics $\mathcal{G}_i$ given by (18). Furthermore, consider the time-varying controller $\mathcal{G}_{\mathrm{s}i}$ given by

$$u_i(t) = \sum_{j=1, j\neq i}^{q} [c_{ij} + a_{ij}(t)]\phi_{ij}(x_i(t), x_j(t)), \tag{22}$$

where $a_{ij} : \mathbb{R} \to \overline{\mathbb{R}}_+$ is Lebesgue measurable, $|a_{ij}(t)|$ is locally Lebesgue integrable, $a_{ij}(t) > -c_{ij}$ for all $t \in \mathbb{R}$, $\operatorname{dist}(a_{ij}(t), I_{ij}) \to 0$ as $t \to \infty$, $I_{ij} \subset \mathbb{R}$ is compact, $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, q$, $i \neq j$, $c_{ij} = c_{ji} > 0$ for all $i, j = 1, \ldots, q$, $i \neq j$, and $\phi_{ij} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and satisfies Assumptions 1 and 2. Furthermore, we assume that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ in Assumption 1.

Proposition 4.1: Consider the multiagent dynamical system (18) and (22) and assume that Assumptions 1 and 2 hold. Then $u_i \equiv 0$ for all $i = 1, \ldots, q$ if and only if $x_1 = \cdots = x_q$. Furthermore, $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, is an equilibrium state of (18) and (22).

Theorem 4.1: Consider the closed-loop system $\tilde{\mathcal{G}}$ given by the multiagent dynamical system (18) and the time-varying controller (22). Assume that Assumptions 1 and 2 hold. Furthermore, assume that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ in Assumption 1. Then for every $\alpha \in \mathbb{R}$, $x_1 = \cdots = x_q = \alpha$ is a uniformly semistable state of $\tilde{\mathcal{G}}$. Furthermore, $x_i(t) \to \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ and $\frac{1}{q}\sum_{i=1}^{q} x_{i0}$ is a uniformly semistable equilibrium state.

Note that Example 2.1 serves as a special case of Theorem 4.1. In [20], the authors prove that the consensus protocol given by the form

$$u_i = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} \mathrm{sign}(x_j - x_i)|x_j - x_i|^{\alpha} \tag{23}$$

is a finite-time consensus protocol for $0 < \alpha < 1$. Next, we show that

$$u_i(t) = \sum_{j=1, j\neq i}^{q} [1 + a_{ij}(t)]\mathcal{C}_{(i,j)} \mathrm{sign}(x_j(t) - x_i(t)) \cdot |x_j(t) - x_i(t)|^{\alpha} \tag{24}$$

is a uniformly finite-time consensus protocol for $0 < \alpha < 1$, where $a_{ij} : \mathbb{R} \to \overline{\mathbb{R}}_+$ is Lebesgue measurable, $|a_{ij}(t)|$ is locally Lebesgue integrable, $a_{ij}(t) \geq 0$ for all $t \in \mathbb{R}$, $\lim_{t\to\infty} a_{ij}(t) = 0$, $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, q$, $i \neq j$.

Theorem 4.2: Consider the closed-loop system $\tilde{\mathcal{G}}$ given by the multiagent dynamical system (18) and the time-varying controller (24). Assume that Assumptions 1 and 2 hold. Furthermore, assume that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ in Assumption 1. Then for every $\alpha \in \mathbb{R}$, $x_1 = \cdots = x_q = \alpha$ is a uniformly finite-time-semistable state of $\tilde{\mathcal{G}}$. Furthermore, $x_i(t) = \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ for $t \geq T(x_{10}, \ldots, x_{q0})$ and $\frac{1}{q}\sum_{i=1}^{q} x_{i0}$ is a uniformly semistable equilibrium state.

Note that Example 2.2 serves as a special case of Theorem 4.2.

V. Conclusion

This paper extends the notions of semistability and finite-time semistability to nonlinear time-varying dynamical systems. In particular, Lyapunov theorems for uniform semistability and uniform finite-time semistability are established. These results are used to develop and analyze information consensus algorithms in dynamical networks with time-dependent communication links.

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