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The continuous closed form controllability Gramian and its inverse

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Abstract— **The continuous controllability Gramian is the solution of an input Lyapunov equation in the controller (companion) form or equivalently the infinite integral of an outer product of a vector containing the impulse response and its derivatives corresponding to a unity numerator transfer function. In this paper we make use of both these viewpoints in order to derive the simple zero plaid structure of this Gramian and present the interesting links that the entries of the Gramian have to the entries of the Routh table. Moreover, an expression for the inverse of the Gramian is derived as a simple function of the coefficients of the characteristic polynomial from the fact that it is the solution of a Riccati equation.**

We show how the controllability Gramian forms the core part of closed form expressions of Gramians of more general MIMO systems as well as solutions of general Sylvester equations. The controllability Gramian also appears in certain zero optimization problems, either in a PID like controller setting or in a model reduction setting. The inverse of the controllability Gramian is a key element in such zero optimization.

While much of the results presented can be found in closely related forms in published papers, we believe that they deserve more attention as an effective tool in numerical computations of small to mid-size systems.

I. Introduction

There exists extensive literature within the fields of ordinary differential equations, difference equations, matrix theory and Laplace transforms on closed form expressions. The majority of such results, however, predates the computer era, and is not presented in a form that has onus on efficient algorithmic implementations. This fact, somewhat surprisingly, is still reflected in modern textbooks, e.g., in control theory, in the area of signals and systems as well as in mathematics. In these textbooks, the corresponding types of results are presented in a restrictive setting, with little or no attention to how they could be implemented in general algorithms. Computer algorithms that have been developed over recent decades, e.g., within control theory and mathematics, on the other hand, are often based on general approaches to numerical solutions of ordinary differential equations and linear equations that do not make specific use of the structure that lies in the closed form expressions.

Naturally, much attention has been given to numerical methods during the past decades with the rapid development of fast computers[1],[2]. Those generally provide approximate solutions which are often applicable to large systems, see e.g., [3] regarding the computation of matrix exponentials and [4] and [5] regarding the solutions of Lyapunov equations. Despite the effectiveness and advantages of such numerical methods, closed form time domain solutions nevertheless provide direct, easy and accurate computation for small to mid-size systems. Further, closed form solutions open a window of opportunities definitely worth exploring, e.g. in the control area for the design of controllers and model reduction, both in their own right for small to mid-size systems and by combining them with numerical methods for large systems.

In this paper we focus on closed form expressions of continuous Gramians. Within this area there are in fact some recent papers that present closed form expressions for symbolic computation[6],[7] or parametric presentation of solutions[8],[9]. Three pioneering papers[10],[11],[12] deal with numerical algorithmic aspects. While the work in these papers was followed up in [13] and [14], it seems however to have received relatively little attention. The controllability Gramian forms the core part of closed form expressions for Gramians of more general MIMO systems. It also appears in certain zero optimization problems, either in a PID like controller setting or in a model reduction setting[15]-[18]. Thus it is important to make use of the special zero-plaid Hankel like structure that it turns out to have, also referred to as alternating Hankel in [6] and a Xiao matrix in [12].

The structure can be derived directly from the Lyapunov equations that they satisfy[10],[12],[14]. However, it is also advantageous to view the controllability Gramian as the infinite integral of an outer product of a vector containing the impulse response and its derivatives corresponding to a unity numerator transfer function. Naturally, we can also view the impulse response as an initial value problem of a homogeneous (unforced) differential equation having only the $(n-1)$-th derivative nonzero, i.e., unity. Closed form expressions for Gramians were derived in [19] from this viewpoint involving the eigenvalues of A, the partial fraction coefficients of the unity numerator transfer function, as well as the coefficients of the characteristic polynomial. In this paper the relationship between both viewpoints is exploited in order to derive and clarify the structure of the controllability Gramian. By making use of the corresponding Lyapunov equation, the derived expression for the controllability Gramian only involves the coefficients of the characteristic polynomial even if the focus remains on the impulse response and its properties. An elementary argument

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for the link between general Gramians and the controllability Gramian is also presented. Some initial results along these lines were presented in [20]. The Hankel-like structure of the controllability Gramian suggests the existence of some regular pattern for its inverse. It is also suggested by [21] where a formula for the discrete controllability Gramian is derived which is expressed as an inverse of a form based on the coefficients of the companion matrix. Here a formula for the inverse of the controllability Gramian is obtained by an elementary derivation from the Riccati equation that it satisfies, involving only the coefficients of the underlying companion matrix, along with a computationally efficient recursive procedure for the evaluation of its elements. An alternative presentation of the formula relates it to the Gohberg-Semencul formulas as well as a number of related formulations of inverses of Hankel matrices, see e.g. [22] and [23].

The link between expressions of the controllability Gramian and the general MIMO Gramians is presented in section II. The zero-plaid structure of the controllability Gramian is derived in section III from the properties of the impulse response. Moreover, it is shown how its elements are effectively determined from the coefficients of the characteristic polynomial for A, revealing an interesting link to the entries of the Routh table, a result originally derived in [10]. The derivation of the inverse of the controllability Gramian from its underlying Riccati equation is contained in section IV. Some concluding remarks can be found in section V.

II. Relationship between the controllability Gramian and general Gramians

Consider the general state space representation of MIMO systems in the minimal form given by

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} \tag{1}$$

where A is an $n \times n$ matrix, B is $n \times p$ and C is $r \times n$. The matrix A has the characteristic equation $\det(sI - A) = \sum_{i=0}^{n} a_i s^i = 0$ where $a_n = 1$ and the rest of the $a_i's$ are real numbers.

Now consider the continuous time Lyapunov equation

$$AP + PA^H + BB^H = 0. \tag{2}$$

For a real symmetric BB^H, this equation has a unique, real symmetric solution an $n \times n$ matrix P iff no sum of any two eigenvalues of A is zero (thus no eigenvalue of A is zero)[24].

Remark 1: Consider the indefinite integral

$$Z(t) = \int e^{tA} BB^H e^{tA^H} dt \tag{3}$$

which does not hold any constant terms. Then

$$\begin{array}{rcl} AZ(t) + Z(t)A^H & = & \int \frac{d}{dt}\left(e^{tA}BB^H e^{tA^H}\right) dt \\ & = & e^{tA}BB^H e^{tA^H} + C, \end{array} \tag{4}$$

where C is a constant matrix. If no sum of any two eigenvalues of A is zero then $\int \frac{d}{dt}\left(e^{tA}BB^H e^{tA^H}\right) dt$ does not contain any constant terms and hence $C = 0$. It then follows that

$$P = -Z(0) \tag{5}$$

in this general case.

For a strictly stable A, the solution is the positive semidefinite input Gramian

$$P = \int_0^\infty e^{tA} BB^H e^{tA^H} dt. \tag{6}$$

If in addition (A, B) is controllable, then P is positive definite ($P > 0$).

Now assume that $p = 1$. Let (A^c, B^c) denote the controller (companion) form, i.e.,

$$A^c = \left[\begin{array}{cc} 0_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \\ -a_0 & -a_1 \ \cdots \ -a_{n-1} \end{array}\right] \tag{7}$$

and

$$B^c = u_e = \left[\begin{array}{cccc} 0 & 0 & \cdots & 1 \end{array}\right]'. \tag{8}$$

Note that the transpose $'$ is used for real valued entities in place of the complex conjugate transpose ^{H} where applicable. We shall refer to

$$P^c = \int_0^\infty e^{tA^c} u_e {u_e}' e^{tA^{c'}} dt \tag{9}$$

which satisfies the Lyapunov equation

$$A^c P^c + P^c A^{c'} + u_e u_e^{'} = 0 \tag{10}$$

as the controllability Gramian.

We now present an elementary derivation of how P can be calculated from P^c. If we have a nonsingular similarity matrix T^c such that

$$AT^c = T^c A^c \text{ and } B = T^c B^c, \tag{11}$$

using (10) and (11) we obtain

$$AT^c P^c (T^c)^H + T^c P^c (T^c)^H A^H + BB^H = 0 \tag{12}$$

and hence from (2) that

$$P = T^c P^c (T^c)^H. \tag{13}$$

The similarity transformation T^c to the controller form can be derived as follows, where $t^c_{\cdot i}$ denotes the i-th column of T^c. We have[26]

$$AT^c = \left[\begin{array}{cccc} At^c_{\cdot 1} & At^c_{\cdot 2} & \cdots & At^c_{\cdot n} \end{array}\right], \tag{14}$$

$$\begin{array}{l} T^c A^c = \big[-a_0 t^c_{\cdot n} \quad t^c_{\cdot 1} - a_1 t^c_{\cdot n} \quad t^c_{\cdot 2} - a_2 t^c_{\cdot n} \quad \cdots \\ \qquad \cdots \quad t^c_{\cdot(n-1)} - a_{n-1} t^c_{\cdot n} \big] \end{array} \tag{15}$$

and

$$B = T^c B^c = T^c \left[\begin{array}{cccc} 0 & \cdots & 0 & 1 \end{array}\right]' = t^c_{\cdot n} \tag{16}$$

It then follows directly from (11) that

$$\begin{array}{rcl} t^c_{\cdot n} & = & B \\ t^c_{\cdot(n-1)} & = & AB + a_{n-1}B \\ t^c_{\cdot(n-2)} & = & At^c_{\cdot(n-1)} + a_{n-2}B \\ & \vdots & \\ t^c_{\cdot 2} & = & At^c_{\cdot 3} + a_2 B \\ t^c_{\cdot 1} & = & At^c_{\cdot 2} + a_1 B. \end{array} \tag{17}$$

This recurrence can also easily be derived from the Faddeeva algorithm (see e.g. [25]) and can also be expressed as

$$T^c = \mathcal{C}\mathcal{H}_u \tag{18}$$

where

$$\mathcal{H}_u = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & & & 1 & \\ \vdots & & & & \\ a_{n-1} & 1 & & 0 & \\ 1 & & & & \end{bmatrix} \tag{19}$$

is an upper Hankel matrix and

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{20}$$

is the controllability matrix. Thus, if we know P^c, we can readily compute P from (13) and (17).

Remark 2: The following expression is derived for the input Gramian P in [20]:

$$P = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \pi_{ij} A^i B B^H \left(A^H\right)^j \tag{21}$$

based on a closed form polynomial expression for e^{tA}. Here,

$$\pi_{ij} = \tilde{a}_{i+1}^{'} P^c \tilde{a}_{j+1}, \tag{22}$$

where $\tilde{a}_i$ denotes the i-th column vector of the $\mathcal{H}_u$ matrix. We can form a matrix from the scalars π_{ij}

$$\Pi = \{\pi_{ij}\}_{n\times n} = \mathcal{H}_u P^c \mathcal{H}_u \tag{23}$$

and then we can rewrite (21) as

$$P = \mathcal{C}\Pi\mathcal{C}^H = \mathcal{C}\mathcal{H}_u P^c \mathcal{H}_u \mathcal{C}^H \tag{24}$$

which agrees with (13) and (18). In the general multiple input (MI) case where $p \geq 1$ we may likewise express

$$P = \mathcal{C}\Pi \otimes I_p \mathcal{C}^H = \mathcal{C}\mathcal{H}_u P^c \mathcal{H}_u \otimes I_p \mathcal{C}^H, \tag{25}$$

where $\otimes$ is the direct matrix product and I_p is an $p \times p$ identity matrix such that

$$\Pi \otimes I_p = \begin{bmatrix} \pi_{11} I_p & \cdots & \pi_{1n} I_p \\ \vdots & & \vdots \\ \pi_{n1} I_p & \cdots & \pi_{nn} I_p \end{bmatrix}_{np\times np}. \tag{26}$$

Hence (25) effectively amounts to treating each of the p columns of B separately.

Remark 3: Note that the solutions (24) and (25) of the Lyapunov equation are still valid if A is not strictly stable, provided no sum of any two eigenvalues of A is zero. In this case we can express $P = -Z(0)$ as in (5) and it is not positive (semi)definite. Further, if (A, B) is not controllable, then P is not of full rank and is therefore not positive definite.

Remark 4: Closed form expressions for the output and the cross Gramians as well as for the solution of the Sylvester equation are also derived in [20] depending on π_{ij}. The implication is that these can thus also be computed from expressions for the controllability Gramian.

III. Structure and Direct Computation of the Controllability Gramian

Let $y_b(t)$ denote the solution of

$$y_b^{(n)}(t) + a_{n-1}y_b^{(n-1)}(t) + \ldots + a_0 y_b(t) = \delta(t), \qquad t > 0, \tag{27}$$

or equivalently the solution of

$$y_b^{(n)}(t) + a_{n-1}y_b^{(n-1)}(t) + \ldots + a_0 y_b(t) = 0 \tag{28}$$

satisfying the initial conditions

$$y_b^{(k)}(0) = 0,\ k = 0, 1, 2, \ldots, n-2,\ y_b^{(n-1)}(0) = 1.$$

Now assume that the system is strictly stable, thus $\lim_{t\to\infty} y_b^{(i)}(t) = 0$ for $i = 0, 1, \ldots, n-1$. The vector $Y_b(t)$ contains the basic response $y_b(t)$ and its derivatives, i.e,

$$Y_b(t) = \begin{bmatrix} y_b(t) & \dot{y}_b(t) & \cdots & y_b^{(n-2)}(t) & y_b^{(n-1)}(t) \end{bmatrix}'. \tag{29}$$

Now observing that $Y_b(t) = e^{tA^c}u_e$ we have that

$$P^c = \int_0^\infty Y_b(t) Y_b(t)' dt. \tag{30}$$

It follows directly by repeated integration by parts that P^c will have the following plaid like structure[18]

$$P^c = \begin{bmatrix} \mathcal{Y}_0 & 0 & -\mathcal{Y}_1 & 0 & \mathcal{Y}_2 & \cdots \\ 0 & \mathcal{Y}_1 & 0 & -\mathcal{Y}_2 & 0 & \\ -\mathcal{Y}_1 & 0 & \mathcal{Y}_2 & 0 & -\mathcal{Y}_3 & \\ 0 & -\mathcal{Y}_2 & 0 & \mathcal{Y}_3 & 0 & \\ \mathcal{Y}_2 & 0 & -\mathcal{Y}_3 & 0 & \ddots & \\ \vdots & & \ddots & & & \mathcal{Y}_{n-1} \end{bmatrix}, \tag{31}$$

where

$$\mathcal{Y}_i \quad = \quad \textstyle\int_0^\infty \left(y_b^{(i)}(t)\right)^2 dt. \tag{32}$$

Thus, in order to evaluate P^c we only have to evaluate $\mathcal{Y}_i$, $i = 0, 1, \ldots, n-1$. The same argument holds true in the more general case where we can express $P = -Z(0)$ as in (5) and therefore the plaid structure remains valid as long as no sum of any two eigenvalues of A is zero.

Remark 5: It follows from Lyapunov's stability theorem, that for a strictly stable system the matrix $P^c = \int_0^\infty Y_b(t)Y_b(t)' dt$ is positive definite, also easily noted by the fact that for any nonzero column vector σ, we have that $\sigma' \int_0^\infty Y_b(t)Y_b'(t)dt\sigma = \int_0^\infty (\sigma' Y_b(t))^2 dt > 0$, since the elements in $Y_b(t)$ are linearly independent.

This structure (31) was derived in [10] and in [12] where it is referred to as a Xiao matrix. In both cases the derivation was an algebraic one based on (10). For a similar derivation, see also [14]. In [6] it is referred to as an alternating Hankel form. Note that by permuting rows and columns so that all the odd numbered rows and columns precede the even numbered ones, the matrix transforms into a 2-block diagonal matrix, each block being a Hankel matrix.

Remark 6: The last line in the Lyapunov equation (10) can be written as

$$(A^c P^c)_{n\cdot} + ((A^c P^c)_{\cdot n})' + \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} = 0_{1\times n}. \tag{33}$$

It readily follows that when n is odd

$$\begin{bmatrix} a_0 & a_2 & \cdots & \cdots & a_{n-1} & 0 & \cdots & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & a_{n-2} & 1 & 0 & \cdots & 0 \\ 0 & a_0 & a_2 & \cdots & \cdots & a_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & \cdots & a_{n-2} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & & & & & \vdots \\ \vdots & \vdots & & 0 & a_1 & a_3 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & 0 & a_0 & a_2 & \cdots & \cdots & a_{n-1} \end{bmatrix}' \times \begin{bmatrix} \mathcal{Y}_0 & -\mathcal{Y}_1 & \mathcal{Y}_2 & -\mathcal{Y}_3 & \cdots & \mathcal{Y}_{(n-1)} \end{bmatrix}' = \begin{bmatrix} 0 & \cdots & 0 & 1/2 \end{bmatrix}' \tag{34}$$

and when n is even:

$$\begin{bmatrix} a_0 & a_2 & \cdots & a_{n-2} & 1 & 0 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & a_{n-1} & 0 & \cdots & 0 \\ 0 & a_0 & a_2 & \cdots & a_{n-2} & 1 & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & \cdots & & \cdots & 0 \\ \vdots & \vdots & \ddots & & & & & \vdots \\ \vdots & \vdots & & a_0 & a_2 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & 0 & a_1 & a_3 & \cdots & a_{n-1} \end{bmatrix}' \times \begin{bmatrix} \mathcal{Y}_0 & -\mathcal{Y}_1 & \mathcal{Y}_2 & -\mathcal{Y}_3 & \cdots & -\mathcal{Y}_{(n-1)} \end{bmatrix}' = \begin{bmatrix} 0 & \cdots & 0 & -1/2 \end{bmatrix}'. \tag{35}$$

Again these equations have a unique, real solution iff no sum of any eigenvalues of A is zero.

Remark 7: Applying Gauss elimination to the system (34), we can

- first replace the $\begin{bmatrix} a_0 & a_2 & \cdots & a_{n-1} \end{bmatrix}$ subvector in rows $3, 5, \ldots, n$ by $\begin{bmatrix} 0 & \beta_{1,1} & \cdots & \beta_{1,\frac{n-1}{2}} \end{bmatrix}$ where $\beta_{1,i} = a_{2i} - \frac{a_0}{a_1} a_{2i+1}$, $i = 1, 2, \ldots, \frac{n-1}{2}$, (setting $a_n = 1$).
- Next we replace the $\begin{bmatrix} a_1 & a_3 & \cdots & a_{n-2} & 1 \end{bmatrix}$ subvector in rows $4, 6, \ldots, n-1$ by $\begin{bmatrix} 0 & \beta_{2,1} & \cdots & \beta_{2,\frac{n-1}{2}} \end{bmatrix}$ where $\beta_{2,i} = a_{2i+1} - \frac{a_1}{\beta_{1,1}} \beta_{1,i+1}$, $i = 1, 2, \ldots, \frac{n-1}{2}$, (setting $\beta_{1,\frac{n+1}{2}} = 1$).
- We then replace the $\begin{bmatrix} \beta_{1,1} & \cdots & \beta_{1,\frac{n-1}{2}} \end{bmatrix}$ subvector in rows $5, 7, \ldots, n$ by $\begin{bmatrix} 0 & \beta_{3,1} & \cdots & \beta_{3,\frac{n-3}{2}} \end{bmatrix}$ where $\beta_{3,i} = \beta_{1,i+1} - \frac{\beta_{1,1}}{\beta_{2,1}} \beta_{2,i+1}$, $i = 1, 2, \ldots, \frac{n-3}{2}$,

etc.. Similarly for system (35). Thus we end up with the upper triangular system

$$\begin{bmatrix} a_0 & a_2 & \cdots & \cdots & a_{n-1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & a_{n-2} & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \beta_{1,1} & \cdots & \cdots & \beta_{1,\frac{n-1}{2}} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \beta_{2,1} & \cdots & \beta_{2,\frac{n-3}{2}} & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & & & & & & \vdots \\ & & & & & & & & \beta_{n-3,1} & & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \beta_{n-2,1} \end{bmatrix} \times \begin{bmatrix} \mathcal{Y}_0 \\ -\mathcal{Y}_1 \\ \mathcal{Y}_2 \\ -\mathcal{Y}_3 \\ \vdots \\ \mathcal{Y}_{(n-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1/2 \end{bmatrix} \tag{36}$$

that is readily solved by backward substitution. Thus the total number of operations required to evaluate the Y_b coefficients from the a_i-coefficients is $\mathcal{O}(n^2)$. It is of interest to note that the nonzero vectors in the upper triangular half are exactly the vectors of the inverse Routh table i.e.

$$\begin{array}{llllll} s^{-n} & a_0 & a_2 & \cdots & \cdots & a_{n-1} \\ s^{-(n-1)} & a_1 & a_3 & \cdots & a_{n-2} & 1 \\ s^{-(n-2)} & \beta_{1,1} & \cdots & \cdots & \beta_{1,\frac{n-1}{2}} & \\ s^{-(n-3)} & \beta_{2,1} & \cdots & \beta_{2,\frac{n-3}{2}} & 1 & \\ \vdots & & & & & \\ s^{-2} & \beta_{n-3,1} & 1 & & & \\ s^{-1} & \beta_{n-2,1} & & & & \\ s^{-0} & 1 & & & & \end{array} \tag{37}$$

Remark 8: When A is not strictly stable, some of the elements in the first column of (37) may become zero. In this case, the procedure in *Remark 7* has to be modified by appropriate row permutations.

IV. Closed form inverse of the controllability Gramian

In optimal zeros problems, the controllability Gramian arises directly, in a linear system of equations of the form $P^c \rho = \sigma$. Thus, it is also of interest to obtain closed form expressions for $X^c = (P^c)^{-1}$.

We now present an elementary derivation of $X^c = (P^c)^{-1}$ in terms of the coefficients a_i, $i = 0, 1, \ldots, n-1$, under the assumption that P^c is nonsingular. Since

$$A^c P^c + P^c (A^c)' + u_e (u_e)' = 0 \tag{38}$$

it follows by multiplying from both sides with $X^c = (P^c)^{-1}$ that X^c satisfies the simple Riccati equation

$$X^c A^c + (A^c)' X^c + (X^c u_e)(X^c u_e)' = 0. \tag{39}$$

We further note that since P^c is symmetric and zero plaid, the same must hold true for X^c, i.e.

$$X^c = \begin{bmatrix} x_{11} & 0 & x_{13} & 0 & x_{15} & \cdots \\ 0 & x_{22} & 0 & x_{24} & 0 & \\ x_{13} & 0 & x_{33} & 0 & x_{35} & \\ 0 & x_{24} & 0 & x_{44} & 0 & \\ x_{15} & 0 & x_{35} & 0 & \ddots & \\ \vdots & & \ddots & & & x_{nn} \end{bmatrix}. \tag{40}$$

Introducing the matrix

$$\hat{I} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \tag{41}$$

and the vector $a = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}'$ we can rewrite (39) as:

$$X^c \hat{I} - x_{\cdot n} a' + \hat{I}' X^c - a x_{\cdot n}' + x_{\cdot n} x_{\cdot n}' = 0, \tag{42}$$

where $x_{\cdot n}$ denotes the last column vector of X^c. Comparing the entries in the last column on each side of (42) and keeping in mind the zero-plaid structure of (40), we get

$$-x_{nn}a_{i-1} + (x_{nn} - a_{n-1})x_{in} = 0 \qquad i = n, n-2, \ldots \tag{43}$$

Assume first that $x_{nn} \neq 0$. This holds in particular when P is positive definite because then $x_{nn} > 0$. Then we can conclude

$$x_{in} = 2a_{i-1} \qquad i = n, n-2, \ldots \tag{44}$$

whereas it follows from the zero plaid structure of (40) that

$$x_{in} = 0 \qquad i = n-1, n-3, \ldots \tag{45}$$

If on the other hand $x_{nn} = 0$, we must have that $a_{n-1} = 0$ otherwise it would follow from (43) that the last column of X would be zero, contradicting the assumption that X is nonsingular. Thus (44) still holds for $i = n$. To see that it must hold for $i = n-2, n-4, \ldots$, we compare in a similar fashion column $n-2$ on each side of (42) resulting in

$$-x_{(n-2)n}a_{i-1} + (x_{(n-2)n} - a_{n-3})x_{in} = 0 \tag{46}$$

for $i = n-2, n-4 \ldots$ If now also $x_{(n-2)n} = 0$ we must have $a_{n-3} = 0$ for (44) to hold, then we go on and compare column $n-4$, etc. Introducing the vectors

$$\gamma = \begin{cases} \begin{bmatrix} a_0 & 0 & a_2 & 0 & \cdots & 0 & a_{n-3} & 0 & a_{n-1} \end{bmatrix}' & n \text{ odd} \\ \begin{bmatrix} 0 & a_1 & 0 & a_3 & \cdots & 0 & a_{n-3} & 0 & a_{n-1} \end{bmatrix}' & n \text{ even} \end{cases} \tag{47}$$

and

$$\hat{\gamma} = a - \gamma \tag{48}$$

we thus have that

$$x_{\cdot n} = 2\gamma. \tag{49}$$

Rewriting (42) as:

$$X^c\hat{I} + \hat{I}'X^c - x_{\cdot n}\left(a - \frac{1}{2}x_{\cdot n}\right)' - \left(a - \frac{1}{2}x_{\cdot n}\right)x_{\cdot n}' = 0, \tag{50}$$

substituting from (44) and (45) we also have from (50) that

$$X^c\hat{I} + \hat{I}'X^c - 2\gamma\hat{\gamma}' - 2\hat{\gamma}\gamma' = 0. \tag{51}$$

Noting that the first row of $\hat{I}'X^c$ is zero whereas the first row of $X^c\hat{I}$ starts with a zero followed by the first $n-1$ elements of the first row of X^c we conclude directly from (51) that

$$x_{1(1\cdots(n-1))} = 2a_0 \times \begin{cases} \begin{bmatrix} a_1 & 0 & a_3 & 0 & \cdots & a_{n-4} & 0 & a_{n-2} & 0 \end{bmatrix} & n \text{ odd} \\ \begin{bmatrix} a_1 & 0 & a_3 & 0 & \cdots & 0 & a_{n-3} & 0 & a_{n-1} \end{bmatrix} & n \text{ even} \end{cases} \tag{52}$$

Note that $a_0 \neq 0$, a consequence of the Gantmacher condition for the existence of a unique, real, symmetric solution P^c.

Finally, noting that for $i = 2, 3, \ldots, n$, the ith row of $\hat{I}'X^c$ will be the $(i-1)$st row of X^c whereas the ith row of $X^c\hat{I}'$ starts with a zero followed by the first $n-1$ elements of the ith row of X^c, we derive from (51) the following recurrence for $i = 2, 3, \ldots, n$

$$x_{i(1\cdots(n-1))} = -x_{(i-1)(2\cdots n)} + 2a_{i-1} \times \begin{cases} \begin{bmatrix} 0 & a_2 & 0 & a_4 & \cdots & 0 & a_{n-1} \end{bmatrix} & n \text{ odd and } i \text{ even} \\ \begin{bmatrix} a_1 & 0 & a_3 & 0 & \cdots & a_{n-2} & 0 \end{bmatrix} & n \text{ odd and } i \text{ odd} \\ \begin{bmatrix} 0 & a_2 & 0 & a_4 & \cdots & a_{n-2} & 0 \end{bmatrix} & n \text{ even and } i \text{ even} \\ \begin{bmatrix} a_1 & 0 & a_3 & 0 & \cdots & 0 & a_{n-1} \end{bmatrix} & n \text{ even and } i \text{ odd.} \end{cases} \tag{53}$$

This recurrence along with the boundary conditions (49) and (52) defines X^c in terms of the coefficients $a_0, a_1, \ldots, a_{n-1}$. Such a recurrence is to be expected from the Hankel-like structure of P_c and the Gohberg-Semencul formulas [22]. Thus e.g. when $n = 6$ we obtain[18]:

$$X^c = 2\begin{bmatrix} a_0a_1 & 0 & a_0a_3 & 0 & a_0a_5 & 0 \\ 0 & a_1a_2 - a_0a_3 & 0 & a_1a_4 - a_0a_5 & 0 & a_1a_6 \\ a_0a_3 & 0 & a_2a_3 - a_1a_4 + a_0a_5 & 0 & a_2a_5 - a_1a_6 & 0 \\ 0 & a_1a_4 - a_0a_5 & 0 & a_3a_4 - a_2a_5 + a_1a_6 & 0 & a_3a_6 \\ a_0a_5 & 0 & a_2a_5 - a_1a_6 & 0 & a_4a_5 - a_3a_6 & 0 \\ 0 & a_1a_6 & 0 & a_3a_6 & 0 & a_5a_6 \end{bmatrix}. \tag{54}$$

Introducing the lower Toeplitz matrix

$$\mathcal{T}_l = \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix}, \tag{55}$$

and the matrix

$$\tilde{I} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & \cdots & 0 & (-1)^{n-1} \end{bmatrix}, \tag{56}$$

we have

$$X^c = \mathcal{T}_l\tilde{I}\mathcal{H}_u + \tilde{I}\mathcal{T}_l\tilde{I}\mathcal{H}_u\tilde{I} \tag{57}$$

since the right hand side of (57) clearly satisfies (49), (52) and (53). Note that adding $\tilde{I}\mathcal{T}_l\tilde{I}\mathcal{H}_u\tilde{I}$ simply imposes the zero pattern onto X^c and doubles the nonzero entries. Note that formula (57) has a close resemblance with some of the formulae that have been derived for inverses of Hankel matrices, see e.g. [23]. Also note that X^c is computed more efficiently by making use of (49), (52) and the recurrence (53) rather than expression (57).

V. CONCLUSIONS AND FUTURE WORK

The continuous controllability Gramian is the solution of an input Lyapunov equation in the controller (companion) form or equivalently the infinite integral of an outer product of a vector containing the impulse response and its derivatives corresponding to a unity numerator transfer function. In this paper we make use of both these viewpoints in order to derive the simple zero plaid structure of this Gramian and present the interesting links that the entries of the Gramian have to the entries of the Routh table. Moreover, an expression for the inverse of the Gramian is derived as a simple function of the coefficients of the characteristic polynomial from the fact that it is the solution of a Riccati equation.

We show how the controllability Gramian form the core part of closed form expressions of Gramians of general MIMO systems, the input Gramian, as well as the output Gramian. Closed form expressions for the cross Gramian and the solution of the general Sylvester equation are also easily computed from the controllability Gramian.

The controllability Gramian also appears in certain zero optimization problems, either in a PID like controller setting or in a model reduction setting. The inverse controllability Gramian is a key element in such zero optimization problems. Further, it is of interest to explore whether similar methods to those used in the derivation of the inverse Gramian can be applied to derive a direct solution of more complicated Riccati equations. Indeed the zero optimization problem is a special case of the LQR problem.

The emphasis in this work has been on the derivation of computationally efficient formulations of closed form expressions. The short term motivation has simply been to provide another tool in the linear systems toolbox to be used along with methods that have already been developed based on numerical approaches. Thus while much of the results presented can be found in older papers in closely related forms, we believe that they deserve more attention as an effective tool in numerical computations of small to mid-size systems. Initial computation tests reveal that solving (35) or (34) for the controllability Gramian by using Matlab's backslash command can handle considerably larger systems than Matlab's standard `lyap` command.

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