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Almost disturbance decoupling: static state feedback solutions with maximal pole assignment

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***Abstract*— This paper is concerned with maximal pole placement solutions for the Almost Disturbance Decoupling Problem (ADDP) by static state feedback. It is shown that, when solving ADDP by means of high gain state feedback, always are present some fixed finite poles and poles close to infinity in the closed loop system, which can not be freely assigned. A class of "optimal" geometric subspaces with which maximal complementary pole placement can be achieved is characterized. A reliable construction algorithm for such an optimal geometric subspace (resp. the state feedback solution) is also proposed for a particular class of linear systems. An example is proposed to illustrate our contributions.**

***Index Terms*— Linear systems; almost disturbance decoupling; fixed pole; infinite pole; pole placement.**

I. Introduction

In classical as well as modern control theory, the problem of (almost) disturbance decoupling occupies a central part. Several important problems, such as robust, decentralized, noninteracting, model reference or tracking control etc., are linked to (almost) disturbance decoupling[1]. Regardless of where the problem arises from, the basic (almost) disturbance decoupling problem can be stated as follows: to design a linear time invariant controller such that the system controlled output is exactly (or approximately in a precise sense) decoupled from the disturbance input while guaranteeing the internal stability of the resulting closed-loop feedback system.

The Almost Disturbance Decoupling Problem by static state feedback (**ADDP**) was first introduced in [2]; it is an alternative to the traditional disturbance decoupling problem by state feedback (**DDP**) when this classical **DDP** is not solvable. **DDP** was solved in [3] and [4] in terms of geometric conditions by introducing the key concept of controlled invariant subspaces; **ADDP** has also been intensively studied in [5] in a theoretical way (without considering numerical solutions).

In many practical situations, almost (or exact) disturbance decoupling is not the only control objective. One possible use of the remaining degrees of freedom is often to add other requirements such as model matching, particular pole placement strategies,$\cdots$ This paper focuses on the pole assignability while solving **ADDP** (**DDP**) by static state feedback. The **DDP** by state feedback with maximal pole placement has been solved by introducing the concept of fixed poles, see [6], [7]. But to the best of our knowledge, for **ADDP** by state feedback and pole placement abilities, there has been no general study, especially for the reliable numerical solution.

The aim of this paper is to show how to get an optimal solution of **ADDP** in the sense of maximal pole placement. In fact, we will show, using the so-called geometric approach, that, there also exist some fixed poles in **ADDP**, i.e. poles which are present in the closed loop system after applying any state feedback solution of **ADDP**. These fixed poles do not depend on the choice of the control law but precisely on the fact that this particular problem is being solved.

II. Notation and geometric preliminaries

We shall consider linear time-invariant disturbed systems $\Sigma(A,B,D,E)$ described by:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dq(t) \\ z(t) = Ex(t) \end{cases}$$

where x, u, q, and z are respectively the state, control input, disturbance input, and output to be controlled. These signals belong to the vector spaces $\mathscr{X}$, $\mathscr{U}$, $\mathscr{Q}$, and $\mathscr{Z}$, respectively.

Vectors are denoted by lower case letters, matrices/maps by capitals and subspaces by script capitals. If A is a square matrix, then $\sigma(A)$ denotes its spectrum. If $A : \mathscr{X} \mapsto \mathscr{Y}$ and $\mathscr{V} \subseteq \mathscr{X}$, the restriction of the map A to $\mathscr{V}$ is denoted by $A|\mathscr{V}$. If $\mathscr{V}_1$ and $\mathscr{V}_2$ are A-invariant subspaces and $\mathscr{V}_2 \subseteq \mathscr{V}_1$, the map induced by A in the quotient space $\mathscr{V}_1/\mathscr{V}_2$ is denoted by $A|\mathscr{V}_1/\mathscr{V}_2$. To simplify, we sometimes use $\mathscr{B}$ in place of ImB, the image of B and $\mathscr{K}$ in place of $KerE$, the kernel of E. $\oplus$ denotes direct sum of subspace, $\uplus$ denotes union of sets with common elements repeated.

Let us denote $\Sigma_{(A,B)x} := \{x(t) : [0,\infty) \to \mathscr{X};\ x(t)$ is **a**.**c**.(absolutely continuous), and $\dot{x}(t) - Ax(t) \in ImB$ **a**.**e**.(almost everywhere)$\}$, and similarly $\Sigma_{(A,[B|D])x} := \{x(t) : [0,\infty) \to \mathscr{X};\ x(t)$ is **a**.**c**., and $\dot{x}(t) - Ax(t) \in ImB + ImD$ **a**.**e**.$\}$.

If $\mathscr{X}$ is a normed vector space, with norm $\|.\|$, and $\mathscr{L}$ a subspace of $\mathscr{X}$, then for any $x \in \mathscr{X}$, its distance to $\mathscr{L}$ is denoted as: $d(x,\mathscr{L}) := \inf_{y \in \mathscr{L}} \|x - y\|$.

For any measurable function, say $W : [0,\infty) \to \mathscr{X}$, we

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say that $W \in L_p[0,\infty)$ if $\|W\|_{L_p} < +\infty$, where:

$$\|W\|_{L_p} := \begin{cases} \left(\int_0^\infty \|W(t)\|^p \, dt\right)^{1/p} & \text{for } 1 \leq p < \infty \\ \text{ess sup}_{t\geq 0} \|W(t)\| & \text{for } p = \infty \end{cases}$$

The reachable space of Σ (by the control u) is denoted by $\langle A|\mathscr{B}\rangle := \mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + \cdots + A^{n-1}\mathscr{B}$, where n is the dimension of $\mathscr{X}$.

A subspace $\mathscr{V} \subset \mathscr{X}$ is called $(A,\mathscr{B})$ (or controlled)-invariant if there exists $F : \mathscr{X} \to \mathscr{U}$ such that $(A+BF)\mathscr{V} \subset \mathscr{V}$. F is called a friend of $\mathscr{V}$ and we denote $\mathscr{F}(\mathscr{V})$ the set of all such F.

A subspace $\mathscr{R} \subset \mathscr{X}$ is called an $(A,\mathscr{B})$ controllability subspace if there exist $F : \mathscr{X} \to \mathscr{U}$, and $G : \mathscr{Y} \to \mathscr{U}$, with $\mathscr{Y} \subset \mathscr{U}$, such that: $\mathscr{R} := \langle A + BF|Im(BG)\rangle$.

A subspace $\mathscr{V}_a \subset \mathscr{X}$ is called an almost $(A,\mathscr{B})$ (or controlled)-invariant subspace if for any $x_0 \in \mathscr{V}_a$ and for any $\epsilon > 0$ there exists a state trajectory $x_\epsilon \in \Sigma_{(A,B)x}$ with the properties that $x_\epsilon(0) = x_0$ and $d(x_\epsilon(t),\mathscr{V}_a) \leq \epsilon$, for any $t \geq 0$.

A subspace $\mathscr{R}_a \subset \mathscr{X}$ is called an almost $(A,\mathscr{B})$ controllability subspace if for any $x_0 \in \mathscr{R}_a$, and any $x_1 \in \mathscr{R}_a$ there exists $T > 0$ such that, for any $\epsilon > 0$ there exists a state trajectory $x_\epsilon \in \Sigma_{(A,B)x}$ with the properties that $x_\epsilon(0) = x_0, x_\epsilon(T) = x_1$ and $d(x_\epsilon(t),\mathscr{R}_a) \leq \epsilon, \forall t \geq 0$.

The supremal $(A,\mathscr{B})$ (or controlled)-invariant subspace contained in $\mathscr{K}$ is denoted by $\mathscr{V}^*$, or by $\mathscr{V}^*(\mathscr{K})$. It is the limit of the following non increasing algorithm, see [4], [3]:

$$\begin{cases} \mathscr{V}^0 = \mathscr{X} \\ \mathscr{V}^{i+1} = \mathscr{K} \cap A^{-1}\left(ImB + \mathscr{V}^i\right) \end{cases}$$

Similarly, $\mathscr{R}^*$, or $\mathscr{R}^*(\mathscr{K})$, the supremal $(A,\mathscr{B})$ controllability subspace contained in $\mathscr{K}$, is the limit of the following non decreasing algorithm, see [3]:

$$\begin{cases} \mathscr{R}^0 = 0 \\ \mathscr{R}^{i+1} = \mathscr{V}^* \cap \left(A\mathscr{R}^i + ImB\right) \end{cases}$$

$\mathscr{R}_a^*$, or $\mathscr{R}_a^*(\mathscr{K})$, the supremal almost $(A,\mathscr{B})$ controllability subspace contained in $\mathscr{K}$, is the limit of the following non decreasing algorithm, see [2]:

$$\begin{cases} \mathscr{R}_a^0 = 0 \\ \mathscr{R}_a^{i+1} = \mathscr{K} \cap \left(A\mathscr{R}_a^i + ImB\right) \end{cases}$$

$\mathscr{V}_a^*$, or $\mathscr{V}_a^*(\mathscr{K})$, the supremal almost $(A,\mathscr{B})$ controlled-invariant subspace contained in $\mathscr{K}$, satisfies: $\mathscr{V}_a^* = \mathscr{V}^* + \mathscr{R}_a^*$, see [2]:

$\mathscr{S}^*$ is usually introduced in the context of $(\mathscr{K}, A)$ invariance (dual to $(A,\mathscr{B})$ invariance). In our present context, we prefer to handle it through its almost controllability properties, as established in [2], and namely: $\mathscr{S}^* = A\mathscr{R}_a^* + ImB$ and $\mathscr{R}_a^* = \mathscr{K} \cap \mathscr{S}^*$.

Note that all these notions of exact/almost controlled invariance or controllability properties, can easily be defined, similarly, for the "composite" system (let $B_c := [B, D]$) , say $\Sigma(A, B_c, 0, E)$, i.e. with $\mathscr{U} \oplus \mathscr{Q}$ in place of $\mathscr{U}$. They will be denoted, respectively, $\mathscr{V}_c^*, \mathscr{R}_c^*, \mathscr{R}_{ca}^*, \mathscr{S}_c^*$.

Definition 1: **ADDP**, the almost disturbance decoupling problem, is said to be solvable if the following holds: $\forall\epsilon > 0$, $\exists F : \mathscr{X} \mapsto \mathscr{U}$ such that in the closed loop system with $x(0) = 0$, $\|z(t)\|_{L_q} \leq \epsilon \|q(t)\|_{L_p}$ for all L_p measurable disturbance input $q(t)$ and for all $1 \leq p \leq q \leq \infty$, see [2]. Equivalently, as shown in [5], **ADDP** is solvable if: $\forall\epsilon > 0$, there exists a sequence $\{F_\epsilon\}$ such that $\left\|Ee^{(A+BF_\epsilon)t}D\right\|_{L_p} \xrightarrow{\epsilon\to 0} 0$ for $p = 1$ and ∞.

It is well known [2] that **ADDP** is solvable by static state feedback if and only if:

$$ImD \subset \mathscr{V}_a^* \tag{1}$$

Definition 2: If $\mathscr{V}_a$ is an almost invariant subspace, the class of all the static state feedbacks $F_\epsilon : \mathscr{X} \mapsto \mathscr{U}$ such that, for any $x_0 \in \mathscr{V}_a$ and for any $t \geq 0$, $d(e^{(A+BF_\epsilon)t}x_0, \mathscr{V}_a) \leq \epsilon$, is denoted by $\mathscr{F}_\epsilon(\mathscr{V}_a)$. We call $F_\epsilon \in \mathscr{F}_\epsilon(\mathscr{V}_a)$ an ϵ-distance friend of the almost invariant subspace $\mathscr{V}_a$.

Some particular system structures play a key role in the solution of control problems, among them are the invariant zeros. The finite invariant zeros of $\Sigma(A, B, 0, E)$, i.e. from u to z, are equal to the dynamics[1] of the system in the quotient space $\mathscr{V}^*/\mathscr{R}^*$:

$$Z(A, B, E) := \sigma(A + BF|(\mathscr{V}^*/\mathscr{R}^*)),$$

for any $F \in \mathscr{F}(\mathscr{V}^*)$.

III. Pole assignability of almost invariant subspace

It is well known, since the seminal paper from [2], that any almost controlled-invariant subspace, say $\mathscr{V}_a$, can be written as the direct sum of a controllability subspace, say $\mathscr{R}$, plus a coasting subspace[2], say $\mathscr{C}$, plus a sliding subspace[3], say $\mathscr{J}$. Moreover, $\mathscr{J}$ can be seen as the limit, say when ϵ tends to zero, of a family of controlled-invariant subspaces $\mathscr{J}_\epsilon$, on which the dynamics are infinitely fast as ϵ tends to zero.

Lemma 1: Let $\mathscr{V}_a$ be an almost invariant subspace, $\mathscr{V}^*(\mathscr{V}_a)$ and $\mathscr{R}^*(\mathscr{V}_a)$ denote, respectively, the supremal $(A,\mathscr{B})$ controlled-invariant (resp. controllability subspace) included in $\mathscr{V}_a$. For any given spectra of ad-hoc lengths, say Λ_1 and Λ_2, for any $\epsilon > 0$ there always exists an ϵ-distance friend of $\mathscr{V}_a$, say F_ϵ, such that:
- the spectrum of $(A + BF_\epsilon)$ in $\mathscr{R}^*(\mathscr{V}_a)$ equals Λ_1 (free)
- the spectrum of $(A+BF_\epsilon)$ in $(\mathscr{V}_a + \langle A|\mathscr{B}\rangle)/\mathscr{V}_a$ equals Λ_2 (free)
- the spectra of $(A + BF_\epsilon)$ in $\mathscr{V}^*(\mathscr{V}_a)/\mathscr{R}^*(\mathscr{V}_a)$ and in $\mathscr{X}/(\mathscr{V}_a + \langle A|\mathscr{B}\rangle)$ are fixed (the same for any F_ϵ)
- the spectrum of $(A+BF_\epsilon)$ in $\mathscr{V}_a/\mathscr{V}^*(\mathscr{V}_a)$ is "infinite" but stable, in the sense that $\mathscr{V}_a/\mathscr{V}^*(\mathscr{V}_a)$ can be identified with

[1] Which indeed are fixed, after having applied any state feedback, say F, i.e. replacing A by $A + BF$. They are also invariant after any change of basis in $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{Z}$ as well as any output rejection $L : \mathscr{Z} \to \mathscr{X}$, i.e. when replacing A by $A + LE$.

[2] a controlled-invariant subspace $\mathscr{C}$ is called a coasting subspace if and only if $\mathscr{R}^*(\mathscr{C}) = \{0\}$, i.e. the supremal controllability subspace in $\mathscr{C}$ is 0

[3] an almost controlled-invariant subspace $\mathscr{J}$ is called a sliding subspace if and only if $\mathscr{V}^*(\mathscr{J}) = \{0\}$, i.e. the supremal controlled-invariant subspace in $\mathscr{J}$ is 0

a sliding subspace for which on any approximation $\mathscr{J}_\epsilon$, all the dynamics tend to “minus infinity” as ϵ tends to zero.
Proof: See [8] and [9].

∎

REMARK 1: The key point used in the proof is how to construct an ϵ-distance friend of $\mathscr{V}_a$, which comes from a sequence of feedback maps $\{F_n\}$. The idea is first to decompose $\mathscr{V}_a$ into a direct sum of $\mathscr{V}^*(\mathscr{V}_a)$ and an almost controllability subspace $\mathscr{R}_a$ (in fact, this $\mathscr{R}_a$ is a sliding subspace). Then decompose $\mathscr{R}_a$ into the direct sum of singly generated almost controllability subspaces.[4] Each of these subspaces is approximated by a sequence of controlled invariant subspaces $L(n)$, as showed in REMARK 2. On each of these approximates we define a feedback. Finally, these are used to define a sequence of feedbacks $\{F_n\}$ on $\mathscr{X}$ which turn out to have the desired properties.

REMARK 2: The convergence of subspaces has to be understood in the usual Grassmannian sense. The convergence algorithm comes from [10], see also [5]: Suppose $\mathscr{L} = span\{b, A_F b, \cdots, A_F^{k-1}\}$ is a singly generated sliding subspace, where $b \in \mathscr{B}$. Without any loss of generality, we can assume that the vectors $b, A_F b, \cdots, A_F^{k-1}$ are linearly independent. For $n \in \mathbb{N}$ sufficiently large, the mapping $(I + \frac{1}{n}A_F)$ is non-singular. Define sequences of vectors $x_i(n), i = 1, 2, \cdots, k$, recursively by

$$\begin{cases} x_1(n) := (I + \dfrac{1}{n}A_F)^{-1}b \\ x_{i+1}(n) := (I + \dfrac{1}{n}A_F)^{-1}A_F x_i(n) \end{cases}$$

Define $\mathscr{L}(n) := span\{x_1(n), x_2(n) \cdots, x_k(n)\}$, then for any $n \in \mathbb{N}$, $\mathscr{L}(n)$ is an $(A, \mathscr{B})$-invariant subspace, $\mathscr{L}(n) \subset \langle A|\mathscr{B}\rangle$, and $\mathscr{L}(n) \to \mathscr{L}$ as $n \to \infty$. Assume $b := Bu$, where $u \in \mathscr{U}$, then define a sequence $F_n : \mathscr{L}(n) \mapsto \mathscr{U}$ by:

$$F_n x_i(n) := -n^i u, \; i = 1, 2, \cdots$$

F_n then turns out to make $\mathscr{L}(n)$ invariant under $A + BF_n$, and it may be seen that $\sigma(A+BF_n|\mathscr{L}(n)) = \{-n, -n, \cdots\}$.

The pole placement freedom related to any almost invariant subspace $\mathscr{V}_a$ is summarized in Fig.1.

IV. Pole assignability of ADDP

Equation (1) means that **ADDP** is solvable if and only if there exists an almost controlled-invariant subspace $\mathscr{V}_a$ included in $\mathscr{K}$ such that: $ImD \subset \mathscr{V}_a$. Since $\mathscr{V}_a \subset \mathscr{K}$, it is obvious that, any ϵ-distance friend of $\mathscr{V}_a$ can be used as a state feedback such that the system state trajectory that starts from any state in ImD can be restricted within the ϵ-distance to subspace $\mathscr{K}$, namely, such an ϵ-distance friend of $\mathscr{V}_a$ solves **ADDP**. Lemma 1 shows that, ϵ-distance friend of $\mathscr{V}_a$ always exists and furthermore, the system dynamics depend on the choice of the almost controlled-invariant subspace $\mathscr{V}_a$. With such a $\mathscr{V}_a$, we can use the algorithm introduced in Lemma 1 to find state feedback solutions of **ADDP** with

[4] An almost controllability subspace $\mathscr{L}$ is called singly generated if it can be noted as $\mathscr{L} := span\{b, A_F b, \cdots, A_F^{k-1}b\}$, where $b \in \mathscr{B}$, $dim(b) = 1$, $A_F := A + BF$ and $k \leq n$, a non negative integer.

$$\begin{array}{cl}
\mathscr{X} & \\
\Big| & \text{fixed finite (uncontrollable)} \\
\mathscr{V}_a + \langle A|\mathscr{B}\rangle & \\
\Big| & \text{free finite} \\
\mathscr{V}_a & \\
\Big| & \text{infinite (but stable)} \\
\mathscr{V}^*(\mathscr{V}_a) & \\
\Big| & \text{fixed finite} \\
\mathscr{R}^*(\mathscr{V}_a) & \\
\Big| & \text{free finite} \\
0 &
\end{array}$$

Fig. 1. Pole placement freedom related to $\mathscr{V}_a$

desired properties. We call such a $\mathscr{V}_a$ a geometric solution of **ADDP**.

Definition 3: Let $\mathscr{V}_a$ be an almost $(A, \mathscr{B})$ invariant subspace, $F_\epsilon \in \mathscr{F}_\epsilon(\mathscr{V}_a)$. Then the closed loop system exhibits some finite fixed poles, independently of the choice of F_ϵ in $\mathscr{F}_\epsilon(\mathscr{V}_a)$. We call them finite fixed poles of the almost $(A, \mathscr{B})$ invariant subspace $\mathscr{V}_a$, $\sigma_{fixed}^{finite}(\mathscr{V}_a)$.

Based on Lemma 1, $\sigma_{fixed}^{finite}(\mathscr{V}_a)$ is given by:

$$\begin{aligned} \sigma_{fixed}^{finite}(\mathscr{V}_a) := &\sigma\left(A + BF_\epsilon \;\middle|\; \frac{\mathscr{X}}{\mathscr{V}_a + \langle A|\mathscr{B}\rangle}\right) \\ &\uplus \sigma\left(A + BF_\epsilon \;\middle|\; \frac{\mathscr{V}^*(\mathscr{V}_a)}{\mathscr{R}^*(\mathscr{V}_a)}\right) \end{aligned} \quad (2)$$

where $\mathscr{V}^*(\mathscr{V}_a)$ and $\mathscr{R}^*(\mathscr{V}_a)$ denote, respectively, the supremal $(A, \mathscr{B})$ controlled-invariant (resp. controllability) subspace contained in $\mathscr{V}_a$.

Definition 4: Suppose that **ADDP** is solvable for $\Sigma(A, B, D, E)$. Then the finite fixed poles of **ADDP** are defined as:

$$\sigma_{fixed}^{\mathbf{ADDP}}(A, B, D, E) := \bigcap_i \sigma_{fixed}^{finite}(\mathscr{V}_{ai})$$

where $\{\ \mathscr{V}_{ai},\ i = 1, 2, \cdots\ \}$ is the set of all almost disturbance invariant subspace contained in $\mathscr{K}$.

$\mathscr{R}_{ca}^*$ plays a key role in the characterization of $\sigma_{fixed}^{\mathbf{ADDP}}(A, B, D, E)$, because of lack of place, we give the following theorems without proofs, they are developed in [9].

Theorem 2: Assume that **ADDP** is solvable,
- Any feedback solution of **ADDP** contains a set of finite fixed poles, $\sigma_{fixed}^{ADDP}(A, B, D, E)$.
- The finite fixed poles of **ADDP** are characterized as $\sigma_{fixed}^{ADDP}(A, B, D, E) = \sigma\left(A + B\Phi \mid \frac{\mathscr{X}}{\mathscr{S}_c^* \cap \mathscr{V}^* + \langle A|\mathscr{B}\rangle}\right) \uplus \sigma\left(A + B\Phi \mid \frac{\mathscr{S}_c^* \cap \mathscr{V}^*}{\mathscr{R}^*}\right)$, where Φ is any map which makes $\mathscr{S}_c^* \cap \mathscr{V}^*$ $(A + B\Phi)$ invariant.
- When using ϵ-distance friends of $\mathscr{R}_{ca}^*$, poles tending to $-\infty$ occur as $\sigma_{stable}^{\infty} = \lim_{\epsilon \to 0}(\sigma(A + BF_\epsilon|\mathscr{J}_\epsilon))$, where F_ϵ is any map which makes $\mathscr{J}_\epsilon$ $(A + BF_\epsilon)$ invariant, and where $\mathscr{J}_\epsilon$ is a controlled-invariant approximation of the sliding part $\mathscr{R}_{ca}^*/(\mathscr{S}_c^* \cap \mathscr{V}^*)$.
- It is possible to find particular feedback solutions for which

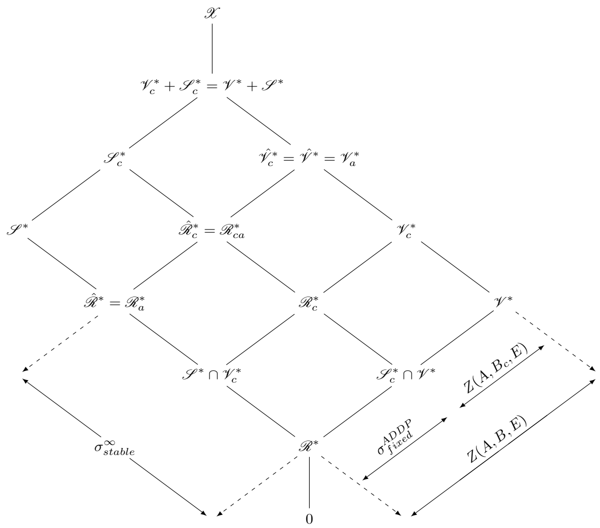

Fig. 2. Lattice properties and location of Fixed Poles of **ADDP** when **ADDP** is solvable

all the other finite poles, other than the finite fixed poles of **ADDP**, can be placed freely.

Theorem 3: Assume that **ADDP** is solvable, and $(A, \mathscr{B})$ is controllable, the finite fixed poles of **ADDP**, are characterized as follows:

$$Z(A,B,E) = \sigma_{fixed}^{ADDP}(A,B,D,E) \uplus Z(A,B_c,E)$$

When **ADDP** is solvable, the lattice properties and location of poles of **ADDP** are summarized in Fig.2.

From now on, we shall assume, for shortness, that $(A, \mathscr{B})$ is controllable.

Assume that $\mathscr{V}_a$ is a geometric solution of **ADDP**, then the number of fixed poles of $\mathscr{V}_a$ equals the dimension of the maximal coasting subspace contained in $\mathscr{V}_a$, the number of infinite poles equals the dimension of maximal sliding subspace contained in $\mathscr{V}_a$. In order to get the maximal freedom in the assignment of the system dynamics, we must choose a geometric solution that makes the above two parts as small as possible.

When the condition (1) holds, in [2] a general procedure to solve the **ADDP** has been given. Then in [5], [10], the authors described in a more detailed way how to get state feedback solutions F_ϵ for **ADDP**, but all based on $\mathscr{V}_a^*$, the supremal almost $(A, \mathscr{B})$ controlled-invariant subspace contained in $\mathscr{K}$. According to the above analysis, this $\mathscr{V}_a^*$ is not the best geometric solution with respect to pole placement (for both the finite and infinite parts). On the contrary. it can be a rather conservative geometric solution with respect to pole placement, since the coasting and sliding parts inside this subspace are both maximal, so it will lead to the minimum freedom when choosing closed-loop system dynamics.

V. Optimal geometric solution of **ADDP**

Our objective is to solve **ADDP** and simultaneously place the maximal number of poles in the closed loop system. From the previous analysis this amounts to finding a class of optimal geometric solutions of **ADDP** that holds minimal fixed finite and infinite poles. To characterize such a class, we go through the following steps: first it will be shown that the pole assignability of any geometric solution of **ADDP** can be improved by adding $\mathscr{R}^*$ in the sense of the minimization of fixed finite and infinite poles, namely, the optimal geometric solutions of **ADDP** must contain $\mathscr{R}^*$. Next, we will show that any geometric solution of **ADDP** that contains $\mathscr{R}^*$ must also contain $\mathscr{S}_c^* \cap \mathscr{V}^*$, which as shown in [11] is an $(A, \mathscr{B})$-invariant subspace contained in $\mathscr{K}$ and contains minimal fixed poles of **ADDP**. Also we will show that the minimization of infinite poles can not be achieved at the cost of the deterioration of finite pole assignability when a geometric solution already contains $\mathscr{S}_c^* \cap \mathscr{V}^*$. Finally, we give the characterization of optimal geometric solutions of **ADDP**.

Lemma 4: In order to minimize the number of infinite poles while solving **ADDP**, an optimal geometric solution of **ADDP** must contain $\mathscr{R}^*$, the supremal $(A, \mathscr{B})$ controllability subspace contained in $\mathscr{K}$, where $\mathscr{K}$ is the kernel of the output matrix .

Proof: Suppose $\mathscr{V}_a$ is a geometric solution of **ADDP**, i.e. $\mathscr{V}_a$ is an almost $(A, \mathscr{B})$ controlled-invariant subspace contained in $\mathscr{K}$ and containing ImD. Obviously $\mathscr{V}_a + \mathscr{R}^*$ is also a geometric solution of **ADDP**, since being an almost $(A, \mathscr{B})$ controlled-invariant subspace contained in $\mathscr{K}$ and containing ImD. Then to prove this Lemma, we just need to show that the dimension of maximal sliding subspaces inside $\mathscr{V}_a + \mathscr{R}^*$ is less than that of $\mathscr{V}_a$, where $\mathscr{V}_a$ is any almost invariant subspace contained in $\mathscr{K}$. This directly comes from the fact that the dimension of the maximal sliding subspaces contained in $\mathscr{J} + \mathscr{R}^*$ is less than that of $\mathscr{J}$, where $\mathscr{J}$ is any sliding subspace contained in $\mathscr{K}$.

Indeed, let $\mathscr{J}$ be any sliding subspace contained in $\mathscr{K}$, define $\mathscr{J} := \overline{\mathscr{J}} \oplus (\mathscr{J} \cap \mathscr{R}^*)$, we claim that $\overline{\mathscr{J}}$ is a supremal sliding subspace contained in $\mathscr{J} + \mathscr{R}^*$. To show this, note that $\mathscr{R}^*$ is the supremal $(A, \mathscr{B})$ controllability subspace contained in $\mathscr{K}$, it is thus the supremal $(A, \mathscr{B})$ controllability subspace contained in $\mathscr{J} + \mathscr{R}^*$.

$$\begin{aligned} dim\left(\frac{\mathscr{J} + \mathscr{R}^*}{\mathscr{R}^*(\mathscr{J} + \mathscr{R}^*)}\right) &= dim\left(\frac{\overline{\mathscr{J}} \oplus \mathscr{R}^*}{\mathscr{R}^*}\right) \\ &= dim\left(\overline{\mathscr{J}}\right) \leq dim\left(\mathscr{J}\right) \end{aligned}$$

■

Lemma 5: Suppose $\mathscr{V}_a$ is a geometric solution of **ADDP**, i.e. $\mathscr{V}_a$ is an almost controlled-inavriant subspace such that $ImD \subset \mathscr{V}_a \subset \mathscr{K}$, the following property holds:

$$\mathscr{V}_a + \mathscr{R}^* \supset (\mathscr{S}_c^* \cap \mathscr{V}^*)$$

where $\mathscr{R}^*$, $\mathscr{S}_c^*$ and $\mathscr{V}^*$ are defined above.

Proof: The proof is by contradiction. To simplify the notation, let $\mathscr{V}_a^1 := \mathscr{V}_a + \mathscr{R}^*$, $\overline{\mathscr{V}_a} := \mathscr{V}_a^1 + \mathscr{R}_a^*$.

If $\mathscr{V}_a^1 \not\supset (\mathscr{S}_c^* \cap \mathscr{V}^*)$, we conclude that $\mathscr{V}^*(\mathscr{V}_a^1) \not\supset (\mathscr{S}_c^* \cap \mathscr{V}^*)$, since $\mathscr{V}^*(\mathscr{V}_a^1)$ is the supremal $(A, \mathscr{B})$-invariant subspace contained in $\mathscr{V}_a^1$ and $(\mathscr{S}_c^* \cap \mathscr{V}^*)$ is also an $(A, \mathscr{B})$-invariant subspace [9].

Using suitable bases, we have:

$$\mathscr{V}_a^* = \mathscr{V}^* + \mathscr{R}_a^* := \mathscr{V}^* \oplus \hat{\mathscr{R}}_a^* = \left[\begin{array}{c|c} V_1 & 0 \\ \hline 0 & S \end{array}\right] \tag{3}$$

where $\mathscr{V}^* := Im \left[\begin{array}{c} V_1 \\ \hline 0 \end{array}\right]$, $\hat{\mathscr{R}}_a^* := Im \left[\begin{array}{c} 0 \\ \hline S \end{array}\right]$, and $\hat{\mathscr{R}}_a^*$ is a sliding subspace contained in $\mathscr{K}$.

Thanks to [2], we can let $\overline{\mathscr{V}_a} := \mathscr{V}_a^1 + \mathscr{R}_a^* = \mathscr{V}^*(\mathscr{V}_a^1) \oplus \tilde{\mathscr{R}}_a^*$, where $\tilde{\mathscr{R}}_a^* = \hat{\mathscr{R}}_a^*$. Moreover $\mathscr{V}^*(\mathscr{V}_a^1) \subset \mathscr{V}^*$ and $(\mathscr{S}_c^* \cap \mathscr{V}^*) \subset \mathscr{V}^*$, since these two subspaces are $(A, \mathscr{B})$-invariant contained in $\mathscr{K}$ and $\mathscr{V}^*$ is the supremal. With the assumption $\mathscr{V}^*(\mathscr{V}_a^1) \not\supset (\mathscr{S}_c^* \cap \mathscr{V}^*)$, we get immediately thanks to Equation (3):

$$\overline{\mathscr{V}_a} = \mathscr{V}^*(\mathscr{V}_a^1) \oplus \tilde{\mathscr{R}}_a^* \not\supset (\mathscr{S}_c^* \cap \mathscr{V}^*) \tag{4}$$

$\mathscr{V}_a$ being a geometric solution of **ADDP**, obviously $\overline{\mathscr{V}_a} := \mathscr{V}_a^1 + \mathscr{R}_a^* = \mathscr{V}_a + \mathscr{R}_a^*$ is also a geometric solution of **ADDP** and contains $\mathscr{R}_a^*$. We have shown in [9] that $\mathscr{R}_{ca}^*$ is the infimum of all geometric solutions of **ADDP** that contain $\mathscr{R}_a^*$, hence $\overline{\mathscr{V}_a} \supset \mathscr{R}_{ca}^* \supset (\mathscr{S}_c^* \cap \mathscr{V}^*)$, see Fig.2. Since this conflicts with Equation (4), we conclude that the assumption is not correct. ■

We have shown in [11] that to minimize the number of fixed poles while solving **ADDP**, we can use $\mathscr{V}_a + \mathscr{R}^*$ instead of $\mathscr{V}_a$ as a new geometric solution of **ADDP**. With the result of Lemma 4, we conclude that an optimal geometric solution of **ADDP** must contain $\mathscr{R}^*$ for maximal pole assignability to be obtained (i.e. to also minimize the number of both fixed finite and infinite poles). Thanks to Lemma. 5, such a $\mathscr{V}_a$ must contain $\mathscr{S}_c^* \cap \mathscr{V}^*$.

The combination of Theorem. 2 and Lemma. 5 immediately yields the following: an optimal geometric solution of **ADDP** with respect to the minimization of the number of fixed poles will be an almost $(A, \mathscr{B})$ controlled-invariant subspace containing ImD, that is the sum of $\mathscr{S}_c^* \cap \mathscr{V}^*$ and of a sliding subspace contained in $\mathscr{K}$, namely, such an optimal geometric solution of **ADDP** can be chosen on the track between $\mathscr{R}_{ca}^*$ and $\mathscr{S}_c^* \cap \mathscr{V}^*$, as shown in Fig.2. Furthermore, when **ADDP** is solvable, such an optimal geometric solution of **ADDP** always exists, since $\mathscr{R}_{ca}^*$ is always a geometric solution of **ADDP**. In the sequel, we will show that this also holds with respect to the minimization of the number of infinite poles of **ADDP**.

Let us first give the following lemma:

Lemma 6: let $\mathscr{V}_a^2$ be an optimal geometric solution of **ADDP** with respect to the minimization of the number of infinite poles, i.e. $\mathscr{V}_a^2$ is an almost $(A, \mathscr{B})$ controlled-invariant subspace contained in $\mathscr{K}$ and containing $\mathscr{S}_c^* \cap \mathscr{V}^*$. Let $\mathscr{V}_a^2$ be denoted as $\mathscr{V}_a^2 := \mathscr{V}^2 \oplus \mathscr{J}_2$, where $\mathscr{V}^2$ is the supremal $(A, \mathscr{B})$ controlled-invariant subspace contained in $\mathscr{V}_a^2$, $\mathscr{J}_2$ is a sliding subspace contained in $\mathscr{K}$. Then $\mathscr{V}_a^1 := \mathscr{R}_{ca}^* \cap \mathscr{V}_a^2 = \mathscr{S}_c^* \cap \mathscr{V}^* + \mathscr{J}_2$ will also be an optimal geometric solution of **ADDP** with respect to the minimization of the number of infinite poles, furthermore $\mathscr{S}_c^* \cap \mathscr{V}^* + \mathscr{J}_2 \subset \mathscr{R}_{ca}^*$. See Fig.3 for the corresponding lattice properties.

Proof: Since $\mathscr{V}^2$ is the supremal $(A, \mathscr{B})$ controlled-invariant subspace contained in $\mathscr{V}_a^2$, it obviously contains $\mathscr{S}_c^* \cap \mathscr{V}^*$

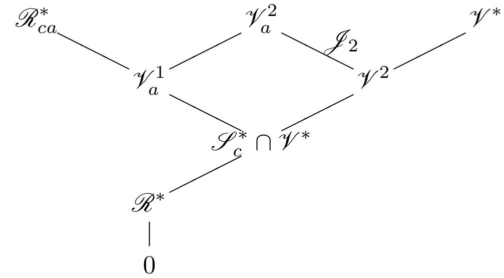

Fig. 3. Lattice properties of optimal geometric solution

thanks to $\mathscr{S}_c^* \cap \mathscr{V}^* \subset \mathscr{V}_a^2$ and $\mathscr{S}_c^* \cap \mathscr{V}^*$ is an $(A, \mathscr{B})$ controlled-invariant subspace; since $\mathscr{J}_2$ is a sliding subspace contained in $\mathscr{K}$, we obviously have $\mathscr{J}_2 \subset \mathscr{R}_{ca}^*$; because $\mathscr{S}_c^* \cap \mathscr{V}^* \subset \mathscr{V}^2$ and $\mathscr{J}_2 \cap \mathscr{V}^2 = 0$, we get $(\mathscr{S}_c^* \cap \mathscr{V}^*) \cap \mathscr{J}_2 = 0$. Based on the above results, we get immediately $\mathscr{R}_{ca}^* \cap \mathscr{V}_a^2 = \mathscr{S}_c^* \cap \mathscr{V}^* + \mathscr{J}_2 =: \mathscr{V}_a^1$, obviously $\mathscr{V}_a^1$ is an almost $(A, \mathscr{B})$ controlled-invariant subspace contained in $\mathscr{K}$.

In [9], we have shown that when **ADDP** is solvable $\mathscr{R}_{ca}^*$ is a geometric solution of **ADDP**, i.e. $ImD \subset \mathscr{R}_{ca}^*$, we conclude $\mathscr{V}_a^1 := \mathscr{R}_{ca}^* \cap \mathscr{V}_a^2 \supset ImD$ since $ImD \subset \mathscr{V}_a^2$.

Now we conclude that $\mathscr{V}_a^1$ is an almost $(A, \mathscr{B})$ controlled-invariant subspace contained in $\mathscr{R}_{ca}^*$ and contains ImD, moreover it is an optimal geometric solution of **ADDP** with respect to the minimization of the number of infinite poles. ■

To conclude this section, we note that the following characterization of the class of optimal geometric solutions of **ADDP** with respect to maximal pole placement is now available, the simple proof is omitted.

Theorem 7: Assume that **ADDP** is solvable and $(A, \mathscr{B})$ is controllable, optimal (in the sense of maximal pole assignment) geometric solutions of **ADDP**, say $\mathscr{V}_a$, can be constructed as the sum of the $(A, \mathscr{B})$ controlled-invariant subspace $\mathscr{S}_c^* \cap \mathscr{V}^*$ and a sliding subspace contained in $\mathscr{K}$ with minimal dimension, say $\mathscr{J}$, such that $ImD \subset \mathscr{S}_c^* \cap \mathscr{V}^* + \mathscr{J}$.

VI. Algorithm for Optimal geometric solution in a particular case

As shown in [12],[13],[5], a subspace $\mathscr{R}_a$ is almost $(A, \mathscr{B})$ controlled-invariant if and only if there exist a map $F : \mathscr{X} \mapsto \mathscr{U}$ and a chain[5] $\{\mathscr{B}_i\}_{i=1}^k$ in $\mathscr{B}$ such that

$$\mathscr{R}_a = \mathscr{B}_1 + (A + BF)\mathscr{B}_2 + \cdots + (A + BF)^{k-1}\mathscr{B}_k$$

Trentelman has shown in [8],[5] that a sliding subspace $\mathscr{J}$ can be obtained as

$$\mathscr{J} = \tilde{\mathscr{B}}_1 \oplus (A + BF)\tilde{\mathscr{B}}_2 \oplus \cdots \oplus (A + BF)^{k-1}\tilde{\mathscr{B}}_k$$

, where $\{\tilde{\mathscr{B}}_i\}_{i=1}^k$ is a chain in $\tilde{\mathscr{B}}$, $\mathscr{B} := (\mathscr{B} \cap \mathscr{V}^*) \oplus \tilde{\mathscr{B}}$.

In the particular case of systems with $dim(\frac{\mathscr{B}}{\mathscr{B} \cap \mathscr{V}^*}) = 1$, i.e. $dim(\tilde{\mathscr{B}}) = 1$, obviously $\mathscr{R}_a$ can be denoted as $\mathscr{R}_a = \mathscr{B}_1 + A\mathscr{B}_2 + \cdots + A^{k-1}\mathscr{B}_k$ (resp. $\mathscr{J}$ can be denoted as

[5]A sequence of subspaces $\{\mathscr{L}_i\}_{i=1}^k$ is call a chain if $\mathscr{L}_1 \supset \mathscr{L}_2 \supset \cdots$

$\mathscr{J} = \tilde{\mathscr{B}} \oplus A\tilde{\mathscr{B}} \oplus \cdots \oplus A^{k-1}\tilde{\mathscr{B}}$). It gives us a simple way to find the sliding subspace $\mathscr{J}$ with minimal dimension such that $ImD \subset \mathscr{S}_c^* \cap \mathscr{V}^* + \mathscr{J}$, as required in Theorem.7.

VII. Illustrative Example

Let us consider the system $\Sigma(A, B, D, E)$ with:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} D = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$E = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we get:

$$\mathscr{R}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathscr{V}^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathscr{S}_c^* \cap \mathscr{V}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathscr{R}_{ca}^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathscr{V}_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

where $\mathscr{V}_a$ is an optimal geometric subspace of **ADDP**.

Since, $ImD \not\subset \mathscr{V}^* + ImB$, $ImD \subset \mathscr{V}_a^*$, **DDP** is not solvable by state feedback (possibly with disturbance feed-forward), while **ADDP** is solvable by high gain state feedback. Because $Z(A, B, E) = \{-1, 1\}$ and $Z(A, B_c, E) = \{1\}$, we get immediately $\sigma_{fixed}^{ADDP} = \{-1\}$, namely: to solve **ADDP**, in the closed-loop system, we must have a fixed pole set $\{-1\}$, and the number of fixed poles is $[dim(\mathscr{S}_c^* \cap \mathscr{V}^*) - dim(\mathscr{R}^*)] = 1$, the number of infinite poles is $[dim(\mathscr{V}_a) - dim(\mathscr{S}_c^* \cap \mathscr{V}^*)] = 2$, our maximal freedom in pole placement is $d = dim(\mathscr{X}) - dim(\mathscr{V}_a) + dim(\mathscr{R}^*) = 3$, since $(A, \mathscr{B})$ is controllable. If we were using $\mathscr{V}_a^*$ as the geometric solution, our freedom in pole placement would be smaller: $d = dim(\mathscr{X}) - dim(\mathscr{V}_a^*) + dim(\mathscr{R}^*) = 1$.

Let us arbitrarily choose three finite poles, e.g. $\{-2, -3, -4\}$. Using the method introduced in this paper, we can easily find a state feedback $F_p = [-24\,p^2,\quad 48\,p - 50\,p^2,\quad -24 + 100\,p - 59\,p^2,\quad -50 + 118\,p - 60\,p^2,\quad -59 + 120\,p - 60\,p^2,\quad -60 + 120\,p - 60\,p^2]$ such that with $u = F_p x$, the system closed-loop poles are $\{-4, -3, -2, -1, p, p\}$. With MAPLE®, let $\epsilon := -\frac{1}{p}$, we can verify that $\left\|Ee^{(A+BF_p)t}D\right\|_{L_1} \overset{\epsilon \to 0}{\longrightarrow} 0$ and $\left\|Ee^{(A+BF_p)t}D\right\|_{L_\infty} \overset{\epsilon \to 0}{\longrightarrow} 0$, namely, sequence $\{F_p\}$ solves **ADDP**. See also Fig.4.

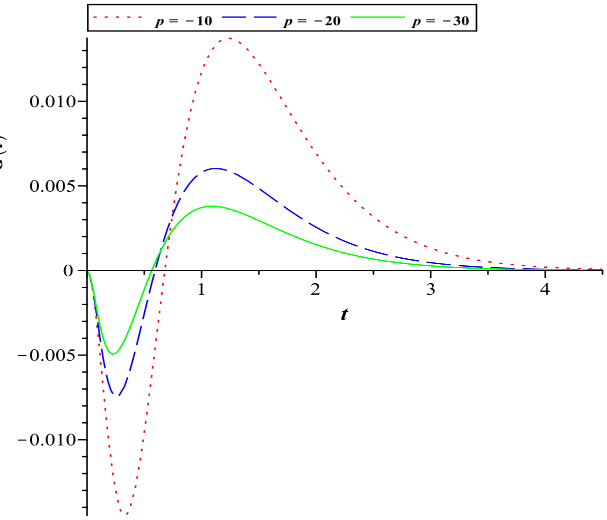

Fig. 4. Disturbance impulse response $G(t)$

VIII. Conclusion

Thanks to the study of the fixed finite and infinite poles of **ADDP**, when **ADDP** is solvable, the optimal geometric subspaces with which **ADDP** can be solved and maximal pole placement can be achieved are characterized. For the first time, we construct a simple but reliable numerical algorithm for such an optimal geometric subspace solution (resp. the state feedback solution) for a particular class of linear systems where $dim(\frac{\mathscr{B}}{\mathscr{B} \cap \mathscr{V}^*}) = 1$. An example illustrates the feasibility.

Our objective is now to construct a reliable algorithm for finding the above optimal geometric subspaces (resp. the state feedback solutions) in the general case.

References

[1] A. Saberi, A. A. Stoorvogel, and P. Sannuti, *Filtering Theory: With Applications to Fault Detection, Isolation, and Estimation*. Birkhäuser Boston, 2006.

[2] J. Willems, "Almost invariant subspaces: an approach to high gain feedback design,part i:almost controlled invariant subspaces," *IEEE Transactions on Automatic Control*, vol. AC-26, pp. 235–252, 1981.

[3] W. Wonham, *Linear multivariable control: a geometric approach(3^{rd} ed.)*. New York: Springer-Verlag, 1985.

[4] G. Basile and G. Marro, *Controlled Invariants and Conditioned Invariants in Linear System Theory*. Englewood Cliffs, New Jersey: Prentice-Hall, 1992.

[5] H. Trentelman, "Almost invariant subspaces and high gain feedback," Ph.D. dissertation, Rijksuniversiteit Groningen, 1985.

[6] M. Malabre, J. Martinez-Garcia, and B. Del-Muro-Cuellar, "On the fixed poles for disturbance rejection," *Automatica*, vol. 33, pp. 1209–1211, June 1997.

[7] D. Chu, "The fixed poles of the disturbance decoupling problem and almost stability subspace $v_{b,g}^*(ker(c))$," *Numerische Mathematik*, vol. 96, pp. 221–252, 2003.

[8] H. Trentelman, "On the assignability of infinite root loci in almost disturbance decoupling." *International Journal of Control*, vol. 38, pp. 147–167, 1983.

[9] R. Zou and M. Malabre, "Almost disturbance decoupling and pole placement," *Submitted to a journal*, 2008, also available from http://www.irccyn.ec-nantes.fr.

[10] J. Schumacher, "Almost stabilizability subspaces and high gain feedback," *IEEE Transactions on Automatic Control*, vol. AC-29, pp. 620–627, 1984.

[11] M. Malabre and R. Zou, "Almost self-bounded controlled invariant subspaces and almost disturbance decoupling," in *17th IFAC World Congress,Seoul*, July 2008.

[12] J. Willems, "Almost *a* (mod $\mathscr{B}$) invariant subspaces," *Astérisque*, vol. 75-76, pp. 239–248, 1980.

[13] M. Malabre, "A complement about almost controllability subspaces." *Systems & Control Letters*, vol. 3, pp. 119–122, 1983.