2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Discrete-Time Quantized H_∞ Filtering with Quantizer Ranges Consideration

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Abstract— **This paper is concerned with the quantized H_∞ filtering problem for discrete-time systems. The quantizer considered here is dynamic and composed of a dynamic scaling and a static quantizer. Motivated by practical transmission channels requirements, the static quantizer ranges are fully considered in this paper. A quantized H_∞ filter design strategy is proposed with taking quantizer errors into account, where a convex optimization method is developed to minimize static quantizer ranges with meeting H_∞ performance requirement for quantized augmented systems. A numerical example is given to illustrate the effectiveness of the proposed filter design method.**

I. INTRODUCTION

The recent interest in quantized systems is the involvement of communication channels of limited bandwidth in data transmission. In this setting, the signal is quantized and then coded for transmission. From the controller or filter design point of view, a fundamental problem is how to design a control strategy composed of a controller and a quantizer (or a filtering strategy composed of a filter and a quantizer) with the minimum information rate in order to guarantee a given control objective or filtering objective. Filters are the most essential building blocks of signals processing. And the problem of filtering with quantized signals has been considered by several researchers [17], [21], ect.

Many contributions have appeared in the literature examining the impact of quantization on linear systems. These quantization policies can be mainly categorized depending on whether the quantizer is static or dynamic. A static quantizer is a memoryless nonlinear function, whereas a dynamic quantizer uses memory and thus can be much more complex and potentially more powerful.

There has been many quantized results using static quantizers, such as [3], [5]- [9], [12] and [22]. The attraction of static quantizers is the simplicity of their coding/decoding schemes. However, the main drawback of static policies is that they require an infinite number of quantization bits to ensure asymptotic stability [5]. In contrast, quantization policies with memory (dynamic quantization policies) have the advantages to scale the quantization levels dynamically so that the region of attraction is increased and the steady state limit cycle is reduced. This is indeed the basic idea behind [2], [16], [18]-[20]. In fact, it is shown in [16] that stabilization of an SISO LTI system (in some stochastic sense) can be achieved using only a finite number of quantization levels. In this setting, the dynamic quantizer effectively consists of two parts: an encoder at the output end and a decoder at the input end [14]. Another type of dynamic quantizers uses dynamic scaling in conjunction with a static quantizer. Noticeable work along this line includes [2], [10], [13], [19], [20]. Note that dynamic scaling is a very popular idea in signal processing for reducing quantization errors [15] but the key difference here is that the system stability and performance are also considered.

Noting that the works outlined above with dynamic quantizers deal with only stability/stabilization problems, whereas the H_∞ control problem has not been addressed in them. As pointed out in [24] that the control strategies of updating the quantizer's parameter are dependent on time in the existing works [2] and [13], and such control strategies cannot be applied for the case of H_∞ control systems since we do not know the value of the disturbance inputs and thus cannot drive the state into an invariant region, as done in [13]. In contrast, in[24], a state or output dependent control strategy is proposed, and by using which the quantized continuous-time H_∞ control problem is solved. Similarly, the latest work [4] studies the networked-based H_∞ control problem with dynamic quantizers. But these works do not consider the minimum number of quantization levels required to assure the H_∞ performance requirements for quantized systems. However, a major question about the quantized systems concerns the minimum number of quantization levels required to assure closed loop system stability and performance, which may have many benefits that include lower cost, higher reliability, and easier maintenance.

Motivated by the above reasons, this paper is concerned with the quantized H_∞ filtering problem for discrete-time linear systems with a type of dynamic quantizers, which are conjuncted with static quantizers via dynamic scalings. By taking quantizer errors into account, a quantized H_∞ filter design strategy is proposed, where a convex optimization method is developed to minimize static quantizer ranges. The resulting design guarantees that the quantized augmented system is asymptotically stable and with a prescribed H_∞ performance bound. The effectiveness of the proposed filter design method is demonstrated by a numerical example. The organization of this paper is as follows. Section II presents the problem under consideration and some preliminaries. Section III gives design methods of quantized H_∞ filtering strategies. In Section IV, an example is presented to illustrate the effectiveness of the proposed methods. Finally, Section V gives some concluding remarks.

Notation: Given a matrix E, E^T and E^{-1} denote its transpose, and inverse when it exists, respectively. The symbol $*$ within a matrix represents the symmetric entries. For a vector $x \in R^k$, the 2-norm of x is defined as $|x| := (x^T x)^{\frac{1}{2}}$, and for a matrix $Q \in R^{m\times n}$, $\|Q\|, \lambda_{max}(Q)$ and $\lambda_{min}(Q)$ is defined as the largest singular value, the maximum eigenvalue and the minimum eigenvalue of matrix Q, respectively. For an appropriate square matrix Y, denote $He(Y) = Y + Y^T$.

II. PROBLEM STATEMENT AND PRELIMINARIES

A. Quantizer

Firstly, we introduce the quantizer considered in this paper. The definition of a quantizer is given with general form as in [13]. Let

This work was supported in part by the Funds for Creative Research Groups of China (No. 60521003), the State Key Program of National Natural Science of China (Grant No. 60534010), National 973 Program of China (Grant No. 2009CB320604), the Funds of National Science of China (Grant No. 60674021), the 111 Project (B08015) and the Funds of PhD program of MOE, China (Grant No. 20060145019).

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$z \in R^l$ be the variable being quantized. By a *quantizer* we mean a piecewise constant function $q : R^l \rightarrow \mathbb{D}$, where $\mathbb{D}$ is a finite subset of R^l. This leads to a partition of R^l into a finite number of quantization regions of the form $\{z \in R^l : q(z) = i\}, i \in \mathbb{D}$. These quantization regions are not assumed to have any particular shapes. When z does not belong to the union of quantization regions of finite size, the quantizer saturates. More precisely, we assume that there exist positive real numbers M and Δ such that the following two conditions hold:

$$|q(z) - z| \quad \leq \Delta, \quad \text{if} \quad |z| \quad \leq M. \tag{1}$$

$$|q(z) - z| > \Delta, \quad \text{if} \quad |z| > M. \tag{2}$$

Condition (1) gives a bound on the quantization error when the quantizer does not saturate. Condition (2) provides a way to detect the possibility of saturation. M and Δ represent the range and the quantization error bound of the quantizer $q(\cdot)$, respectively. Assume that $q(x) = 0$ for x in some neighborhood of the origin, i.e., the origin lies in the interior of the set $\{x : q(x) = 0\}$. In the filtering strategy to be developed below, we consider the one-parameter family of quantizers

$$q_\mu(z) = \mu q(\frac{z}{\mu}), \quad \mu > 0, \tag{3}$$

where μ is the quantizer's parameter. The range and the error of this quantizer is $M\mu$ and $\Delta\mu$, respectively. μ can be seen as the "zoom" variable: increasing μ corresponds to zooming out and essentially obtaining a new quantizer with larger range and larger quantization error such that any signals can be adequately measured, while decreasing μ corresponds to zooming in and obtaining a quantizer with smaller range but also smaller quantization error such that the signals can be driven to 0.

B. Problem statement

Consider an LTI model described by

$$\begin{aligned} x(k+1) &= Ax(k) + B_1\omega(k), \\ z(k) &= C_1x(k), \\ y(k) &= C_2x(k), \end{aligned} \tag{4}$$

where $x(k) \in R^n$ is the state, $y(k) \in R^p$ is the measured output, $\omega(k) \in R^m$ is the disturbance and $z(k) \in R^q$ is the regulated output, respectively. A, B_1, C_1 and C_2 are known constant matrices of appropriate dimensions.

With the quantizer defined by (1)-(3), the quantized filter is given as

$$\begin{aligned} \xi(k+1) &= A_F\xi + B_F q_{\mu_1}(y) \\ &= A_F\xi + B_F\mu_1 q_1(\tfrac{y}{\mu_1}), \\ z_F(k) &= C_F\xi + D_F q_{\mu_1}(y) \\ &= C_F\xi + D_F\mu_1 q_1(\tfrac{y}{\mu_1}), \end{aligned} \tag{5}$$

where $\xi(k) \in R^n$ is the filter state, $z_F(k)$ is the estimation of $z(k)$, and the constant matrices A_F, B_F, C_F and D_F are filter matrices to be designed. $q_{\mu_1}(\cdot)$ is a dynamic quantizer defined by (3) and is composed of dynamic scaling μ_1 and static quantizer $q_1(\cdot)$ defined by (1) with range M_1 and error Δ_1. The main technical difficulty is that there is no separate communication channel to communicate the gain value. One approach is that both sides of the communication channel compute the same μ_1 independently. This is possible only when the gain μ_1 can be computed using only the quantized signal because this signal is available to both sides of the communication channel.

Applying (5) to (4), the following quantized augmented system is obtained as

$$\begin{aligned} x_e(k+1) &= A_e x_e(k) + B_e\omega(k) + \bar{B}_1\bar{e}, \\ z_e(k) &= C_e x_e(k) + D_e\bar{e}, \end{aligned} \tag{6}$$

where $x_e(k) = [x(k)^T, \xi(k)^T]^T$, $z_e(k) = z(k) - z_F(k)$ is the estimation error, and

$$\begin{aligned} A_e &= \begin{bmatrix} A & 0 \\ B_F C_2 & A_F \end{bmatrix}, B_e = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\ C_e &= \begin{bmatrix} C_1 - D_F C_2 & -C_F \end{bmatrix}, \bar{B}_1 = \begin{pmatrix} 0 \\ B_F \end{pmatrix}, \end{aligned}$$

and $D_e = -D_F$, $\bar{e} = \mu_1 e$ with $e = q_1(\frac{y}{\mu_1}) - \frac{y}{\mu_1}$.

Due to the effect of quantization errors, the problem addressed in this paper is as follows:

Quantized H_∞ Filtering Problem: Find a method to optimize the static quantizer range M_1, and then design a quantized H_∞ filtering strategy with the minimized quantizer range such that the augmented system (6) is asymptotically stable and with a prescribed H_∞ performance bound.

C. Preliminaries

The following lemma presented will be used in this paper.

Lemma 1: Let $\gamma > 0$ be a given constant. Then the following statements are equivalent:

(i) there exists a symmetric positive matrix $X > 0$ such that the following LMI holds

$$He(\Omega_1) < 0; \tag{7}$$

where

$$\Omega_1 = \begin{bmatrix} -\frac{X}{2} & 0 & A_e^T X & C_e^T & 0 \\ 0 & -\frac{\gamma^2 I}{2} & B_e^T X & 0 & B_e^T X \\ 0 & 0 & -\frac{X}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{I}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{X}{2} \end{bmatrix}.$$

(ii) there exist a nonsingular matrix T, and a symmetric matrix $P > 0$ with

$$P = \begin{bmatrix} Y & N \\ N & -N \end{bmatrix}, \tag{8}$$

such that the following LMI holds

$$He(\Omega_2) < 0, \tag{9}$$

where

$$\Omega_2 = \begin{bmatrix} -\frac{P}{2} & 0 & A_{ea}^T P & C_{ea}^T & 0 \\ 0 & -\frac{\gamma^2 I}{2} & B_{ea}^T P & 0 & B_{ea}^T P \\ 0 & 0 & -\frac{P}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{I}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{P}{2} \end{bmatrix},$$

with

$$\begin{aligned} A_{ea} = \begin{bmatrix} A & 0 \\ B_{Fa} C_2 & A_{Fa} \end{bmatrix}&, \quad B_{ea} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\ C_{ea} &= \begin{bmatrix} C_1 - D_{Fa} C_2 & -C_{Fa} \end{bmatrix}, \end{aligned} \tag{10}$$

and

$$A_{Fa} = T^{-1} A_F T, \; B_{Fa} = T^{-1} B_F, \; C_{Fa} = C_F T. \tag{11}$$

Proof: Due to the limit of the space, it is omitted. ■

III. Quantized H_∞ filtering strategy design

In this section, firstly, taking quantizer errors into account, a quantized H_∞ filtering strategy design method is presented in subsection A without the consideration of optimizing static quantizer ranges. Then, in subsection B, a convex optimization method is developed to optimize static quantizer ranges, and a quantized H_∞ filtering strategy is proposed to solve the Quantized H_∞ Filtering Problem.

A. Quantized H_∞ filter design without considering static quantizer ranges

Firstly, taking quantizer errors into account, the following lemma is presented to design filters and matrix variables P, Q, which will be used in the sequel.

Lemma 2: For plant (6), $\gamma > 0$ is a given scalar, assume that there exist matrices F_A, F_B, C_F, D_F and $S > 0,\ N < 0$ such that the following LMI holds

$$He(\Omega_3) < 0, \tag{12}$$

where

$$\Omega_3 = \begin{bmatrix} -\frac{S}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -S & \frac{-S+N}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\gamma^2 I}{2} & 0 & 0 & 0 & 0 & 0 \\ SA & SA & SB_1 & -\frac{S}{2} & 0 & 0 & 0 & 0 \\ \Gamma_1 & \Gamma_3 & (S-N)B_1 & -S & \frac{-S+N}{2} & 0 & 0 & 0 \\ \Gamma_2 & \Gamma_4 & 0 & 0 & 0 & -\frac{I}{2} & 0 & 0 \\ 0 & 0 & SB_1 & 0 & 0 & 0 & -\frac{S}{2} & 0 \\ 0 & 0 & (S-N)B_1 & 0 & 0 & 0 & -S & \frac{-S+N}{2} \end{bmatrix},$$

with

$$\begin{aligned} &\Gamma_1 = (S-N)A + F_B C_2 + F_A, \\ &\Gamma_2 = C_1 - D_F C_2 - C_F, \Gamma_4 = C_1 - D_F C_2, \\ &\Gamma_3 = (S-N)A + F_B C_2. \end{aligned} \tag{13}$$

Denote

$$\begin{aligned} A_F &= N^{-1} F_A,\ B_F = N^{-1} F_B, \\ C_F &= C_F,\ D_F = D_F. \end{aligned} \tag{14}$$

Then there exist two positive definite matrices $P > 0$ and

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} > 0, \tag{15}$$

such that the following LMI holds

$$\begin{bmatrix} \Theta_1 + Q_{11} & A_e^T P B_e + Q_{12} \\ B_e^T P A_e + Q_{12}^T & \Theta_2 + Q_{22} \end{bmatrix} < 0, \tag{16}$$

where $\Theta_1 = A_e^T P A_e - P + C_e^T C_e, \Theta_2 = 2B_e^T P B_e - \gamma^2 I$, and A_e, B_e, C_e are defined in (6).

Proof: Due to the limit of the space, it is omitted. ■

Remark 1: It is well-known that the standard H_∞ filter design for discrete-time systems is reduced to solving the following inequality

$$\begin{bmatrix} \Theta_1 & A_e^T P B_e \\ B_e^T P A_e & -\gamma^2 I + B_e^T P B_e \end{bmatrix} < 0. \tag{17}$$

Compare (16) with (17), except the added matrix variable Q, the block $(2,2)$ in the left of inequality (16) is with an additional term $B_e^T P B_e$, which reflects the effectiveness of the quantization errors.

To facilitate the presentation of Theorem 1, the following algorithm is given first based on Lemma 2.

Algorithm 1:

Step 1. By using Lemma 2, design the filter gains A_F, B_F, C_F, D_F and matrix variables P, Q satisfying (16).

Step 2. Compute the value of $\frac{\eta \|C_2\| \Delta_1}{\lambda_{min}(Q)}$, where $\eta = \phi + \sqrt{\phi^2 + \varphi \lambda_{min}(Q)}$ with $\phi = \|A_e^T P \bar{B}_1 + C_e^T D_e\|$ and $\varphi = \|2\bar{B}_1^T P \bar{B}_1 + D_e^T D_e\|$.

Now, the following result for H_∞ filtering is proposed based on Algorithm 1 without the consideration of optimizing static quantizer ranges, which guarantees that system (6) is global asymptotic stability and with the H_∞ performance attenuation level γ.

Theorem 1: Consider system (4), assume that M_1 is chosen large enough such that

$$M_1 > \frac{\eta \|C_2\| \Delta_1}{\lambda_{min}(Q)}. \tag{18}$$

Then, filtering strategy (5) with the designed filter gains A_F, B_F, C_F, D_F and with the dynamic scaling

$$\mu_1 = \frac{2|y|}{M_1 + \frac{\eta \Delta_1 \|C_2\|}{\lambda_{min}(Q)}}, \tag{19}$$

renders the augmented system (6) asymptotically stable and with the H_∞ disturbance attenuation level γ.

Proof: By using the properties of (1) for the quantizer $q_1(\cdot)$, it is easy to check that whenever $|y| \le M_1 \mu_1$,

$$|\bar{e}| = \mu_1 |q_1(\frac{y}{\mu}) - \frac{y}{\mu_1}| \le \mu_1 \Delta_1. \tag{20}$$

Then consider the Lyapunov function candidate $V(k) = x_e^T(k) P x_e(k)$ for the quantized augmented system (6), and by using (16), the difference of $V(k)$ along solutions of (6) is computed as

$$\begin{aligned} &\Delta V(k) = V(k+1) - V(k) \\ &= (A_e x_e + B_e \omega + \bar{B}_1 \bar{e})^T P (A_e x_e + B_e \omega + \bar{B}_1 \bar{e}) \\ &\quad - x_e^T P x_e \\ &\le \gamma^2 \omega^T \omega - z_e^T z_e + x_e^T (A_e^T P A_e - P + C_e^T C_e) x_e \\ &\quad + 2 x_e^T A_e^T P B_e \omega + 2 x_e^T (A_e^T P \bar{B}_1 + C_e^T D_e) \bar{e} \\ &\quad + 2 \omega^T B_e^T P B_e \omega + \bar{e}^T (2\bar{B}_1^T P \bar{B}_1 + D_e^T D_e) \bar{e} \\ &\quad - \gamma^2 \omega^T \omega \\ &\le -z_e^T z_e + \gamma^2 \omega^T \omega - \begin{bmatrix} x_e \\ \omega \end{bmatrix}^T Q \begin{bmatrix} x_e \\ \omega \end{bmatrix} \\ &\quad + 2|x_e|\phi|\bar{e}| + |\bar{e}|^2 \varphi \\ &\le -z_e^T z_e + \gamma^2 \omega^T \omega - \lambda_{min}(Q)|x_e|^2 \\ &\quad + 2|x_e|\phi|\bar{e}| + |\bar{e}|^2 \varphi \\ &\le -z_e^T z_e + \gamma^2 \omega^T \omega - \lambda_{min}(Q)(|x_e| - \tfrac{\eta |\bar{e}|}{\lambda_{min}(Q)}) \\ &\times (|x_e| - \tfrac{(\phi - \sqrt{\phi^2 + \varphi \lambda_{min}(Q)})|\bar{e}|}{\lambda_{min}(Q)}) \end{aligned} \tag{21}$$

According to (18), there always exists a sufficiently small scalar $\varepsilon \in (0,1)$ such that

$$M_1 > \frac{\eta \|C_2\| \Delta_1}{\lambda_{min}(Q)(1-\varepsilon)}. \tag{22}$$

Combining (22) with (19), we obtain that there always exists a sufficiently small scalar $\varepsilon \in (0,1)$ such that the following holds

$$\begin{aligned} &\frac{\eta \|C_2\| \Delta_1}{\lambda_{min}(Q)(1-\varepsilon)} \mu_1 \le |y| \\ &\quad = (M_1 + \frac{\eta \|C_2\| \Delta_1}{\lambda_{min}(Q)}) \mu_1 / 2 \le M_1 \mu_1. \end{aligned} \tag{23}$$

In other words, if we always choose μ_1 satisfying (23), then (21) holds and thus

$$\Delta V(k) \le -z_e^T z_e + \gamma^2 \omega^T \omega - \varepsilon \frac{\lambda_{min}(Q)}{\|C_2\|} |x_e||y|. \tag{24}$$

By setting $\omega = 0$, obviously, $\Delta V(k) < 0$, i.e., the system is asymptotically stable.

In addition, for any $k > 0$, we can obtain

$$V(k) - V(0) \leq -\sum_{k=0}^{n}(z_e^T(k)z_e(k) - \gamma^2\omega^T(k)\omega(k)). \tag{25}$$

Using $V(k) \geq 0$, $\Delta V(k) < 0$ and zero initial condition, the following is obtained

$$\|z_e(k)\|_2^2 < \gamma^2\|\omega(k)\|_2^2, \tag{26}$$

which implies that the H_∞ disturbance attention level γ is achieved. ■

Remark 2: Theorem 1 offers a quantized H_∞ filtering strategy for discrete-time systems such that the required H_∞ performance can be guaranteed. A similar result for continuous-time state feedback systems has been obtained in [24]. The difference between them is that in [24], the controller gain used in the quantized control strategy is firstly designed by the standard state-feedback H_∞ controller design, whereas in this paper, the filter gains used in the quantized filtering strategy of Theorem 1 is firstly designed by Lemma 2 with the consideration of the effect of the quantization errors. In fact, the additional term $B_e^TPB_e$ in the block $(2,2)$ of inequality (16) compared to (17), which has been mentioned in Remark 1, is resulted from the term $2\omega^TB_e^TP\bar{B}_1e$ in (21) by using $2\omega^TB_e^TP\bar{B}_1e \leq \omega^TB_e^TPB_e\omega + e^T\bar{B}_1^TP\bar{B}_1e$, which renders the effect of the quantization errors involved in the proposed design conditions.

Remark 3: In Theorem 1, the static quantizer range M_1 for the existence condition of the quantized H_∞ filtering strategy is given based on Algorithm 1. But the static quantizer range achieved by this method may be very large and does not accord with practical communication channel requirements. In the following subsection, a method will be developed to minimize the static quantizer ranges.

B. Quantized H_∞ filter design with considering static quantizer ranges

In this subsection, we will develop a convex optimization method to optimize M_1, and further, give a quantized H_∞ filtering strategy to solve the Quantized H_∞ Filtering Problem.

According to (18), we can minimize M_1 by minimizing the value of $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)}$. For the fact that $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)}$ is complicated because it depends on design parameters $\|C_e^TD_e + A_e^TP\bar{B}_1\|, \lambda_{min}(Q)$ and $\|2\bar{B}_1^TP\bar{B}_1 + D_e^TD_e\|$. In the sequel, we aim to optimize these parameters, and consequently minimize M_1 indirectly.

Let $\beta_1 > 0, \beta_2 > 0, \alpha > 0, \delta > 0$ and $\varepsilon > 0$ be scalars and

$$\|A_e^TP^{\frac{1}{2}}\| < \beta_1, \tag{27}$$

$$\|P^{\frac{1}{2}}\bar{B}_1\| < \beta_2, \tag{28}$$

$$\|C_e^T\| < \alpha, \tag{29}$$

$$\|D_e\| < \delta, \tag{30}$$

$$\|2\bar{B}_1^TP\bar{B}_1 + D_e^TD_e\| < \varepsilon. \tag{31}$$

Then, according to (27)-(31), we can optimize the values of $\|A_e^TP^{\frac{1}{2}}\|, \|P^{\frac{1}{2}}\bar{B}_1\|, \|C_e^T\|, \|D_e\|$ and $\|2\bar{B}_1^TP\bar{B}_1 + D_e^TD_e\|$ by optimizing scalars $\beta_1, \beta_2, \alpha, \delta$ and ε, respectively. Obviously, inequalities (27)-(31) are, respectively, equivalent to

$$\begin{bmatrix} -\beta_1^2I & A_e^TP \\ PA_e & -P \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} -\beta_2^2I & \bar{B}_1^TP \\ P\bar{B}_1 & -P \end{bmatrix} < 0, \tag{33}$$

$$\begin{bmatrix} -\alpha^2I & C_e \\ C_e^T & -I \end{bmatrix} < 0, \tag{34}$$

$$\begin{bmatrix} -\delta^2I & D_e^T \\ D_e & -I \end{bmatrix} < 0, \tag{35}$$

$$\begin{bmatrix} -\varepsilon^2I & \Theta_3 \\ \Theta_3 & -I \end{bmatrix} < 0, \tag{36}$$

where $\Theta_3 = 2\bar{B}_1^TP\bar{B}_1 + D_e^TD_e$, and $C_e, \bar{B}_1$ and D_e are defined in (6).

Now, in order to solve the Quantized H_∞ Filtering Problem, we need to solve inequalities (32)- (36) combined with (16). However, inequalities (16), (32), (33) and (36) are not convex and cannot be solved directly. Thus, the following lemma is presented to convert them to convex ones.

Lemma 3: Let $\gamma > 0$ be a given scalar, for scalars $\beta_1 > 0, \beta_2 > 0, \alpha > 0, \delta > 0$ and $\varepsilon > 0$, matrix variables F_A, F_B, C_F, D_F, $S > 0$, $N < 0$ and Q with structure

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^T & Q_{22} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{bmatrix} > 0, \tag{37}$$

the following statements hold:

(i) (32) holds if and only if the following LMI holds

$$He(\Omega_4) < 0 \tag{38}$$

where

$$\Omega_4 = \begin{bmatrix} -\frac{\beta_1^2I}{2} & 0 & 0 & 0 \\ 0 & -\frac{\beta_1^2I}{2} & 0 & 0 \\ SA & 0 & -\frac{S}{2} & 0 \\ (S-N)A + F_BC_2 & F_A & -S & \frac{-S+N}{2} \end{bmatrix}.$$

(ii) (33) holds if and only if the following LMI holds

$$\begin{bmatrix} -\beta_2^2I & 0 & F_B^T \\ 0 & -S & -S \\ F_B & -S & -S+N \end{bmatrix} < 0. \tag{39}$$

(iii) (36) holds if the following LMI holds

$$He(\Omega_5) < 0 \tag{40}$$

where

$$\Omega_5 = \begin{bmatrix} -\frac{\varepsilon^2I}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{I}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -S & 0 & 0 & 0 & 0 & 0 \\ 2F_B & 0 & -2S & -S+N & 0 & 0 & 0 & 0 \\ -D_F & 0 & 0 & 0 & -\frac{I}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -S & 0 & 0 \\ 0 & 2F_B & 0 & 0 & 0 & -2S & -S+N & 0 \\ 0 & -D_F & 0 & 0 & 0 & 0 & 0 & -\frac{I}{2} \end{bmatrix}.$$

(iv) (16) holds if and only if the following LMI holds

$$He(\Omega_6) < 0 \tag{41}$$

where

$$\Omega_6 = \begin{bmatrix} \frac{\Xi_1}{2} & 0 & 0 & 0 \\ \Xi_2^T & \frac{\Xi_3}{2} & 0 & 0 \\ Q_{13}^T + Q_{23}^T & Q_{13}^T & \frac{-\gamma^2 I + Q_{33}}{2} & 0 \\ SA & SA & SB_1 & -\frac{S}{2} \\ \Gamma_1^T & \Gamma_3^T & (S-N)B_1 & -S \\ \Gamma_2^T & \Gamma_4^T & 0 & 0 \\ 0 & 0 & SB_1 & 0 \\ 0 & 0 & (S-N)B_1 & 0 \end{bmatrix.$$
$$\left. \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-S+N}{2} & 0 & 0 & 0 \\ 0 & -\frac{I}{2} & 0 & 0 \\ 0 & 0 & -\frac{S}{2} & 0 \\ 0 & 0 & -S & \frac{-S+N}{2} \end{matrix} \right],$$

with $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are defined by (13) and

$$\Xi_1 = -S + Q_{11} + Q_{12} + Q_{12}^T + Q_{22}$$
$$\Xi_2 = -S + Q_{11} + Q_{12},$$
$$\Xi_3 = -S + N + Q_{11}.$$

Proof: Due to the limit of the space, it is omitted. ∎

Let $\rho = c_0\beta_1^2 + c_1\beta_2^2 + c_2\alpha^2 + c_3\delta^2 + c_4\varepsilon^2$, where c_0, c_1, c_2, c_3 and c_4 are constants to be chosen. Then, based on Lemma 3, by optimizing ρ, the following algorithm is presented to give a convex optimization method to design the filter gain matrices and matrix variables P, Q with the consideration of system performance and quantizer ranges at the same time.
Denote

$$Q(\xi) = \begin{bmatrix} Q_{11} - \xi & Q_{12} & Q_{13} \\ * & Q_{22} - \xi & Q_{23} \\ * & * & Q_{33} - \xi \end{bmatrix} > 0, \quad (42)$$

Algorithm 2:
Step 1. Solving the following optimization problem

$$\min \quad \rho$$
$$F_A, F_B, C_F, D_F, S, N, \beta_1, \beta_2, \alpha, \delta, \varepsilon,$$
$$\xi, Q_{11}, Q_{12}, Q_{13}, Q_{22}, Q_{23}, Q_{33}$$

subject to (34), (35), (38)-(42), output the optimal solutions as $N = N_{opt}$, $S = S_{opt}$, $F_A = F_{Aopt}$, $F_B = F_{Bopt}$, $C_F = C_{Fopt}$, $D_F = D_{Fopt}$, Q_{opt}.

Step 2. Compute

$$\begin{aligned} A_{F_{opt}} &= N_{opt}^{-1} F_{Aopt},\ B_{F_{opt}} = N_{opt}^{-1} F_{Bopt}, \\ C_{F_{opt}} &= C_{Fopt}, D_{Fopt} = D_{Fopt}, \\ P_{opt} &= \begin{bmatrix} S_{opt} - N_{opt} & N_{opt} \\ N_{opt} & -N_{opt} \end{bmatrix}, \bar{B}_{1opt} = \begin{bmatrix} 0 \\ B_{Fopt} \end{bmatrix}. \end{aligned} \quad (43)$$

The resulting $A_{Fopt}, B_{Fopt}, C_{Fopt}$ and D_{Fopt} will form the optimized filter gains, and P_{opt}, Q_{opt} are optimized matrices.
Step 3. Compute the value of $\frac{\eta_{opt}\|C_2\|\Delta_1}{\lambda_{min}(Q_{opt})}$, where $\eta_{opt} = \phi_{opt} + \sqrt{\phi_{opt}^2 + \varphi_{opt}\lambda_{min}(Q_{opt})}$ with $\phi_{opt} = \|A_{eopt}P_{opt}^{\frac{1}{2}}\|\|P_{opt}^{\frac{1}{2}}\bar{B}_{1opt}\| + \|C_{eopt}^T\|\|D_{eopt}\|$ and $\varphi_{opt} = \|2\bar{B}_{1opt}^T P \bar{B}_{1opt} + D_{eopt}^T D_{eopt}\|$.

Remark 4: Because $\lambda_{min}(Q)$ has a significant effect on the value of $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)}$, condition (42) is introduced to restrict the value of $\lambda_{min}(Q)$, such that $\lambda_{min}(Q) \geq \xi$.

Remark 5: Algorithm 2 presents a convex optimization method to design the filter gains A_F, B_F, C_F, D_F and matrix variables P, Q with the consideration of optimizing the value of $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)}$ (realized by conditions (34), (35), (38), (39), (40), (42)) as well as guaranteeing the H_∞ performance (realized by condition (41)).
Now, based on Algorithm 2, the following corollary similar to Theorem 1 can be obtained:

Corollary 1: Consider system (4), assume that M_{1min} is chosen large enough such that

$$M_{1min} > \frac{\eta_{opt}\|C_2\|\Delta_1}{\lambda_{min}(Q_{opt})}. \quad (44)$$

Then, filtering strategy (5) with the designed filter gains $A_{Fopt}, B_{Fopt}, C_{Fopt}, D_{Fopt}$ and with the dynamic scaling

$$\mu_1 = \frac{2|y|}{M_{1min} + \frac{\eta_{opt}\Delta_1\|C_2\|}{\lambda_{min}(Q_{opt})}}, \quad (45)$$

solves the Quantized H_∞ Filtering Problem.

IV. Example

To illustrate the effectiveness of the proposed optimized H_∞ filtering strategy, using MATLAB via the LMI Control Toolbox [11], an example is given to provide a comparison between our design method with the consideration of optimizing M_1 and the design method without the consideration of optimizing M_1.

Example 1: Consider a linearized model of an F-404 engine from [1], which is also studied in [7] and [23]. Using the zero-order hold equivalent method, with a sample period T = 0.1s, a discrete-time model of the system is obtained as

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.8673 & 0 & 0.2022 \\ 0.0293 & 0.9763 & -0.0301 \\ 0.0259 & 0 & 0.8032 \end{bmatrix} x(k) \\ &+ \begin{bmatrix} 0.1 & 0 \\ 0.5 & 0 \\ -0.2 & 0 \end{bmatrix} \omega(k), \\ z(k) &= \begin{bmatrix} 0 & 0 & 4 \end{bmatrix} x(k), \\ y(k) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k). \end{aligned} \quad (46)$$

A. Quantized filter design by Algorithm 1

In this part, we design a quantized H_∞ filtering strategy based on Algorithm 1 without the consideration of optimizing M_1.
By using Lemma 2, we design a filter with the scalar $\gamma_{ini} = 3.3175$.
By Algorithm 1 with the designed filter parameters, and $\xi = 0.001, \Delta_1 = 0.01$ for $\gamma = 3.9875$ and $\gamma = 3.8875$, respectively, we obtain $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)} = 2799.2$ and $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)} = 2796.2$. For these two cases, by Theorem 1, let $M_1 = 2799.3 > 2799.2$ and $M_1 = 2796.3 > 2796.2$, respectively, filtering strategy (5) guarantees the H_∞ filtering objective.

B. Quantized filter design by Algorithm 2

In this subsection, we design a quantized H_∞ filtering strategy based on Algorithm 2 with the consideration of optimizing M_1.
Let $c_0 = 1, c_1 = 0.1, c_2 = 1, c_3 = 0.001, c_4 = 0.1$. For one case, let $\gamma = 3.9875$, by Algorithm 2 with $\xi = 0.001$ and $\Delta_1 = 0.01$, we obtain the filter parameters and matrices P_{opt}, Q_{opt}.
It is easy to compute $\frac{\eta_{opt}\|C_2\|\Delta_1}{\lambda_{min}(Q_{opt})} = 91.3768$. By Corollary 1, let $M_{1min} = 91.5 > 91.3768$, then, filtering strategy (5) with the corresponding filter parameters and dynamic scaling (45) solves the Quantized H_∞ Filtering Problem.
For the second case, let $\gamma = 3.8875$, we obtain the filter parameters and matrices P_{opt}, Q_{opt}. It is easy to compute $\frac{\eta_{opt}\|C_2\|\Delta_1}{\lambda_{min}(Q_{opt})} = 135.9763$. For this case, by Corollary 1, let $M_{1min} = 136.1 > 135.9763$, then filtering strategy (5) with the corresponding filter parameters and dynamic scaling (45) solves the Quantized H_∞ Filtering Problem.

C. Comparison

Table 1 is given to compare the quantizer ranges obtained based on Algorithm 1 and the optimization method given in Algorithm 2.

	Algorithm 1	Algorithm 2
$M_1(\gamma = 3.9875)$	2796.3	91.5
$M_1(\gamma = 3.8875)$	2799.3	136.1

TABLE I

COMPARISON OF THE QUANTIZER RANGES

From this table, we can see that compared with the quantizer ranges obtained based on Algorithm 1, the optimized quantizer ranges obtained based on Algorithm 2 are much more improved. This phenomenon shows the effectiveness of our optimization method. On the other hand, for the example, we see that the tighter the H_∞ performance bound to γ_{ini} is, the larger quantizer range is needed.

D. Simulation

Given initial condition $x_0 = \begin{bmatrix} 0.5 & -0.5 & 0.5 \end{bmatrix}^T$, $\xi_0 = \begin{bmatrix} -0.5 & 0.5 & 1 \end{bmatrix}^T$, for $\gamma = 3.9875$, let the disturbance $\omega(k)$ be

$$\omega(k) = \begin{cases} 2sin(k), & 50 \le k \le 60, \\ 0, & \text{otherwise.} \end{cases}$$

Then Figure 1 shows the estimation error responses of system (6) with the quantized H_∞ filtering strategies designed with the consideration of optimizing M_1 (solid curve) and without the consideration of optimizing M_1 (deashed curve), respectively.

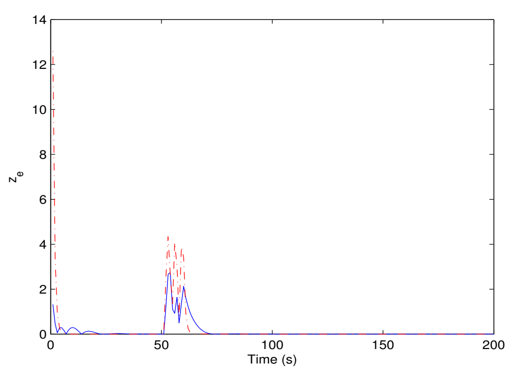

Fig. 1. Trajectories of estimation errors.

This example shows the superiority and the necessity of our design method with the consideration of optimizing static quantizer ranges.

V. CONCLUSION

The quantized H_∞ filtering problem of discrete-time LTI systems has been investigated. In particular, the static quantizer ranges are fully considered for their practical importance. By taking quantizer errors into account, a quantized H_∞ filter design strategy is proposed, where a convex optimization method is developed to minimize static quantizer ranges. The resulting design guarantees that the quantized augmented system is asymptotically stable and with a prescribed H_∞ performance bound. A numerical example has been presented to illustrated the effectiveness of the proposed H_∞ filtering strategy.

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