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Consensus in Networked Multi-Agent Systems via Sampled Control: Switching Topology Case

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Abstract— **In this paper, consensus problems of continuous-time networked multi-agent systems via sampled control are investigated. The sampled control protocols are induced from continuous-time linear consensus protocol by using periodic sampling technology and zero-order hold circuit. A sufficient condition for reaching average-consensus under undirected networks with sampling delay and switching topology is obtained. Some numerical simulations are presented to illustrate the utility of our theoretical results.**

I. INTRODUCTION

Consensus problem in networked multi-agent systems has been attracted increasing attention in recent years. It is a comprehensive interdisciplinary subject including control theory, mathematics, biology, physics, computer science, robotics, artificial intelligence and so on. The applications of multi-agent systems are extensive, ranging from multiple space-craft alignment, heading direction in flocking behavior, average in distributed computation and rendezvous of multiple vehicles. Based on certain quantities of interest, consensus problems of multi-agent systems have been studied by many researchers(see [1],[2] and the references therein).

In the field of system and control, the development of consensus theory was mainly impelled by the particles swarm model mentioned in [3]. Vicsek et al. proposed a discrete model of finite autonomous agents all moving in the plane with same speed but with different headings. Every agent's heading was updated using a local rule based on the average of its own heading plus the heading of its neighbors. Moreover, the concept of neighbors of agents was introduced. Some simulation results which demonstrated the nearest neighbor rule were obtained.

In [4], Jadbabaie et al. provided a theoretical explanation of the consensus behavior of the Vicsek's model and derived convergence results for several similarly inspired models. They proved that Vicsek's model was still valid under switching topology, but for which there did not exist a common quadratic Lyapunov function.

A systematical framework of consensus problem in networks of dynamic agents with fixed/switching topology and communication time-delays was established in [5] by Olfati-Saber and Murray. In their paper, directed networks with fixed topology, directed networks with switching topology, and undirected networks with communication time-delays and fixed topology were considered under the assumption that the dynamics of each agent was a simple scalar continuous-time integrator $\dot{x} = u$. Moreover, a disagreement function was introduced for disagreement dynamics of a directed network with switching topology.

Following [4] and [5], in [6], Ren and Beard investigated more comprehensive discrete-time and continuous-time consensus scheme which included Jadbabaie's result as a special case. Compared with the results in [4] and [5], they provided a milder condition that guaranteed consensus achieving for all the agents.

In [7], a simple but compelling model of network of agents interacting via time-dependent communication links was studied. The analysis in this paper was integrated within a formal framework of set-valued Lyapunov theory. They also showed that more communication did not necessarily lead to faster convergence and may eventually led to a loss of convergence, even for the simple models.

In [8], two problems were considered: the state agreement problem for coupled nonlinear differential equations and the rendezvous problem for kinematic point-mass mobile robots. Their theory based on vector field and non-smooth analysis. Under the assumption that the vector fields satisfied a certain sub-tangentiality condition, they proved that asymptotic state agreement was achieved if and only if the dynamic interaction digraph had the property of being sufficiently connected over time.

In the past few years, consensus problems of multi-agent systems have been developed vary fast and several research topics have been addressed.

Because communication links among agents might be unreliable due to interaction among agents or external disturbances, the information-exchange topologies are often dynamic. Meanwhile, owing to long distance or the confine of medium, communication delays are ubiquitous. Therefore, consensus problems with switching topologies and time-varying delays have received general attention [9], [10], [11], [12], [13].

In [14], [15], [16], [17], consensus problems of multi-agent systems with higher order dynamics were considered. Part or all of the agents updated their states according to second-order or high-order dynamics.

Asynchronous consensus problem were considered in [18], [19], [20]. The asynchronism may led to negative affection such as induced delays and time-varying topologies, so it was more difficult to analyze such systems.

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There are also some results investigating stochastic consensus problems [21], [22], [23], [24], where the communicating channel between agents was stochastic. Finite-time consensus problem is another interesting topic [25], [26]. Compared with asymptotic consensus, the finite-time consensus systems have faster convergence rate, higher control accuracy and better disturbance rejection. Some other interesting results can be seen in [27], [28], [29], [30] and so on.

With the development of digital sensors and controllers, in many cases, though the system itself is a continuous process, the synthesis of control law can only use the data sampled at the discrete sampling instants. Compare to continuous-time system with continuous-time controller or direct discrete-time system, continuous-time system via sampled control has many advantages. On the one hand, the digital controller which is designed based on the sampled controller has obvious advantages in control accuracy , control speed, performance and price, and has better generality. On the other hand, in engineering applications, continuous signals will require broad bandwidth of networks, and in most cases, will not be available in practice. Therefore, sampled control for continuous-time system is more coincident with applications in our real life. Sampled control is applied extensively nowadays. Robots, vehicles, airplanes, satellites, and almost all of modern artificial products are controlled by digital controller where continuous signals are transferred into discrete ones.

For consensus problems of continuous-time multi-agent systems via sampled control, there are only a few relevant results. In [31] and [32], formation control of multi-agent systems with intermittent information exchange between the agents was considered. They both derived stability conditions under a predetermined sampling period. In [33], sampled-data based average-consensus control for networks consisting of continuous-time first-order integrator agents under a noisy distributed communication environment was considered. They proved that when the sampling size was sufficiently small, the static mean square error between the individual state and the average initial states of all nodes was arbitrarily small. However, in the real applications, we always want to know how large the sampling period would be chosen to guarantee the system run well. This requires us to find an upper bound of sampling period. Moreover, sampling delay can not be ignored and sometimes may play a key role in the stability analysis of the whole network. Therefore, we will also consider the case when sampling delay exists and is less than a sampling period.

The main contribution of this work is that sampled control is introduced into consensus problem of multi-agent systems. Consensus problems with sampled data and sampling delay are considered. We only consider undirected networks with sampling delay and switching topology. The consensus protocol for networks with sampling delay are introduced. We will establish conditions in which case all the agents can achieve consensus. Finally, numerical examples are given to illustrate the utility of our results.

An outline of the rest of this paper is as follows. In Section II, we review graph theory and the consensus problems on networks. Section III introduces the sampled-data based control protocol for networks. Section IV presents the main results. The simulation results are presented in Section V. Finally, Section VI concludes the whole paper.

In this paper, notation $\mathbf{1}_M$ is the column vector $[1,\cdots,1]^T$ with M-dimension. Notation $diag\{a_1,\cdots,a_M\}$ represents the diagonal matrix $\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_M \end{bmatrix}$.

II. Brief Review of Graph and Consensus Problem in Networks

In this section, we introduce algebraic graph theory and consensus problems. Let $\mathscr{G}=(\mathscr{V},\mathscr{E},\mathscr{A})$ be an undirected graph with the set of vertices $\mathscr{V}=\{1,2,\cdots,M\}$ and the set of edges $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}$, and a weighted adjacency matrix $\mathscr{A}=[a_{ij}]$ with nonnegative adjacency elements a_{ij}. An edge of $\mathscr{G}$ is denoted by $e_{ij}=(j,i)$. The adjacency elements associated with the edges are positive, i.e., $e_{ij}\in\mathscr{E}\iff a_{ij}>0$. Moreover, we assume $a_{ii}=0$ for all $i\in\mathscr{V}$. The set of *neighbors* of node i is denoted by $\mathscr{N}_i=\{j\in\mathscr{V}:(j,i)\in\mathscr{E}\}$. Since the graph considered is undirected, it means once e_{ij} is an edge of $\mathscr{G}$, e_{ji} is an edge of $\mathscr{G}$ as well. As a result, the adjacency matrix $\mathscr{A}$ is a symmetric nonnegative matrix.

A *cluster* is any subset $J\in\mathscr{V}$ of the nodes of the graph. The set of neighbors of a cluster N_J is defined by

$$N_J=\bigcup_{i\in J}\mathscr{N}_i=\{j\in\mathscr{V}:i\in J,(j,i)\in\mathscr{E}\}$$

The degree of node i is the number of its neighbors $\mathscr{N}_i$ and is denoted by $\deg(i)$. The *degree* of node i is given by

$$\deg(i)=\sum_{j=1}^{N}a_{ij} \tag{1}$$

The *degree matrix* is defined as $\Delta=\text{diag}\{\deg(1),\deg(2),\cdots,\deg(M)\}$. The *Laplacian* of graph $\mathscr{G}$ is defined by

$$L=\Delta-\mathscr{A} \tag{2}$$

An important fact of L is that all the row sums of L are zero and thus $\mathbf{1}_M=[1,1,\cdots,1]^T\in\mathbb{R}^M$ is an eigenvector of L associated with the zero eigenvalue.

A *path* between each distinct vertices i and j is meant a finite ordered sequence of distinct edges of $\mathscr{G}$ of the form $(i,k_1),(k_1,k_2),\cdots,(k_l,j)$. A graph is called *connected* if there exist a path between any two distinct vertices of the graph.

Lemma 1: [34] Graph $\mathscr{G}$ is connected if and only if $\text{rank}(L)=M-1$.

By Lemma 1, for a connected graph, there is only one zero eigenvalue of L, all the other ones are positive and real.

Given a graph $\mathscr{G}$, denote $\Lambda^+(L)$ as the set of nonzero eigenvalues of the Laplacian L of $\mathscr{G}$.

Given a graph $\mathscr{G}$, let $x_i\in\mathbb{R}$ denote the state of node i. We refer to $(\mathscr{G},x)$ with $x=[x_1,x_2,\cdots,x_M]^T\in\mathbb{R}^M$ as a *network* with state x and communication topology $\mathscr{G}$. Suppose each node of a graph is a dynamic agent with dynamics

$$\dot{x}_i(t) = u_i(t) \tag{3}$$

where x_i is aforementioned state of node i and u_i is the control input that will be used for consensus problem.

Let $\chi : \mathbb{R}^M \to \mathbb{R}$ be a function of the state of the network $x(t)$. The *χ-consensus problem* in a network $(\mathscr{G}, x)$ is a distributed way to calculated $\chi(x_0)$ by applying inputs u_i that only depend on the states of itself and its neighbors. We say a feedback

$$u_i(t) = k_i(x_{j_1}(t), x_{j_2}(t), \cdots, x_{j_{L_i}}(t)) \tag{4}$$

is a *control protocol* with topology $\mathscr{G}$ if the cluster $\{j_1, \cdots, j_{L_i}\} = \{i\} \bigcup \mathscr{N}_i$, $i = 1, \cdots, M$.

We say protocol (4) asymptotically solves the χ-consensus problem if and only if there exists an asymptotically stable equilibrium x^* of the network satisfying $x_i^* = \chi(x(0)), i = 1, \cdots, M$. Whenever the nodes of a network are all in consensus, the common value of all nodes is called the *network decision value*. A special case with $\chi(x) = \mathrm{Ave}(x) = 1/M(\sum_{i=1}^M x_i)$ is called *average-consensus* problem.

III. SAMPLED-DATA CONTROL PROTOCOL AND INDUCED NETWORK DYNAMICS

In this section, we investigate distributed solutions of the consensus problem via sampled-data linear control. In [5] the following continuous-time linear consensus protocol was introduced:

$$u_i(t) = \sum_{j \in \mathscr{N}_i} a_{ij}(x_j(t) - x_i(t)), i = 1, \cdots, M. \tag{5}$$

Here a sampled-data control protocol is induced from (5) by using period sampling technology and zero-order hold circuit. Let $h > 0$ be the sampling period, the obtained protocol is given as:

$$\begin{array}{l} u_i(t) = \sum_{j \in \mathscr{N}_i} a_{ij}(x_j(kh) - x_i(kh)), \\ \text{if } t \in [kh, kh+h), k = 0, 1, 2, \cdots; i = 1, \cdots, M. \end{array} \tag{6}$$

By using the protocol (6), the network dynamics is summarized as follows:

$$x(kh+h) = \Phi x(kh), k = 0, 1, 2, \cdots. \tag{7}$$

where

$$\Phi = I - hL \tag{8}$$

with L the aforementioned Laplacian associate with the graph $\mathscr{G}$.

If sampling induced time delay is concerned, the situation becomes complicated. We assume that the sampling delay τ is fixed and less than the sampling period, i.e., $0 < \tau < h$. In this situation, the protocol becomes

$$u_i(t) = \begin{cases} \sum\limits_{j \in \mathscr{N}_i} a_{ij}(x_j(kh-h) - x_i(kh-h)), \text{if } t \in [kh, kh+\tau) \\ \sum\limits_{j \in \mathscr{N}_i} a_{ij}(x_j(kh) - x_i(kh)), \text{if } t \in [kh+\tau, kh+h) \end{cases}$$
$$k = 0, 1, 2, \cdots; i = 1, \cdots, M. \tag{9}$$

Then the network dynamics is given as follows:

$$\begin{bmatrix} x(kh+h) \\ x(kh) \end{bmatrix} = \Psi \begin{bmatrix} x(kh) \\ x(kh-h) \end{bmatrix}, k = 0, 1, 2, \cdots. \tag{10}$$

where

$$\Psi = \begin{bmatrix} I - (h-\tau)L, & -\tau L \\ I, & 0 \end{bmatrix}. \tag{11}$$

IV. CONVERGENCE ANALYSIS

In this section, we provide the convergence analysis of the average-consensus problem for networks with switching topology. First, we give the set of admissible graphs which will be used. We assume that the graph $\mathscr{G}$ belongs to a collection of undirected and connected graphs given by

$$\mathscr{G}_c = \{\mathscr{G} | rank(L_{\mathscr{G}}) = M-1, \mathbf{1}_M^T L_{\mathscr{G}} = 0\} \tag{12}$$

with

$$\lambda_{max} = \max_{\mathscr{G} \in \mathscr{G}_c} \max_{\lambda(L_{\mathscr{G}}) \in \Lambda^+(L_{\mathscr{G}})} \lambda(L_{\mathscr{G}}) < +\infty \tag{13}$$

In this situation, the protocol becomes

$$u_i(t) = \begin{cases} \sum\limits_{j \in \mathscr{N}_i(t)} a_{ij}(t)(x_j(kh-h) - x_i(kh-h)), \text{if } t \in [kh, kh+\tau) \\ \sum\limits_{j \in \mathscr{N}_i(t)} a_{ij}(t)(x_j(kh) - x_i(kh)), \text{if } t \in [kh+\tau, kh+h) \end{cases}$$
$$k = 0, 1, 2, \cdots; i = 1, \cdots, M. \tag{14}$$

where $\mathscr{A}(t) = [a_{ij}(t)]$ is the adjacency matrix of the graph $\mathscr{G}(t)$ at time t.

Then the network dynamics is summarized as follows:

$$\begin{bmatrix} x(kh+h) \\ x(kh) \end{bmatrix} = \Psi_{\mathscr{G}(t)} \begin{bmatrix} x(kh) \\ x(kh-h) \end{bmatrix}, k = 0, 1, 2, \cdots. \tag{15}$$

where

$$\Psi_{\mathscr{G}(t)} = \begin{bmatrix} I - (h-\tau)L_{\mathscr{G}(t)}, & -\tau L_{\mathscr{G}(t)} \\ I, & 0 \end{bmatrix}. \tag{16}$$

with $L_{\mathscr{G}(t)}$ the aforementioned Laplacian associate with the graph $\mathscr{G}(t)$ at instant t.

We write $[x^T(kh), x^T(kh-h)]$ as

$$[x^T(kh), x^T(kh-h)] = \mathbf{1}_{2M}^T Ave(x(0)) + [\delta^T(kh), \delta^T(kh-h)] \tag{17}$$

where δ is called the *disagreement vector*. It is easy to verify that δ satisfies the following *disagreement dynamics*

$$\zeta(kh+h) = \Psi_{\mathscr{G}(t)} \zeta(kh) \tag{18}$$

where $\zeta(kh) = [\delta^T(kh), \delta^T(kh-h)]^T$.

Before giving the main theorem, we present the following two lemmas.

Lemma 2: Given a set of Schur stable matrix

$$\mathscr{A}_S = \{A | A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}, -\frac{3}{2}b < a < 1, -\frac{2}{3} < b \leq 0\} \tag{19}$$

and a positive definite matrix

$$P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} > 0, \tag{20}$$

then we have

$$\begin{aligned} Q &= P - A^T P A \\ &= \begin{bmatrix} 1+2a-2a^2 & b-1-2ab \\ b-1-2ab & 1-2b^2 \end{bmatrix} > 0, \forall A \in \mathscr{A}_S. \end{aligned}$$

Proof: The matrix $Q > 0$ if and only if $1-2b^2 > 0$ and $\det(Q) > 0$.

On the one hand, noticing $1-2b^2 > 0 \iff -\frac{1}{\sqrt{2}} < b < \frac{1}{\sqrt{2}}$, since $-\frac{1}{\sqrt{2}} < -\frac{2}{3}$, we get $1-2b^2 > 0$ holds, for all $b \in (-\frac{2}{3}, 0]$.

On the other hand, we have

$$\begin{aligned} \det(Q) &= -2a^2 - 4ab - 3b^2 + 2a + 2b \\ &= -2(a+b-1/2)^2 - b^2 + 1/2 \end{aligned}$$

Let

$$f(a,b) = -2(a+b-1/2)^2 - b^2 + 1/2 \tag{21}$$

Then the curve $f(a,b) = 0$ is an ellipsoid. Moreover, we have(See Fig. 1)

$$\{(a,b)| -\frac{3}{2}b < a < 1, -\frac{2}{3} < b \leq 0\} \subset \{(a,b)| f(a,b) > 0\} \tag{22}$$

Thus, $\det(Q) > 0$ holds, for all (a,b) satisfying $-\frac{3}{2}b < a < 1, -\frac{2}{3} < b \leq 0$. ■

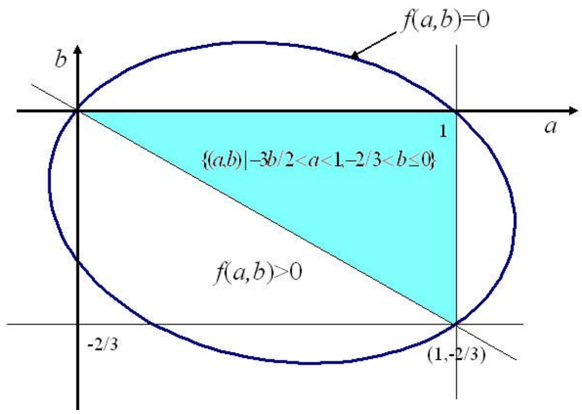

Fig. 1: The illustration for Lemma 2.

Lemma 3: Given a connected graph $\mathscr{G}$, we have

$$\begin{aligned} &\{\zeta \in \mathbb{R}^{2M} | (I_2 \otimes \mathbf{1}_M^T)\zeta = 0\} \\ &= ((I_2 \otimes W_{\mathscr{G}})E_P)\{\eta \in \mathbb{R}^{2M} | ([1,0,\cdots,0] \otimes I_2)\eta = 0\} \end{aligned} \tag{23}$$

where $W_{\mathscr{G}}$ is an orthogonal matrix such that $W_{\mathscr{G}}^{-1} L_{\mathscr{G}} W_{\mathscr{G}} = diag\{0, \lambda_2(L_{\mathscr{G}}), \cdots, \lambda_M(L_{\mathscr{G}})\}$, and E_P is a permutation matrix given by

$$E_P = [e_1, e_{M+1}, e_2, e_{M+2}, \ldots, e_M, e_{2M}]^T \in \mathbb{R}^{2M \times 2M} \tag{24}$$

where e_i is the column vector with 1 in its i'th row and zeros elsewhere, $i = 1, \cdots, 2M$. It is obvious that E_P is an orthogonal matrix as well, i.e., $E_P^T E_P = I_{2M}$.

Proof: For a connected graph $\mathscr{G}$, since $L_{\mathscr{G}}$ is a real and symmetric matrix, there exists an orthogonal matrix $W_{\mathscr{G}}$ such that

$$W_{\mathscr{G}}^{-1} L_{\mathscr{G}} W_{\mathscr{G}} = diag\{0, \lambda_2(L_{\mathscr{G}}), \cdots, \lambda_M(L_{\mathscr{G}})\}$$

It follows that

$$(I_2 \otimes W_{\mathscr{G}}^{-1})\Psi_{\mathscr{G}}(I_2 \otimes W_{\mathscr{G}}) = \begin{bmatrix} \Psi_1 & \Psi_2 \\ I & 0 \end{bmatrix}$$

where

$$\Psi_1 = diag\{1, 1-(h-\tau)\lambda_2, \cdots, 1-(h-\tau)\lambda_M\}$$

and

$$\Psi_2 = diag\{0, -\tau\lambda_2, \cdots, -\tau\lambda_M\}$$

Moreover, we have

$$\begin{aligned} &E_P^T (I_2 \otimes W_{\mathscr{G}}^{-1})\Psi_{\mathscr{G}}(I_2 \otimes W_{\mathscr{G}})E_P \\ &= diag\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1-(h-\tau)\lambda_2 & -\tau\lambda_2 \\ 1 & 0 \end{bmatrix}, \cdots, \\ &\begin{bmatrix} 1-(h-\tau)\lambda_M & -\tau\lambda_M \\ 1 & 0 \end{bmatrix}\} \end{aligned}$$

For undirected connected graph $\mathscr{G}$, we also have

$$L_{\mathscr{G}} \mathbf{1}_M = 0$$

It follows that

$$W_{\mathscr{G}} diag\{0, \lambda_2(L_{\mathscr{G}}), \cdots, \lambda_M(L_{\mathscr{G}})\} W_{\mathscr{G}}^{-1} \mathbf{1}_M = 0$$

Since $W_{\mathscr{G}}$ is orthogonal, we have

$$W_{\mathscr{G}}^{-1} \mathbf{1}_M = c[1, 0, \cdots, 0]^T \in \mathbb{R}^M$$

where c is a nonzero constant scalar. Then we have

$$\begin{aligned} &(I_2 \otimes W_{\mathscr{G}}^{-1})(I_2 \otimes \mathbf{1}_M) = c(I_2 \otimes [1, 0, \cdots, 0]^T) \\ &= c \begin{bmatrix} 1, 0, \cdots, 0, 0, 0, \cdots, 0 \\ 0, 0, \cdots, 0, 1, 0, \cdots, 0 \end{bmatrix}^T \in \mathbb{R}^{2M \times 2} \end{aligned}$$

Notice that

$$E_P^T (I_2 \otimes [1, 0, \cdots, 0]^T) = [1, 0, \cdots, 0]^T \otimes I_2$$

It follows that

$$\begin{aligned} &E_P^T (I_2 \otimes W_{\mathscr{G}}^{-1})(I_2 \otimes \mathbf{1}_M) = c[1, 0, \cdots, 0]^T \otimes I_2 \\ &= c \begin{bmatrix} 1, 0, \cdots, 0, 0, 0, \cdots, 0 \\ 0, 1, \cdots, 0, 0, 0, \cdots, 0 \end{bmatrix}^T \in \mathbb{R}^{2M \times 2} \end{aligned}$$

This implies that

$$E_P^T (I_2 \otimes W_{\mathscr{G}}^{-1}) span\{I_2 \otimes \mathbf{1}_M\} = span\{[1, 0, \cdots, 0]^T \otimes I_2\}$$

We can rewrite it as

$$span\{I_2 \otimes \mathbf{1}_M\} = (I_2 \otimes W_{\mathscr{G}})E_P \ span\{[1, 0, \cdots, 0]^T \otimes I_2\}$$

Since $(I_2 \otimes W_{\mathscr{G}})E_P$ is orthogonal as well, it follows that

$$span\{I_2 \otimes \mathbf{1}_M\}_{\perp} = (I_2 \otimes W_{\mathscr{G}})E_P \ span\{[1, 0, \cdots, 0]^T \otimes I_2\}_{\perp}$$

This means that (23) holds. ■

Theorem 1: : For any switching signal $\mathscr{G}(t) : \mathbb{R}^+ \longrightarrow \mathscr{G}_c$, if the sampling delay and sampling period satisfy

$$0 \leq \tau < \frac{2}{3\lambda_{max}} \tag{25}$$

and

$$\tau < h < \frac{1}{\lambda_{max}} - \frac{\tau}{2}. \tag{26}$$

then the switching protocol (14) globally asymptotically solves the average-consensus problem. Moreover, the following quadratic positive-definite function

$$V(\zeta) = \zeta^T (P \otimes I_M)\zeta \tag{27}$$

is a common Lyapunov function for the disagreement dynamics (18), where P is given by (20).

Proof: Denote

$$R_{\mathscr{G}(t)} = (E_P^T (I_2 \otimes W_{\mathscr{G}(t)}^{-1}))\Psi_{\mathscr{G}(t)}((I_2 \otimes W_{\mathscr{G}(t)})E_P) \tag{28}$$

where $W_{\mathscr{G}(t)}$ is the orthogonal matrix such

$$W_{\mathscr{G}(t)}^{-1} L_{\mathscr{G}(t)} W_{\mathscr{G}(t)} = diag\{0, \lambda_2(L_{\mathscr{G}(t)}), \cdots, \lambda_M(L_{\mathscr{G}(t)})\}$$

Then we have

$$\begin{aligned} R_{\mathscr{G}(t)} = \ & diag\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \\ & \begin{bmatrix} 1-(h-\tau)\lambda_2(L_{\mathscr{G}(t)}) & -\tau\lambda_2(L_{\mathscr{G}(t)}) \\ 1 & 0 \end{bmatrix}, \\ & \cdots, \begin{bmatrix} 1-(h-\tau)\lambda_M(L_{\mathscr{G}(t)}) & -\tau\lambda_M(L_{\mathscr{G}(t)}) \\ 1 & 0 \end{bmatrix}\} \end{aligned} \tag{29}$$

Set $a = 1 - h\lambda_m(L_{\mathscr{G}(t)}) + \tau\lambda_m(L_{\mathscr{G}(t)}), b = -\tau\lambda_m(L_{\mathscr{G}(t)}), m = 2, \cdots, M$. By simple verification, from (25) and (26), we get that $-\frac{3}{2}b < a < 1$ and $-\frac{2}{3} < b \leq 0$ hold. By Lemma 2, we have

$$\begin{aligned} P - & \begin{bmatrix} 1-(h-\tau)\lambda_m(L_{\mathscr{G}(t)}) & -\tau\lambda_m(L_{\mathscr{G}(t)}) \\ 1 & 0 \end{bmatrix}^T \times \\ \times P & \begin{bmatrix} 1-(h-\tau)\lambda_m(L_{\mathscr{G}(t)}) & -\tau\lambda_m(L_{\mathscr{G}(t)}) \\ 1 & 0 \end{bmatrix} > 0 \end{aligned} \tag{30}$$

holds for any $m = 2, \cdots, M$ and any $t \geq 0$.

It follows that

$$\eta^T (I_M \otimes P - R_{\mathscr{G}(t)}^T (I_M \otimes P) R_{\mathscr{G}(t)})\eta > 0 \tag{31}$$

for any nonzero $\eta \in \mathbb{R}^{2M}$ satisfying $([1, 0, \cdots, 0] \otimes I_2)\eta = 0$.

It is easy to verify that

$$E_P (I_M \otimes P) E_P^T = P \otimes I_M, \tag{32}$$

then we have

$$\begin{aligned} & ((I_2 \otimes W_{\mathscr{G}(t)})E_P\eta)^T (P \otimes I_M - \Psi_{\mathscr{G}(t)}^T (P \otimes I_M)\Psi_{\mathscr{G}(t)}) \times \\ & \times ((I_2 \otimes W_{\mathscr{G}(t)})E_P\eta) > 0 \end{aligned} \tag{33}$$

for any nonzero $\eta \in \mathbb{R}^{2M}$ satisfying $([1, 0, \cdots, 0] \otimes I_2)\eta = 0$. By Lemma 3, we have

$$\zeta^T (P \otimes I_M - \Psi_{\mathscr{G}(t)}^T (P \otimes I_M)\Psi_{\mathscr{G}(t)})\zeta > 0 \tag{34}$$

for any nonzero ζ satisfying $(I_2 \otimes \mathbf{1}_M^T)\zeta = 0$. This implies that for any switching signal, we always have

$$\begin{aligned} & V(\zeta(kh+h)) - V(\zeta(kh)) \\ = \ & \zeta^T (kh+h)(P \otimes I_M)\zeta(kh+h) - \zeta^T (kh)(P \otimes I_M)\zeta(kh) \\ = \ & \zeta^T (kh)\Psi_{\mathscr{G}(t)}^T (P \otimes I_M)\Psi_{\mathscr{G}(t)}\zeta(kh) \\ & -\zeta^T (kh)(P \otimes I_M)\zeta(kh) \\ = \ & \zeta^T (kh)(\Psi_{\mathscr{G}(t)}^T (P \otimes I_M)\Psi_{\mathscr{G}(t)} - P \otimes I_M)\zeta(kh) < 0 \end{aligned}$$

Therefore, $V(\zeta)$ is really a common Lyapunov function for the disagreement dynamics (18). This completes the proof. ■

V. SIMULATIONS

In this section, numerical simulations will be given to illustrate the theoretical results obtained in the previous sections. All graphs in our simulations have $0-1$ weights.

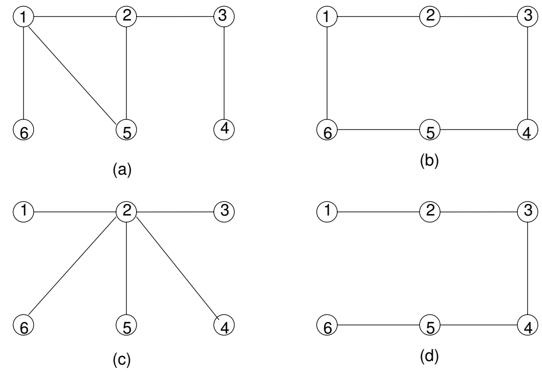

Fig. 2: The graphs in Example 1.

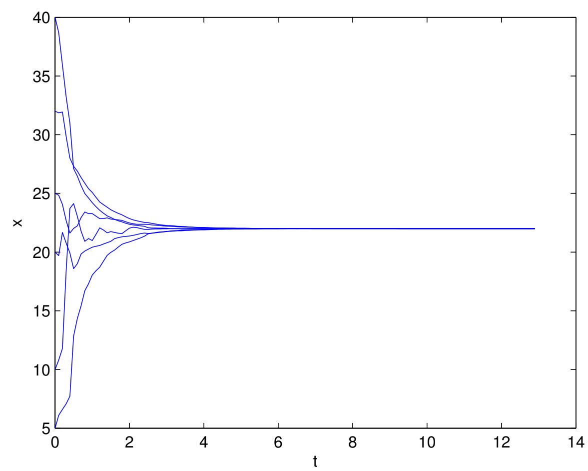

Fig. 3: Simulation result when $\tau = 0.01, h = 0.08$

Example 1 (Network with switching topology): Consider a network with 4 undirected connected graphs shown in Fig.2. By calculation, we obtain $\lambda_{max} = 6$. From (25) and (26), it is easy to see that the upper bound of sampling delay τ and sampling period h is $1/9$. Fig.3 and Fig.4 are the simulation results with different sampling delay and sampling period. We can see that the switching protocol (14) globally asymptotically solves the average-consensus problem indeed.

VI. CONCLUSION

In this paper, convergence analysis of consensus control for networks of multi-agent systems via sampled control has been investigated. Our analysis relies on several tools from algebraic graph theory, matrix theory and stability theory. For undirected networks with switching topology, we have presented a sufficient condition for reaching average-consensus. The result for fixed topology case is contained in

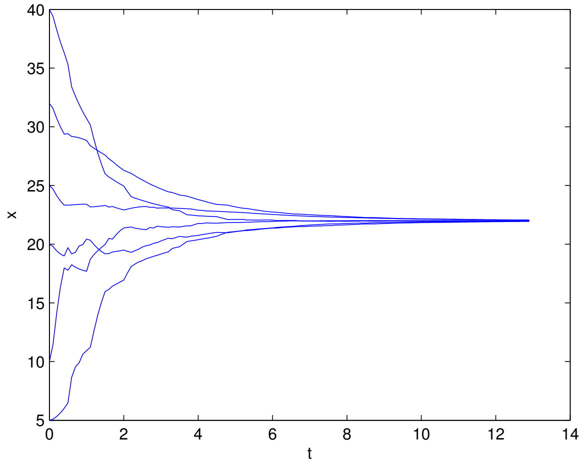

Fig. 4: Simulation result when $\tau = 0.04, h = 0.02$

[36]. The future work include large sampling delay case and time-varying sampling delay case.

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