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Central Suboptimal H_∞ Filter Design for Nonlinear Polynomial Systems

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***Abstract*— This paper presents the central finite-dimensional H_∞ filter for nonlinear polynomial systems, that is suboptimal for a given threshold γ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. In contrast to the previously obtained results, the paper reduces the original H_∞ filtering problem to the corresponding optimal H_2 filtering problem, using the technique proposed in [1]. The paper designs the central suboptimal H_∞ filter for the general case of nonlinear polynomial systems, based on the optimal H_2 filter given in [24]. The central suboptimal H_∞ filter is also derived in a closed finite-dimensional form for third (and less) degree polynomial system states. Numerical simulations are conducted to verify performance of the designed central suboptimal filter for nonlinear polynomial systems against the central suboptimal H_∞ filter available for the corresponding linearized system.**

I. Introduction

Over the past two decades, the considerable attention has been paid to the H_∞ estimation problems for linear and nonlinear systems. The seminal papers in H_∞ control [1] and estimation [2], [3], [4] established a background for consistent treatment of filtering/controller problems in the H_∞-framework. The H_∞ filter design implies that the resulting closed-loop filtering system is robustly stable and achieves a prescribed level of attenuation from the disturbance input to the output estimation error in L_2/l_2-norm. A large number of results on this subject has been reported for systems in the general situation, linear or nonlinear (see ([5]–[14]). The sufficient conditions for existence of an H_∞ filter, where the filter gain matrices satisfy Riccati equations, were obtained for linear systems in [4] and linear systems with state delay in [15] or with measurement delay in [16]. However, the criteria of existence and suboptimality of solution for the central H_∞ filtering problems based on the reduction of the original H_∞ problem to the induced H_2 one, similar to those obtained in [1], [4] for linear systems, remain yet unknown for nonlinear polynomial systems.

Although the general optimal solution of the filtering problem for nonlinear state and observation equations confused with white Gaussian noises is given by the equation for the conditional density of an unobserved state with respect to observations [17], there are a very few known examples of nonlinear systems where that equation can be reduced to a finite-dimensional closed system of filtering equations for a certain number of lower conditional moments (see [18]–[20] for more details). The complete classification of the "general situation" cases (this means that there are no special assumptions on the structure of state and observation equations and the initial conditions), where the optimal nonlinear finite-dimensional filter exists, is given in [21]. Apart from the "general situation," the optimal finite-dimensional filters have been designed for certain classes of polynomial system states over linear observations ([22]–[25]).

This paper presents the central (see [1] for definition) finite-dimensional H_∞ filter for nonlinear polynomial systems, that is suboptimal for a given threshold γ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. In contrast to the results previously obtained for linear systems [4], [15], [16], the paper reduces the original H_∞ filtering problem to the corresponding H_2 filtering problem, using the technique proposed in [1]. To the best authors' knowledge, this is the first paper which applies the reduction technique of [1] to certain classes of nonlinear systems. Indeed, application of the reduction technique makes sense, since the optimal filtering equations solving the H_2 filtering problems have been obtained for certain classes of nonlinear polynomial systems [22], [23], [24]. Designing the central suboptimal H_∞ filter for nonlinear polynomial systems presents a significant advantage in the filtering theory and practice, since (1) it enables one to address filtering problems for non-autonomous nonlinear polynomial systems, where the LMI technique is hardly applicable and the HJB equation-based methods fail to provide a closed-form solution, (2) the obtained H_∞ filter is suboptimal, that is, optimal for any fixed γ with respect to the H_∞ noise attenuation criterion, and (3) the obtained H_∞ filter is finite-dimensional and has the same structure of the estimate and gain matrix equations as the corresponding optimal H_2 filter.

It should be commented that the proposed design of the central suboptimal H_∞ filters for nonlinear polynomial systems with integral-quadratically bounded disturbances naturally carries over from the design of the optimal H_2 filters for nonlinear polynomial systems with unbounded disturbances (white noises). The entire design approach creates a complete filtering algorithm of handling the nonlinear polynomial systems with unbounded or integral-quadratically bounded disturbances optimally for all thresholds γ uniformly or for any fixed γ separately. A similar algorithm for linear systems was developed in [1].

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The paper presents the central suboptimal H_∞ filter for the general case of nonlinear polynomial systems, based on the optimal H_2 filter given in [24]. The central suboptimal H_∞ filter is also derived in a closed finite-dimensional form for third (and less) degree polynomial system states. In doing so, the standard H_∞ filtering conditions of stabilizability, detectability, and noise orthonormality (see [4]) are assumed. Finally, to relax the standard conditions, the paper presents the generalized version of the designed H_∞ filter in the absence of the noise orthonormality, using the technique of handling non-orthonormal noises carried over from [16].

Numerical simulations are conducted to verify performance of the designed central suboptimal filter for nonlinear polynomial system against the central suboptimal H_∞ filter available for the corresponding linearized system. The simulation results show a definite advantage in the values of the noise-output transfer function H_∞ norm in favor of the designed filter.

The paper is organized as follows. Section 2 presents the H_∞ filtering problem statement for nonlinear polynomial systems. The central suboptimal H_∞ filter for nonlinear polynomial systems is designed in Section 3. An example verifying performance of the H_∞ filter designed in Section 3 against the central suboptimal H_∞ filter available for the corresponding linearized system is given in Section 4. The obtained results are generalized to the case of non-orthonormal noises in Section 5. Section 6 presents conclusions to this study.

II. H_∞ Filtering Problem Statement for Polynomial Systems

Consider the following continuous-time polynomial system:

$$\begin{aligned} \mathscr{S}_1:\ \dot{x}(t) &= f(x,t)+B(t)\omega(t), &(1)\\ y(t) &= C(t)x(t)+D(t)\omega(t), &(2)\\ z(t) &= L(t)x(t), &(3)\\ x(t_0) &= x_0, &(4)\end{aligned}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $z(t)\in\mathbb{R}^q$ is the signal to be estimated, $y(t)\in\mathbb{R}^m$ is the measured output, $\omega(t)\in\mathscr{L}_2^p[0,\infty)$ is the disturbance input. $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, and $L(\cdot)$ are known continuous functions.

The nonlinear function $f(x,t)\in R^n$ is considered a polynomial of n variables, components of the state vector $x(t)\in R^n$, with time-dependent coefficients. Since $x(t)\in R^n$ is a vector, this requires a special definition of the polynomial for $n>1$. In accordance with [24], a p-degree polynomial of a vector $x(t)\in R^n$ is regarded as a p-linear form of n components of $x(t)$

$$f(x,t)=\alpha_0(t)+\alpha_1(t)x+\alpha_2(t)xx^T+\ldots+\alpha_p(t)x\ldots_{p\ times}\ldots x,$$

where $\alpha_0(t)$ is a vector of dimension n, α_1 is a matrix of dimension $n\times n$, α_2 is a 3D tensor of dimension $n\times n\times n$, α_p is an $(p+1)$D tensor of dimension $n\times\ldots_{(p+1)\ times}\ldots\times n$, and $x\times\ldots_{p\ times}\ldots\times x$ is a pD tensor of dimension $n\times\ldots_{p\ times}\ldots\times n$ obtained by p times spatial multiplication of the vector $x(t)$ by itself (see [24] for more definition). Such a polynomial can also be expressed in the summation form

$$f_k(x,t)=\alpha_{0\ k}(t)+\sum_i\alpha_{1\ ki}(t)x_i+\sum_{ij}\alpha_{2\ kij}(t)x_ix_j+\ldots$$
$$+\sum_{i_1\ldots i_p}\alpha_{p\ ki_1\ldots i_p}(t)x_{i_1}\ldots x_{i_p},\quad k,i,j,i_1,\ldots,i_p=1,\ldots,n.$$

For the system (1)–(4), the following standard conditions (see [4] for linear systems) are assumed:

- the state $x(t)$ governed by (1) is uniformly stabilizable; $(\mathscr{C}_1)$
- the state $x(t)$ governed by (1) is uniformly detectable through the observations $y(t)$ satisfying (2); $(\mathscr{C}_2)$
- $D(t)B^T(t)=0$ and $D(t)D^T(t)=I_m$. $(\mathscr{C}_3)$

Here, I_m is the identity matrix of dimension $m\times m$. The definitions of uniform stabilizability and detectability for nonlinear systems can be found in [26]. As usual, the first two conditions ensure that the estimation error, provided by the designed H_∞ filter, converge to zero ([27]). The last noise orthonormality condition is technical and corresponds to the condition of independence of the standard Wiener processes (Gaussian white noises) in stochastic filtering problems [28].

Now, consider a full-order $\mathscr{H}_\infty$ filter in the following form $(\mathscr{S}_2)$:

$$\begin{aligned} \mathscr{S}_2:\ \dot{x}_f(t) &= f(x_f,t)+K_f(t)[y(t)-C(t)x_f(t)], &(5)\\ z_f(t) &= L(t)x_f(t), &(6)\end{aligned}$$

where $x_f(t)$ is the filter state. The gain matrix $K_f(t)$ is to be determined.

Upon transforming the model (1)-(3) to include the states of the filter, the following filtering error system is obtained $(\mathscr{S}_3)$:

$$\begin{aligned} \mathscr{S}_3:\ \dot{e}(t) &= \bar{f}(x,x_f,t)+B(t)\omega(t)-K_f(t)\tilde{y}(t), &(7)\\ \tilde{y}(t) &= C(t)e(t)+D(t)\omega(t), &(8)\\ \tilde{z}(t) &= L(t)e(t), &(9)\end{aligned}$$

where $e(t)=x(t)-x_f(t)$, $\tilde{y}(t)=y(t)-C(t)x_f(t)$, $\tilde{z}(t)=z(t)-z_f(t)$, and $\bar{f}(x,x_f,t)=f(x,t)-f(x_f,t)$. Note that since the function $f(x,t)$ is polynomial, the equality $\bar{f}(x,x_f,t)=0$ holds, if $x=x_f$. In other words, $\bar{f}(z,z,t)=0$.

Therefore, the problem to be addressed is as follows: develop a robust $\mathscr{H}_\infty$ filter of the form (5)-(6) for the polynomial system $(\mathscr{S}_1)$, such that the following two requirements are satisfied

1) The resulting filtering error dynamics $(\mathscr{S}_3)$ is robustly asymptotically stable in the absence of disturbances, $\omega(t)\equiv 0$;
2) The filtering error dynamics $(\mathscr{S}_3)$ ensures a noise attenuation level γ in an $\mathscr{H}_\infty$ sense. More specifically, for all nonzero $\omega(t)\in\mathscr{L}_2^p[0,\infty)$, the inequality

$$\|\tilde{z}(t)\|_2^2<\gamma^2\left\{\|\omega(t)\|_2^2+x^T(t_0)Rx(t_0)\right\}\qquad(10)$$

holds, where $\|f(t)\|_2^2:=\int_{t_0}^{T_1}f^T(t)f(t)dt$, T_1 is the rightmost point of a time interval where the solution of (7) exists and is bounded, R is a positive definite symmetric matrix, and γ is a given real positive scalar.

Remark 1. Hereinafter, the formulated $\mathcal{H}_\infty$ filtering problem is considered in a time interval $[t_0, T_1]$, where the solution of the error equation (7) still exists and is bounded. Note that although the solution of the nonlinear state equation (1) may diverge to infinity as T_1 approaches an escape time T^* for the system (1), the solution of the error equation (7) may remain bounded in any interval $[t_0, T_1]$, where $T_1 < T^*$. Thus, the $\mathcal{H}_\infty$ filtering problem still makes sense for the nonlinear state (1) in $[t_0, T_1]$, $T_1 < T^*$.

III. Design of Central H_∞ Filter for Polynomial Systems

The proposed design of the central H_∞ filter (see Theorem 4 in [1]) for polynomial systems is based on the general result (see Theorem 3 in [1]) reducing the H_∞ controller problem to the corresponding optimal H_2 controller problem. In this paper, only the filtering part of this result, valid for the entire controller problem, is used. Then, the optimal filter of the Kalman-Bucy type for polynomial systems ([24]) is employed to obtain the desired result, which is given by the following theorem.

Theorem 1. *The central H_∞ filter for the unobserved state (1) over the observations (2), ensuring the H_∞ noise attenuation condition (10) for the output estimate $z_f(t)$, is given by the equations for the state estimate $x_f(t)$ and the output estimate $z_f(t)$*

$$\dot{x}_f(t) = \hat{f}(x_f,t) + P(t)C^T(t)[y(t) - C(t)x_f(t)], \quad (11)$$

$$z_f(t) = L(t)x_f(t), \quad (12)$$

with the initial condition $x_f(t_0) = 0$, and the equation for the filter gain matrix $P(t)$

$$dP(t) = \hat{g}(x_f,t) + B(t)B^T(t) - \quad (13)$$
$$P(t)[C^T(t)C(t) - \gamma^{-2}L^T(t)L(t)]P(t))dt,$$

with the initial condition $P(t_0) = R^{-1}$. The structure of the functions $\hat{f}(x,t)$ and $\hat{g}(x,t)$ depend on the degree of the polynomial $f(x,t)$ in (1). The functions $\hat{f}(x,t)$ and $\hat{g}(x,t)$ corresponding to lower degrees of $f(x,t)$ can be obtained from the functions $\hat{f}(x,t)$ and $\hat{g}(x,t)$ corresponding to higher degrees of $f(x,t)$ upon setting to zero the coefficients corresponding to the excessive superior degrees. In particular, the functions $\hat{f}(x,t)$ and $\hat{g}(x,t)$ corresponding to the third degree polynomial

$$f_3(x,t) = a_0(t) + a_1(t)x + a_2(t)xx^T + a_3(t)xxx^T,$$

where x is an n-dimensional vector, $a_0(t)$ is an n-dimensional vector, $a_1(t)$ is a $n \times n$-dimensional matrix, $a_2(t)$ is a 3D tensor of dimension $n \times n \times n$, $a_3(t)$ is a 4D tensor of dimension $n \times n \times n \times n$, are given by

$$\hat{f}_3(x_f,t) = a_0(t) + a_1(t)x_f + a_2(t)x_f x_f^T + a_2(t)P(t) + \quad (14)$$
$$3a_3(t)x_f P(t) + a_3(t)x_f x_f x_f^T,$$

$$\hat{g}_3(x_f,t) = a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)x_f P(t) + \quad (15)$$
$$2(a_2(t)x_f P(t))^T + 3(a_3(t)[P(t)P(t) + x_f x_f^T P(t)]) +$$
$$3(a_3(t)[P(t)P(t) + x_f x_f^T P(t)])^T.$$

The functions $\hat{f}(x_f,t)$ and $\hat{g}(x_f,t)$ corresponding to the second or first degree polynomials $f(x,t)$ are obtained setting to zero a_3 or both, a_3 and a_2, respectively.

Proof. First of all, note that the filtering error system (7)-(9) already has the structure used in Theorem 3 from [1], where a linear term is replaced by a polynomial function $\bar{f}(x,x_f,t)$. Hence, according to Theorem 3 from [1], the H_∞ filtering part of this H_∞ controller problem would be equivalent to the H_2 optimal filtering problem, where the worst disturbance $w_{worst}(t) = \gamma^{-2}B^T(t)Q(t)e(t)$ is realized, and $Q(t)$ is the solution of the equation for the corresponding H_2 optimal control gain. Therefore, the system, for which the equivalent H_2 optimal filtering problem is stated, takes the form

$$\mathscr{S}_4: \ \dot{e}(t) = \bar{f}(e(t) + x_f(t), x_f(t), t) \quad (16)$$
$$+ \ \gamma^{-2}B(t)B^T(t)Q(t)e(t) - K_f(t)\tilde{y}(t),$$
$$\tilde{y}(t) = C(t)e(t) + \gamma^{-2}D(t)B^T(t)Q(t)e(t), \quad (17)$$
$$\tilde{z}(t) = L(t)e(t). \quad (18)$$

Note that $\bar{f}(e(t) + x_f(t), x_f(t), t)$ is a polynomial function of the estimation error $e(t) = x(t) - x_f(t)$, and $e(t) = 0$ implies $\bar{f}(0 + x_f(t), x_f(t), t) = 0$.

As follows from Theorem 3 from [1] and Theorem 1 in [24], the H_2 optimal estimate equations for the error states (16) and (18) are given by

$$\mathscr{S}_5: \ \dot{e_f}(t) = \hat{\bar{f}}(e(t) + x_f(t), x_f(t), t) \quad (19)$$
$$- \ K_f(t)\tilde{y}(t) + P(t)C^T(t)[\tilde{y}(t) - C(t)e_f(t)],$$
$$\tilde{z}_f(t) = L(t)e_f(t), \quad (20)$$

where $e_f(t)$ and $\tilde{z}_f(t)$ are the H_2 optimal estimates for $e(t)$ and $\tilde{z}(t)$, respectively. The function $\hat{\bar{f}}(e(t) + x_f(t), x_f(t), t)$ is the H_2 optimal estimate for $\bar{f}(e(t) + x_f(t), x_f(t), t)$, whose specific forms for polynomials $f(x,t)$ of various degrees can be obtained following the procedure established in [24]. In case of a third degree polynomial $f_3(x,t)$ defined in the theorem statement, the function $\hat{\bar{f}}_3(x,y,t)$ is given by

$$\hat{\bar{f}}_3(x,y,t) = a_0(t) + a_1(t)(x-y) + a_2(t)xx^T - a_2(t)yy^T - \quad (21)$$
$$3a_3(t)(x-y)P(t) + a_3(t)xxx^T - a_3(t)yyy^T.$$

In the equations (19),(21), $P(t)$ is the solution of the equation for the corresponding H_2 optimal filter gain, where, according to Theorem 3 from [1], the observation matrix $C(t)$ should be changed to $C(t) - \gamma^{-1}L(t)$.

It should be noted that, in contrast to Theorem 3 from [1], no correction matrix $Z_\infty(t) = [I_n - \gamma^{-2}P(t)Q(t)]^{-1}$ appears in the last innovations term in the right-hand side of the equation (19), since there is no need to make the correction related to estimation of the worst disturbance $w_{worst}(t)$ in the error equation (16). Indeed, as stated in ([4]), the desired estimator must be unbiased, that is, $\tilde{z}_f(t) = 0$. Since the same output error $\tilde{z}(t)$ stands in the criterion (10) and should be minimized as much as possible, the worst disturbance $w_{worst}(t)$ in the error equation (16) should be plainly rejected and, therefore, does not need to be estimated. Thus, the corresponding H_2 optimal filter gain would not include any correction matrix $Z_\infty(t)$. The same situation can be observed in Theorems 1–4 in [4]. However, if not the output error $\tilde{z}(t)$ but the output $z(t)$ itself would stand in the criterion (10), the correction matrix $Z_\infty(t)=[I_n-\gamma^{-2}P(t)Q(t)]^{-1}$ should be included.

Taking into account the unbiasedness of the estimator (19)-(20), it can be readily concluded that the equality $K_f(t)=P(t)C^T(t)$ must hold for the gain matrix $K_f(t)$ in (5). Thus, applying the procedure from [24] to derive specific forms of $\hat{f}(x,t)$ for higher degree polynomials $f(x,t)$ in (1), the filtering equations (5)-(6) take the final form (11)-(12), with the initial condition $x_f(t_0)=0$, which corresponds to the central H_∞ filter (see Theorem 4 in [1]). In case of a third degree polynomial $f_3(x,t)$ defined in the theorem statement, the function $\hat{f}_3(x,t)$ is given by (14). It is still necessary to indicate the equations for the corresponding H_2 optimal filter gain matrix $P(t)$. In accordance with Theorem 1 from [24], the filter gain matrix $P(t)$ is given by the equation (13), with the initial condition $P(t_0)=R^{-1}$, which corresponds to the central H_∞ filter (see Theorems 3 and 4 in [4]). Note that the observation matrix $C(t)$ is changed to $C(t)-\gamma^{-1}L(t)$ according to Theorem 3 from [1]. Specific forms of the function $\hat{g}(x,t)$ in (13) for higher degree polynomials $f(x,t)$ in (1) are derived applying the indicated procedure from [24]. In case of a third degree polynomial $f_3(x,t)$ defined in the theorem statement, the function $\hat{g}_3(x,t)$ is given by (15). Validity of the statement that "the functions $\hat{f}(x,t)$ and $\hat{g}(x,t)$ corresponding to lower degrees of $f(x,t)$ can be obtained from the functions $\hat{f}(x,t)$ and $\hat{g}(x,t)$ corresponding to higher degrees of $f(x,t)$ upon setting to zero the coefficients corresponding to the excessive superior degrees" also follows from Theorem 1 in [24]. ■

Remark 2. The boundedness of the system state $x(t)$ and its estimate $x_f(t)$, as well as the filter gain matrix $P(t)$, is determined by the definiteness of the most superior polynomial term in the right-hand side of (13). If this term is stable, then $x(t)$, $x_f(t)$, and $P(t)$ remain bounded for all $t\in[t_0,\infty)$. The filter gain matrix $P(t)$ also remains positive definite, provided that the initial condition matrix R is positive definite. In the latter case, it makes sense to consider the H_∞ noise-output attenuation problem with a certain level γ in the infinite interval $[t_0,\infty)$. Otherwise, if the most superior polynomial term in (13) is unstable, then $x(t)$, $x_f(t)$, and $P(t)$ diverge to infinity for a finite time and the designed filter does not work properly for all $t\in[t_0,\infty)$. However, even in this case, the designed central suboptimal H_∞ filter for polynomial systems yields the least possible value of the output H_∞ norm in those finite time intervals where the solution of (7) exists and is bounded.

Remark 3. According to the comments in Subsection V.G in [1], the obtained central H_∞ filter (11)–(13) presents a natural choice for H_∞ filter design among all admissible H_∞ filters satisfying the inequality (10) for a given threshold γ, since it does not involve any additional actuator loop (i.e., any additional external state variable) in constructing the filter gain matrix. Moreover, the obtained central H_∞ filter (11)–(13) has the suboptimality property, i.e., it minimizes the criterion

$$J=\|\tilde{z}(t)\|_2^2-\gamma^2\left\{\|\omega(t)\|_2^2+x^T(t_0)Rx(t_0)\right\}$$

for such positive $\gamma>0$ that the inequality $C^T(t)C(t)-\gamma^{-2}L^T(t)L(t)>0$ holds.

Remark 4. Following the discussion in Subsection V.G in [1], note that the complementarity condition always holds for the obtained H_∞ filter (11)–(13), since the positive definiteness of the initial condition matrix R implies the positive definiteness of the filter gain matrix gain $P(t)$ as the solution of (13). Therefore, the stability failure is the only reason why the obtained filter can stop working.

IV. Example: Central H_∞ Filter for Polynomial System

This section presents an example of designing the central H_∞ filter for a third degree polynomial state over linear observations and comparing it to the central H_∞ filter available for the corresponding linearized system, that is the filter obtained in Theorems 3 and 4 from [4].

Let the unmeasured state $x(t)=[x_1(t),x_2(t)]\in R^2$ be given by

$$\dot{x}_1(t)=x_2(t), \tag{22}$$
$$\dot{x}_2(t)=0.1x_2^2(t)+(0.1+0.1\sin(50t))x_2^3(t)+w_1(t),$$

with an unknown initial condition $x(0)=x_0=[x_{10},x_{20}]$, the scalar observation process satisfy the equation

$$y(t)=x_1(t)+w_2(t), \tag{23}$$

and the output $z(t)=[z_1(t),z_2(t)]\in R^2$ be represented as

$$z_1(t)=x_1(t), \tag{24}$$
$$z_2(t)=x_2(t).$$

Here, $w(t)=[w_1(t),w_2(t)]$ is an L_2^2 disturbance input. It can be readily verified that the noise orthonormality condition (see Section 2) holds for the system (22)–(24).

The filtering problem is to find the H_∞ estimate for the linear state with delay (22) over direct linear observations (23), which satisfies the noise attenuation condition (10) for a given γ, using the designed H_∞ filter (11)–(15) for third degree polynomial states. Note that the third degree coefficient in (22) is positive, i.e., the superior polynomial term is unstable (see Remark 1 in Section 3). The filtering horizon is set to $T=1.4$, prior to the escape time for the system state (22).

The filtering equations (11)–(15) take the following particular form for the system (22),(23)

$$\dot{x}_{f_1}(t)=x_{f_2}(t)+P_{11}(t)[y(t)-x_{f_1}(t)], \tag{25}$$
$$\dot{x}_{f_2}(t)=0.1x_{f_2}^2(t)+(0.3+0.3\sin(50t))P_{22}(t)x_{f_2}(t)+$$
$$0.1P_{22}(t)+(0.1+0.1\sin(50t))x_{f_2}^3(t)$$

$$+P_{12}(t)[y(t)-x_{f_1}(t)],$$

with the initial condition $x_f(0)=0$,

$$\dot{P}_{11}(t)=2P_{12}(t)-(1-\gamma^{-2})P_{11}^2(t), \quad (26)$$

$$\dot{P}_{12}(t)=P_{22}(t)+(0.3+0.3\sin(50t))x_{f_2}^2(t)P_{12}(t)$$

$$+0.2xf_2(t)P_{12}(t)+(0.3+0.3\sin(50t))P_{22}(t)P_{12}(t)$$

$$-(1-\gamma^{-2})P_{11}(t)P_{12}(t),$$

$$\dot{P}_{22}(t)=1+(0.6+0.6\sin(50t))x_{f_2}^2(t)P_{22}(t)$$

$$0.4xf_2(t)P_{22}(t)+(0.6+0.6\sin(50t))P_{22}^2(t)-(1-\gamma^{-2})P_{12}^2(t),$$

with the initial condition $P(0)=R^{-1}$.

The estimates obtained upon solving the equations (25),(26) are compared to the central H_∞ linear filter estimates given by Theorems 3 and 4 in [4]. The central H_∞ linear filter, applied to the linearized system (22), yields the following equations:

$$\dot{m}_{K1}(t)=m_{K2}(t)+P_{K11}(t)[y(t)-m_{K1}(t)], \quad (27)$$

$$\dot{m}_{K2}(t)=0.2m_{K2}^2(t)+(0.3+0.3\sin(50t))m_{K2}^3(t)$$

$$+P_{K12}(t)[y(t)-m_{K1}(t)],$$

with the initial condition $m_f(0)=0$,

$$\dot{P}_{K11}(t)=2P_{K12}(t)-(1-\gamma^{-2})P_{K11}^2(t), \quad (28)$$

$$\dot{P}_{K12}(t)=P_{K22}(t)+0.2m_{K2}(t)P_{K12}(t)$$

$$+(0.3+0.3\sin(50t))m_{K2}^2(t)P_{K12}(t)$$

$$-(1-\gamma^{-2})P_{K11}(t)P_{K12}(t),$$

$$\dot{P}_{K22}(t)=1+0.4m_{K2}(t)P_{K22}(t)$$

$$+(0.6+0.6\sin(50t))m_{K2}^2(t)P_{K22}(t)-(1-\gamma^{-2})P_{K12}^2(t),$$

with the initial condition $P(0)=R^{-1}$.

Numerical simulation results are obtained solving the systems of filtering equations (25),(26), and (27),(28). The obtained estimate values are compared to the real values of the state vector $x(t)$ in (22). For each of the filters (25),(26), and (27),(28) and the reference system (22) involved in simulation, the following initial values are assigned: $x_{10}=1$, $x_{20}=1$, $R=I_2=diag[1\ \ 1]$. The L_2 disturbance $w(t)=[w_1(t),w_2(t)]$ is realized as $w_1(t)=1/(1+t)^2$, $w_2(t)=2/(2+t)^2$. The attenuation level value is set to $\gamma=1.05$.

The following graphs are obtained: graphs of the output H_∞ estimation error $z(t)-z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (25),(26); graphs of the output H_∞ estimation error $z(t)-z_f(t)$ corresponding to the conventional estimate $m_K(t)$ satisfying the equations (27),(28) (Fig. 1). The graphs of the output estimation errors are shown in the entire simulation interval from $t_0=0$ to $T=1.4$. Figure 1 also demonstrates the dynamics of the noise-output H_∞ norms corresponding to the shown output H_∞ estimation errors in each case.

The following values of the noise-output H_∞ norm $\|T_{zw}\|^2=\|z(t)-z_f(t)\|_2^2/(\|\omega(t)\|_2^2+x^T(t_0)Rx(t_0))$ are obtained at the final simulation time $T=1.4$: $\|T_{zw}\|=0.952$ for the H_∞ estimation error $z(t)-z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (25),(26); $\|T_{zw}\|=1.079$ for H_∞ estimation error $z(t)-z_f(t)$ corresponding to the conventional estimate $m_K(t)$ satisfying the equations (27),(28).

It can be concluded that the central suboptimal H_∞ filter (25),(26) provides reliably convergent behavior of the output estimation error, yielding very small values of the corresponding H_∞ norm, even in comparison to the assigned threshold value $\gamma=1.05$, and almost zero difference in the output values in the final time. In contrast, the conventional central H_∞ filter (27),(28) provides divergent behavior of the output estimation error, yielding a larger value of the corresponding H_∞ norm, which exceeds the assigned threshold. Thus, the simulation results show definite advantages of the designed central suboptimal H_∞ filter for polynomial systems, in comparison to the previously known conventional H_∞ filter.

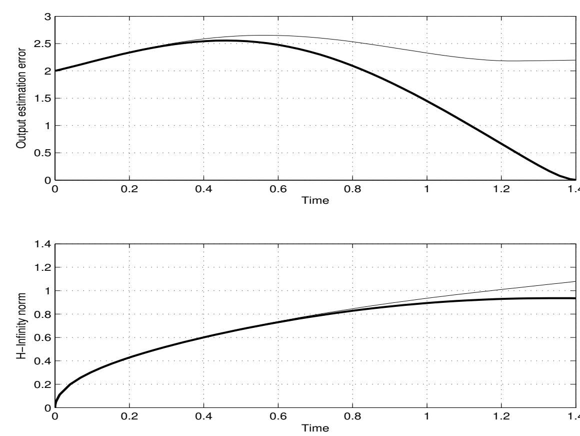

Fig. 1. **Above.** Graphs of the output H_∞ estimation error $\|z(t)-z_f(t)\|$ corresponding to the estimate $x_f(t)$ satisfying the equations (25),(26) (thick line) and the estimate $x_f(t)$ satisfying the equations (27),(28) (thin line), in the simulation interval [0,1.4]. **Below.** Graph of the noise-output H_∞ norm corresponding to the shown output H_∞ estimation errors corresponding to the estimate $x_f(t)$ satisfying the equations (25),(26) (thick line) and the estimate $x_f(t)$ satisfying the equations (27),(28) (thin line), in the simulation interval [0,1.4].

V. GENERALIZATIONS

As shown in [16], the noise orthonormality condition ($\mathscr{C}_3$), third standard condition from Section 2 (see also [1], [4]), can be omitted. This leads to appearance of additional terms in all H_∞ filtering equations. The corresponding generalization of the obtained H_∞ filter is given in the following propositions.

Corollary 1. In the absence of the noise orthonormality condition ($\mathscr{C}_3$), the central H_∞ filter for the unobserved state (1) over the observations (2), ensuring the H_∞ noise attenuation condition (10) for the output estimate $z_f(t)$, is given by the following equations for the state estimate $x_f(t)$ and the output estimate $z_f(t)$

$$\dot{x}_f(t)=\hat{f}(x_f,t)+[P(t)C^T(t)+B(t)D^T(t)]\times \quad (29)$$

$$(D(t)D^T(t))^{-1}[y(t)-C(t)x_f(t)],$$

$$z_f(t)=L(t)x_f(t), \tag{30}$$

with the initial condition $x_f(\theta)=0$ for $\forall\theta\in[t_0-h,t_0]$, and the equation for the filter gain matrix $P(t)$

$$dP(t)=(\hat{g}(x_f,t)+B(t)B^T(t)- \tag{31}$$

$$[P(t)C^T(t)+B(t)D^T(t)](D(t)D^T(t))^{-1}\times$$

$$[D(t)B^T(t)+C(t)P(t)]+\gamma^{-2}P(t)L^T(t)L(t)P(t))dt,$$

with the initial condition $P(t_0)=R^{-1}$. In particular, the functions $\hat{f}(x,t)$ and $\hat{g}(x,t)$ corresponding to the third degree polynomial $f_3(x,t)=a_0(t)+a_1(t)x+a_2(t)xx^T+a_3(t)xxx^T$, where x is an n-dimensional vector, $a_0(t)$ is an n-dimensional vector, $a_1(t)$ is a $n\times n$-dimensional matrix, $a_2(t)$ is a 3D tensor of dimension $n\times n\times n$, $a_3(t)$ is a 4D tensor of dimension $n\times n\times n\times n$, are given by

$$\hat{f}_3(x_f,t)=a_0(t)+a_1(t)x_f+a_2(t)x_fx_f^T+a_2(t)P(t)+ \tag{32}$$

$$3a_3(t)x_fP(t)+a_3(t)x_fx_fx_f^T,$$

$$\hat{g}_3(x_f,t)=a_1(t)P(t)+P(t)a_1^T(t)+2a_2(t)x_fP(t)+ \tag{33}$$

$$2(a_2(t)x_fP(t))^T+3(a_3(t)[P(t)P(t)+x_fx_f^TP(t)])+$$

$$3(a_3(t)[P(t)P(t)+x_fx_f^TP(t)])^T.$$

Proof. The proof is straightforwardly delivered using the technique of handling the H_∞ filtering problems for systems with non-orthonormal noises, which can be found in [16].

Remark 5. Since the H_∞ filter designed in Corollary 1 is based on the corresponding H_2 optimal filter, which is optimal with respect to mean square criteria, Remarks 2–4 remain valid.

VI. Conclusions

This paper designs the central finite-dimensional H_∞ filter for nonlinear polynomial systems, that is suboptimal for a given threshold γ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. The specific form of the central suboptimal H_∞ filter for third (and less) order polynomial systems is derived explicitly. Finally, the generalized version of the H_∞ filter is obtained in the absence of the standard noise orthonormality condition.

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