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Analysis of linear systems using truncated ellipsoids *

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Abstract— **The objective of this paper is to develop numerically simple and effective methods for system analysis using truncated ellipsoids. The paper studies linear time-invariant systems subject to persistent disturbance and state constraint. The maximal output under a set of initial conditions and the overshoot under a given step input are estimated. Attempts are also made to detect an invariant set, as large as possible, within state/output constraint. The results are based on the set invariance condition for the truncated ellipsoid, and a characterization of the set where the output possibly reaches a local extreme.**

Keywords: maximal output, overshoot, state constraint, invariant set, truncated ellipsoid

I. Introduction

We consider the following linear system subject to persistent disturbance:

$$\dot{x} = Ax + Bu + Ew, \quad y = Cx, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is a step input, $w \in \mathbb{R}^q$ is the persistent disturbance bounded by $w^{\mathrm{T}}w \leq 1$ and $y \in \mathbb{R}^p$ is the output. Assume that A is Hurwitz. We examine analysis problems including estimating the maximal output and overshoot, and searching for an invariant set, as large as possible, within a certain state constraint.

Characterizing the maximal output and overshoot under step input and/or persistent disturbance is a traditional problem in control theory. It has been studied under different frameworks, such as the L_1 performance framework (e.g., [6]) and the invariant set framework (e.g., [1], [3], [4], [8]). The L_1 framework exactly characterizes the worst case maximal output under persistent disturbance for linear time-invariant systems. It allows for dynamic uncertainties, but it is not clear how the methods can be used for general parametric, possibly time varying, uncertainties. On the other hand, the invariant set framework usually provides an estimate for the maximal output by using various Lyapunov functions, and the methods can be readily extended to handle systems with general parametric uncertainties and time-varying nonlinearities that can be described with linear differential inclusions.

Two typical types of invariant sets are the invariant ellipsoids [4] and the invariant polytopes[1], [3], [8], corresponding to quadratic Lyapunov functions and polyhedral Lyapunov functions, respectively. The analysis methods resulting from quadratic functions can be conservative but are still widely used due to computational efficiency via LMI technique. The methods based on invariant polytopes may theoretically yield non-conservative results, if the number of vertices is allowed to be arbitrarily large. However, for systems of order greater than 3, the number of vertices quickly grows out of reach for any numerical methods. In recent years, other types of non-quadratic Lyapunov functions have been developed for uncertain systems, constrained control systems and hybrid systems (see e.g.,[5], [7], [9], [11], [12], [13]). The Lyapunov functions in these works pertain to or are composed from several quadratic functions. Thus they lead to optimization problems with matrix inequality constraints, usually a mixture of LMIs and BMIs.

An interesting invariant set is considered in [14] for systems with input and state constraint. The set is formed by cutting off parts of an ellipsoid with several pairs of hyperplanes, representing the state and input constraint. The resulting invariant set is called "semi-ellipsoidal set" in [14]. In this paper, we will call it a truncated ellipsoid. In [14], the truncated ellipsoid is used as a viability set, or an admissible set in [8]: if the initial condition x_0 starts from the set, the response $x(t)$ will satisfy the input and state constraint for all $t \geq 0$.

The truncated ellipsoid is actually an intersection of an ellipsoid and a polytope. So part of its boundary is from an ellipsoid and the rest from a polytope. In terms of Lyapunov function, the truncated ellipsoid is the level set of a function of the form

$$V(x) = \max\{x^{\mathrm{T}}Px, x^{\mathrm{T}}C_1^{\mathrm{T}}C_1x, \cdots, x^{\mathrm{T}}C_p^{\mathrm{T}}C_px\}. \tag{2}$$

When $P = 0$, $V(x)^{1/2}$ is a polyhedral function. Thus $V(x)$ can be considered as the mix of a quadratic function and a polyhedral function.

In this paper, we will use V similar to that in (2) and invariant truncated ellipsoid to estimate a bound for the maximal output under step input and persistent disturbance, as well as to find an invariant set, as large as possible, within state constraint. Since the function V incorporates the structure of the output and the constraint, the resulting optimization problems involves only a few bilinear terms and the constraint becomes LMIs when one or two variables are fixed. Thus the computational burden is just a little heavier than the corresponding algorithm resulting from applying quadratic functions, but the improvement is significant, as will be demonstrated with examples. Some miscellaneous analysis problems were considered in [16] using truncated ellipsoid. This paper deals with more general problems in a systematic way.

Notation We use co$S$ to denote the convex hull of a set S

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and ∂S to denote the boundary of S. We use L_V to denote the 1-level set of a positive definite function V, $L_V := \{x \in \mathbb{R}^n : V(x) \leq 1\}$. For $P = P^{\mathrm{T}} \geq 0$, $\mathcal{E}(P) = \{x \in \mathbb{R}^n : x^{\mathrm{T}}Px \leq 1\}$.

II. Condition of set invariance for intersection of ellipsoids

Consider the following system:

$$\dot{x} = Ax + Ew, \tag{3}$$

where $x \in \mathbb{R}^n, w \in \mathbb{R}^q$. Assume that A is Hurwitz and $w(t)^{\mathrm{T}}w(t) \leq 1$ for all t. A set S is said to be invariant for this system if every $x(0) \in S$ implies $x(t) \in S$ for all $t \geq 0$ and all possible $w(\cdot)$. Invariant ellipsoids for such a system are characterized via a matrix inequality (in [4]), which becomes an LMI when one scalar variable is fixed. This matrix inequality is used to evaluate several input-state and input-output properties including the maximal output and the overshoot. As acknowledged in [4], the estimation of these quantities can be conservative. In this paper, we would like to use the intersection of several ellipsoids as invariant set to reduce the conservatism.

The intersection of ellipsoids can be described as a level set of the pointwise maximum of a family of quadratic functions:

$$V(x) = \max\{x^{\mathrm{T}}P_jx : j = 1, 2, \cdots, J\}, \tag{4}$$

where $P_j = P_j^{\mathrm{T}} \geq 0$. This function has been used in [7], [9], [11] as a Lyapunov function to analyze robust stability and performance under finite energy disturbances for linear differential inclusions and saturated systems. In [7], [9], [11], V is simply called the max function. Denote the one level set of V as

$$L_V = \{x \in \mathbb{R}^n : x^{\mathrm{T}}P_jx \leq 1, j = 1, \cdots, J\},$$

and denote its boundary as ∂L_V.

For $P = P^{\mathrm{T}} \geq 0$, denote $\mathcal{E}(P) = \{x \in \mathbb{R}^n : x^{\mathrm{T}}Px \leq 1\}$. If $P > 0$, then $\mathcal{E}(P)$ is an ellipsoid; if $P \geq 0$ and the rank of P is one, i.e., $P = C^{\mathrm{T}}C$ for a certain row vector C, then $\mathcal{E}(P) = \mathcal{E}(C^{\mathrm{T}}C)$ is the region between two hyperplanes $Cx = \pm 1$, which is unbounded. We may regard $\mathcal{E}(P)$ for a singular P as a degenerated ellipsoid. To unify all the cases, we may simply call L_V the intersection of ellipsoids, allowing some of the ellipsoids to be degenerated.

For the special case where $P_1 > 0$ and the rest $P_j = C_j^{\mathrm{T}}C_j, j = 2, \cdots, J$, all have rank one, L_V is part of the ellipsoid $\mathcal{E}(P_1)$ after truncated by the planes $C_jx = \pm 1$. For simplicity, we call such an L_V a truncated ellipsoid. It is actually the intersection of an ellipsoid and a polytope.

The set L_V is invariant for (3) if and only if $\dot{x}$ points inward of L_V at each $x \in \partial L_V$, for all possible $w, w^{\mathrm{T}}w \leq 1$. Since V is not everywhere differentiable, we need to use directional derivative to describe this property. A general result about the directional derivative of this type of functions can be found in [11].

For a function $V(x)$, the one sided directional derivative is defined ([15], page 213) with respect to two variables: x and a vector ζ specifying the direction of motion. In particular, the one-sided directional derivative of V, at x along ζ is defined as

$$\dot{V}(x;\zeta) := \lim_{\Delta t \to 0, \Delta t > 0} \frac{V(x + \zeta\Delta t) - V(x)}{\Delta t}.$$

For $x \in \mathbb{R}^n$, let

$$I_{\max}(x) := \{j : x^{\mathrm{T}}P_jx \geq x^{\mathrm{T}}P_kx \ \forall k\}.$$

Then by [11], the directional derivative of V at x along ζ is

$$\dot{V}(x;\zeta) = \max\{2x^{\mathrm{T}}P_j\zeta : j \in I_{\max}(x)\}. \tag{5}$$

With directional derivative, the set L_V is invariant if and only if,

$$\dot{V}(x; Ax + Ew) \leq 0 \quad \forall x \in \partial L_V, w^{\mathrm{T}}w \leq 1.$$

In what follows, we give a condition for the invariance of L_V in terms of some bilinear matrix inequalities.

Proposition 1: The level set L_V is invariant for system (3) if there exist $\lambda_{jk} \geq 0, \beta_j \geq 0$, $j, k = 1, \cdots, J$, such that

$$\begin{bmatrix} M_j & P_jE \\ E^{\mathrm{T}}P_j & -\beta_jI \end{bmatrix} \leq 0, j = 1, \cdots, J, \tag{6}$$

where $M_j = A^{\mathrm{T}}P_j + P_jA - \Sigma_{k=1}^J\lambda_{jk}(P_k - P_j) + \beta_jP$.

When $J = 1$, (6) reduces to one matrix inequality which is the one appears in [4]. It becomes an LMI when β_1 is fixed. For the general case, we need to fix $J \times J$ scalar variables λ_{jk}, β_j, to make the J inequalities LMIs. (Notice that λ_{jk} has no effect when $j = k$.)

In the next two sections, we will use the condition for the invariance of the intersection of ellipsoids to address several performance analysis problems. We will choose some of the P_j's as $P_j = C_j^{\mathrm{T}}C_j$, where C_j is from the output matrix C or a certain state constraint $|Cx|_\infty \leq 1$. By doing so, we incorporate the structure of the output and/or constraint into the Lypunov function V. Moreover, with some algebraic manipulation, we can reduce the number of bilinear terms, so that the matrix inequalities become LMIs when a few variables are fixed. If we define the optimal value of the performance index as the function of these few variables, we can use Matlab function such as "fminbnd" or "fminsearch" to optimize these variables. Thus the problem is reduced to a low dimensional optimization via a certain LMI solvers.

III. Estimation of output bound and overshoot

In this section, we use the truncated ellipsoid and the max function to estimate the bound of output due to a set of initial conditions and a step input, respectively.

A. Output bound under a set of initial conditions

Consider the system (3) with the output $y = Cx$ where C is a row vector. If there are several output channels, we may consider each one separately. Assume that the initial condition belongs to a set $X_0 = \mathrm{co}\{x_1, x_2, \cdots, x_K\}$. Our objective is to estimate an upper bound of $|y(t)|$ for all possible $w(\cdot)$, $w^{\mathrm{T}}(t)w(t) \leq 1$ and $x(0) \in X_0$. This will be achieved by using a truncated ellipsoid, which is the 1-level set of

$$V(x) = \max\{x^{\mathrm{T}}(C^{\mathrm{T}}C/\gamma^2)x, x^{\mathrm{T}}(P/\gamma^2)x\}.$$

If L_V is invariant and contains X_0, then we have $x^{\mathrm{T}}(t)C^{\mathrm{T}}Cx(t) \le \gamma^2$, i.e., $|y(t)| \le \gamma$ for all t. The condition for invariance of L_V follows from Proposition 1 by taking $P_1 = C^{\mathrm{T}}C/\gamma^2, P_2 = P/\gamma^2$: there exist $\lambda_1, \lambda_2 \ge 0, \bar\beta_1, \bar\beta_2 \ge 0$, such that

$$\begin{bmatrix} M_1 & C^{\mathrm{T}}CE \\ E^{\mathrm{T}}C^{\mathrm{T}}C & -\bar\beta_1\gamma^2 \end{bmatrix} \le 0,$$

$$\begin{bmatrix} M_2 & PE \\ E^{\mathrm{T}}P & -\bar\beta_2\gamma^2 \end{bmatrix} \le 0,$$

where

$$M_1 = A^{\mathrm{T}}C^{\mathrm{T}}C + C^{\mathrm{T}}CA - \lambda_1(P - C^{\mathrm{T}}C) + \bar\beta_1 C^{\mathrm{T}}C$$
$$M_2 = A^{\mathrm{T}}P + PA - \lambda_2(C^{\mathrm{T}}C - P) + \bar\beta_2 P.$$

The objective is to minimize γ^2 subject to the above inequalities and that $X_0 \subset L_V$, i.e., $x_i^{\mathrm{T}}Px_i \le \gamma^2, x_i^{\mathrm{T}}C^{\mathrm{T}}Cx_i \le \gamma^2, i = 1, \cdots, K$. It seems that we need to fix 4 variables, $\lambda_1, \lambda_2, \bar\beta_1, \bar\beta_2$ to make all the constraint LMIs. However, with a change of variables, we can turn the problem into a standard "gevp" problem by fixing two variables. Let $\alpha_1 = 1/\lambda_1, \alpha_2 = \lambda_2$, $\beta_1 = \bar\beta_1\gamma^2/\lambda_1$ and $\beta_2 = \bar\beta_2\gamma^2$, the two matrix inequalities can be rearranged as:

$$\begin{bmatrix} \beta_1 C^{\mathrm{T}}C & 0 \\ 0 & 0 \end{bmatrix} \le \gamma^2 \begin{bmatrix} -\alpha_1(A^{\mathrm{T}}C^{\mathrm{T}}C + C^{\mathrm{T}}CA) + P - C^{\mathrm{T}}C & \alpha_1 C^{\mathrm{T}}CE \\ \alpha_1 E^{\mathrm{T}}C^{\mathrm{T}}C & \beta_1 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} \beta_2 P & 0 \\ 0 & 0 \end{bmatrix} \le \gamma^2 \begin{bmatrix} -A^{\mathrm{T}}P - PA + \alpha_2(C^{\mathrm{T}}C - P) & PE \\ E^{\mathrm{T}}P & \beta_2 \end{bmatrix} \tag{8}$$

Note that the two matrices in (7) are linear with respect to all variables and the two matrices in (8) are linear in P for fixed α_2 and β_2. When α_2, β_2 are fixed, the minimal γ can be obtained by solving a "gevp" problem. If we define the minimal γ for the "gevp" problem as a function of α_2 and β_2, $\gamma_1(\alpha_2, \beta_2)$, we may use "fminsearch" in Matlab to find the minimal γ_1 over $\alpha_2, \beta_2 \in [0, \infty)$.

Also note that the optimization problem reduces to the corresponding problem in [4] if $\alpha_1 = \alpha_2 = \beta_1 = 0$.

In the absence of disturbance, i.e., $w = 0$ or $E = 0$, we can take $\beta_1 = \beta_2 = 0$ and the two inequalities reduce to

$$\alpha_1(C^{\mathrm{T}}CA + A^{\mathrm{T}}C^{\mathrm{T}}C) \le P - C^{\mathrm{T}}C,$$
$$PA + A^{\mathrm{T}}P \le \alpha_2(C^{\mathrm{T}}C - P)$$

which become LMIs when α_2 is fixed.

We use two simple examples to demonstrate the improvement.

Example 1: Consider a second-order system with

$$A = \begin{bmatrix} 0 & 1 \\ -0.1 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{9}$$

The maximal output estimated with invariant ellipsoid is 1.2252. Using the truncated ellipsoid, we obtain a smaller bound as 0.9161. The actual value for the maximal output is 0.8374.

Example 2: Consider a third-order system with disturbance, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -4 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},$$

and $x_0 = 0$. The bound on the output obtained via invariant ellipsoid is 0.9023. The bound obtained via the truncated ellipsoid (constraints (7) and (8)) is 0.6789. The actual maximal output for this case equals the L_1 norm of the system, which is $\int_0^\infty |Ce^{At}E|dt = 0.6$.

B. Estimation of the maximal output and overshoot under step input

Consider the system

$$\dot z = Az + Bu + Ew, \quad y = Cz, \quad z_0 = 0, \tag{10}$$

where $z \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}$ is a scalar output, $u \in \mathbb{R}^m$ is a step input with final value u_f and w is the disturbance bounded by $w^{\mathrm{T}}w \le 1$. Assume A is Hurwitz and w is piecewise continuous. We'd like to estimate the maximal y that will be reached during the transient response. To do this, we shift the origin to the steady state value of z for $w = 0$ by defining $x = z + A^{-1}Bu_f$. Then

$$\dot x = Ax + Ew, \quad y = Cx - CA^{-1}Bu_f, \quad x_0 = A^{-1}Bu_f. \tag{11}$$

Let $y_1(t) = Cx(t)$ and y_∞ be the steady state value of the output y in case of $w = 0$, i.e., $y_\infty = -CA^{-1}Bu_f$. For simplicity, assume $y_\infty > 0$. Then $y_1(0) = -y_\infty < 0$. Denote

$$y_{1,\max} = \sup\{y_1(t) : t \ge 0, w^{\mathrm{T}}w \le 1\}.$$

It is clear that $y_{1,\max} \ge 0$ since $y_1(\infty) = 0$ with $w = 0$. If $y_{1,\max} > 0$, then the original output y has an overshoot equaling this value and the maximal value of y is $y_{1,\max} + y_\infty$. Unlike the problem of estimating the maximal absolute value of the output $|y(t)|$ in Section III-A, we intend to estimate the maximal value of $y_1(t)$.

Suppose that we have an invariant set L_V including x_0 for (11). The maximal value of $y_1(t)$ for all $t \ge 0$ can be estimated by evaluating the maximal Cx over the entire L_V. This might be too conservative since it is the same as the maximal $|Cx|$ over L_V (a symmetric set), which is probably reached at $t = 0$ with the negative value $y_1(0) = -y_\infty$. Since $y_{1,\max}$ is clearly reached at a certain $t > 0$ instead of $t = 0$, we can restrict our attention to a subset of L_V, where a local extreme of y_1 is possible in the presence of w.

Proposition 2: Let $V(x) = \max\{x^{\mathrm{T}}P_jx : j = 1, \cdots, J\}$ and suppose that L_V is an invariant set for (11) that includes x_0. If there exist $\alpha_j \ge 0, j = 1, \cdots, J$, $\alpha_w \ge 0$ and $\alpha_c \in \mathbb{R}$ such that

$$\begin{bmatrix} M_1 & -\alpha_c A^{\mathrm{T}}C^{\mathrm{T}}CE \\ -\alpha_c E^{\mathrm{T}}C^{\mathrm{T}}CA & -\alpha_c E^{\mathrm{T}}C^{\mathrm{T}}CE - \alpha_w I \end{bmatrix} \le 0, \tag{12}$$

where

$$M_1 = C^{\mathrm{T}}C - \Sigma_{j=1}^{J}\alpha_j P_j - \alpha_c A^{\mathrm{T}}C^{\mathrm{T}}CA,$$

then $y_1(t_e)^2 \le \alpha_1 + \cdots + \alpha_J + \alpha_w$ for every $t_e > 0$ where a local extreme of y_1 is reached. Thus $y_{1,max} \le \alpha_1 + \cdots + \alpha_J + \alpha_w$.

Remark 1: The main idea in Proposition 2 is to exclude $t = 0$, since $|y_1(0)|$ is usually the maximal value of $|y_1(t)|$ but $y_1(0)$ is not the maximal value of $y_1(t)$. If we extend the time to $t < 0$, $y_1(0)$ may not be a local extreme of $y_1(t)$ since there may exist no w such that $C(Ax_0 + Ew) = 0$. ◦

Remark 2: The matrix inequality (12) can be replaced with

$$\begin{bmatrix} M_2 & -\alpha_c F^{\mathrm{T}}CE \\ -\alpha_c E^{\mathrm{T}}C^{\mathrm{T}}F & -\alpha_w I \end{bmatrix} \le 0,$$

where

$$M_2 = C^{\mathrm{T}}C - \Sigma_{j=1}^{J}\alpha_j P_j - \alpha_c(F^{\mathrm{T}}CA + A^{\mathrm{T}}C^{\mathrm{T}}F)$$

and $F \in \mathbb{R}^{1\times n}$ is any row vector. This can be shown by following the same procedure as the proof of Proposition 2. Instead of using $(Ax + Ew)^{\mathrm{T}}C^{\mathrm{T}}C(Ax + Ew) = 0$ as a consequence of $C(Ax+Ew) = 0$, we can use $x^{\mathrm{T}}F^{\mathrm{T}}C(Ax+Ew) = 0$. This would introduce an additional variable F for optimization. We may pick $F = C$ for simplicity. The resulting matrix inequality would be linear in A and the result can be extended to handle linear differential inclusions. ◦

Combining the above discussion and the condition for the invariance of the set L_V in Section II, we can formulate an optimization problem to estimate a bound on the local extrema of y_1:

$$\begin{aligned} \alpha^* = \inf \quad & \alpha_1 + \cdots + \alpha_J + \alpha_w \\ s.t. \quad & (6), (12) \\ & x_0^{\mathrm{T}}P_j x_0 \le 1; j = 1, \cdots, J, \\ & \alpha_w \ge 0, \alpha_j, \beta_j \ge 0, \lambda_{jk} \ge 0, j, k = 1, \cdots, J, \\ & P_j = P_j^{\mathrm{T}} > 0, \; j = 1, \cdots, J, \end{aligned} \tag{13}$$

where (6) ensures that L_V is an invariant set. Then all the local extrema of y_1 and $y_{1,\max}$ are bounded by $\sqrt{\alpha^*}$.

For the special case where $V(x) = x^{\mathrm{T}}Px$, the constraints (6), (12) reduce to

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + \beta_1 P & PE \\ E^{\mathrm{T}}P & -\beta_1 I \end{bmatrix} \le 0,$$

$$\begin{bmatrix} C^{\mathrm{T}}C - \alpha_1 P - \alpha_c A^{\mathrm{T}}C^{\mathrm{T}}CA & -\alpha_c A^{\mathrm{T}}C^{\mathrm{T}}CE \\ -\alpha_c E^{\mathrm{T}}C^{\mathrm{T}}CA & -\alpha_c E^{\mathrm{T}}C^{\mathrm{T}}CE - \alpha_w I \end{bmatrix} \le 0.$$

When α_1 and β_1 are fixed, the constraint becomes LMIs. So we can use "fminsearch" to perform a two dimensional optimization on α_1 and β_1. In the absence of disturbance ($E = 0$), with a change of variable, $\alpha_1 P \to P$, the optimization problem can be further reduced to

$$\begin{aligned} \inf_{P>0,\alpha_c} \quad & \alpha, \\ s.t. \quad & C^{\mathrm{T}}C - P - \alpha_c A^{\mathrm{T}}C^{\mathrm{T}}CA \le 0; \\ & A^{\mathrm{T}}P + PA \le 0; \\ & x_0^{\mathrm{T}}Px_0 \le \alpha. \end{aligned}$$

where all the constraints are linear.

Example 3: A second-order system is described as

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} w,$$
$$y = [1 \;\; 0]z, \quad z(0) = 0.$$

Under a unit step input, the steady state output for $w = 0$ is $y_\infty = 0.6667$. When transformed to the state $x = z + A^{-1}B$, we have $x_0 = \begin{bmatrix} -0.6667 \\ 1 \end{bmatrix}$ and $y_1(0) = -0.6667$.

We first consider the case where $w = 0$. With a quadratic function, a bound for the maximal $y_1(t)$ is obtained as 0.5971 (89.5%). With $V(x) = \max\{x^{\mathrm{T}}P_1x, x^{\mathrm{T}}P_2x\}$, the bound is reduced to 0.5268 (79%). The resulting invariant sets are plotted in Fig. 1, where the outer curve is the boundary of the invariant ellipsoid from quadratic function and the inner closed curve in dash-dotted curve is the boundary of the invariant set as the intersection of two ellipsoids. The horizontal dotted line is $CAx = 0$, where the local extreme of y_1 is obtained. The actual overshoot determined from simulation is 0.3718 (55.8%). Fig. 1 plots the trajectory x starting from the initial condition x_0. It should be noted that

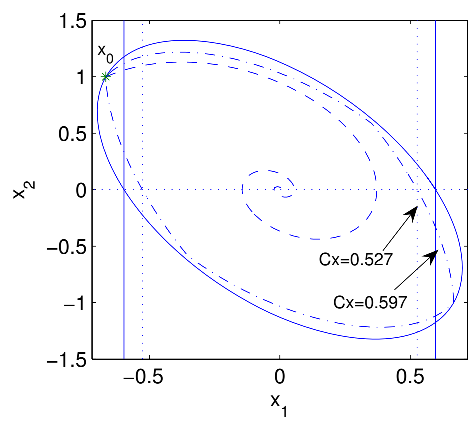

Fig. 1. Two invariant sets for estimating overshoot for $w = 0$.

a method for estimating the overshoot is briefly mentioned in [4] on page 97. With that method, a bound for the maximal y is obtained as 2.0399, corresponding to a bound for y_1 as 1.3732.

Next we consider the case where $w^{\mathrm{T}}w \le 1$. With $V(x) = \max\{x^{\mathrm{T}}P_1x, x^{\mathrm{T}}P_2x\}$, a bound for the maximal $y_1(t)$ is obtained as 0.5757. The actual $y_{1,\max}$ is obtained as follows. Denote $y_{1,w,\max}(t)$ as the maximal y_1 that can be reached at t due to w only (zero initial condition and zero input u). Then

$$y_{1,w,\max}(t) = \int_0^{\mathrm{T}} |Ce^{A(t-\tau)}E| d\tau$$

can be numerically computed. Note that $y_{1,w,\max}(t)$ is not an actual response due to any $w(\cdot)$, but the worst value of $y_1(t)$ for each t. Let $y_{1,w,0}(t) = Ce^{At}x_0$ be the response under x_0. Then $y_{1,\max} = \sup_{t>0} y_{1,w,\max}(t) + y_{1,w,0}(t) = 0.4288$. The three functions $y_{1,w,0}, y_{1,w,\max}$ and $y_{1,w,0}+y_{1,w,\max}$ are plotted in Fig. 2 with dash-dotted, dashed and solid curves, respectively. It should be mentioned that the maximal y_1 due

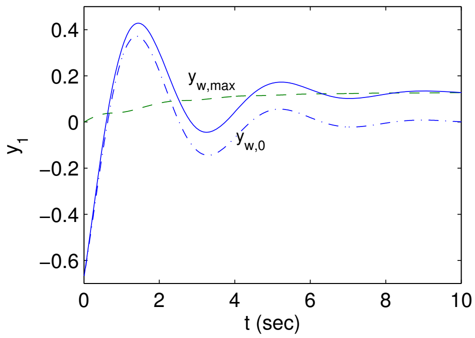

Fig. 2. Estimation of overshoot for $w^{\mathrm{T}}w \leq 1$.

to w only is $y_{1,w,\max}(\infty) = 0.1278$, which is much greater than the increase of output bound (0.4288-0.3718=0.057) and the increase of the estimate of output bound (0.5757-0.5268=0.0498). It is expected that the estimate of $y_{1,\max}$ can be improved by using V composed from more quadratic functions, at the cost of increased numerical complexity.

IV. MAXIMAL INVARIANT SET UNDER STATE CONSTRAINT

Consider the linear system

$$\dot{x} = Ax + Ew, \quad y = Cx, \tag{14}$$

where $x \in \mathbb{R}^n, w \in \mathbb{R}^q$, $y \in \mathbb{R}^p$ and $w^{\mathrm{T}}w \leq 1$. Each output is constrained within a given bound. For simplicity, assume that the bound for each output is 1, i.e., $|y_i(t)| \leq 1$ for all $i = 1, 2, \cdots, p$. Denote the corresponding state constraint set as

$$X_c = \{x \in \mathbb{R}^n : \ |C_i x| \leq 1, \ i = 1, \cdots, p.\}.$$

A set $X_0 \subset X_c$ is said to be admissible if all trajectories starting from X_0 will stay within X_c for all $t > 0$. We would like to determine an admissible set which is as large as possible. Such a problem has been extensively studied under the constrained control framework (e.g., [2], [8], [14]). In this section, we focus on the analysis aspect of the problem where the control is not explicitly involved. Although the method can be used for constructing a linear state feedback by replacing A with $A + BK$.

The problem of determining the largest admissible set is usually converted into one of finding the largest invariant set inside X_c. A simple solution is to find a maximal invariant ellipsoid inside the constraint set X_c, which can be formulated as an LMI problem. To reduce the conservatism, the recent work [14] proposed an interesting invariant set as the intersection of X_c and an invariant ellipsoid. Condition for the invariance of the intersection is derived as nonlinear matrix inequalities and the size of the invariant set is maximized with a modified Newton's method.

Here we consider the same type of invariant set as in [14]. It is actually the 1-level set of the Lyapunov function

$$V(x) = \max\{x^{\mathrm{T}}Px, x^{\mathrm{T}}C_j^{\mathrm{T}}C_j x : j = 1, 2, \cdots, p\}. \tag{15}$$

As we mentioned earlier, the 1-level set $L_V = \{x \in \mathbb{R}^n : V(x) \leq 1\}$ is formed by truncating the ellipsoid $\mathcal{E}(P)$ with planes $C_i x = \pm 1$ and thus lies within the state constraint X_c.

In this paper, we will take a quite different approach to ensure the invariance of L_V as compared to the method in [14]. An important relaxation is that we don't require the ellipsoid $\mathcal{E}(P)$ to be invariant. Instead, we directly give a condition for the invariance of L_V by using Proposition 1, which can be considered as the result of the S procedure. Moreover, our method can be used to handle persistent disturbances.

By Proposition 1, a sufficient condition for L_V to be invariant is: there exist $\lambda_{jk} \geq 0, \beta_j \geq 0, \ j,k = 0, \cdots, p$ such that

$$\begin{bmatrix} M_0 & PE \\ E^{\mathrm{T}}P & -\beta_0 I \end{bmatrix} \leq 0,$$
$$\begin{bmatrix} M_j & P_j E \\ E^{\mathrm{T}}P_j & -\beta_j I \end{bmatrix} \leq 0, j = 1, \cdots, p,$$

where

$$M_0 = A^{\mathrm{T}}P + PA - \sum_{k=1}^{p} \lambda_{0k} P_k + \left(\sum_{k=1}^{p} \lambda_{0k}\right) P + \beta_0 P$$
$$M_j = A^{\mathrm{T}}P_j + P_j A - \lambda_{j0}(P - P_j) - \sum_{k=1}^{p} \lambda_{jk}(P_k - P_j) + \beta_j P_j$$

and $P_j = C_j^{\mathrm{T}}C_j$. The above become LMIs when $\Sigma_{k=1}^{p}\lambda_{0k}, \beta_0$ and λ_{j0} are fixed. In the absence of w or $E = 0$, we can set $\beta_j = 0$. With some manipulation, the condition can be further reduced to: there exist $a_j > 0, b_{jk} \geq 0, \ \alpha_0 \geq 0, \alpha_j \geq 0$, $j,k = 1, 2, \cdots, p$, such that $\Sigma_{j=1}^{p}\alpha_j = \alpha_0$, and

$$A^{\mathrm{T}}P + PA \leq -\alpha_0 P + \sum_{j=1}^{p} \alpha_j C_j^{\mathrm{T}}C_j, \tag{16}$$

$$a_j(A^{\mathrm{T}}C_j^{\mathrm{T}}C_j + C_j^{\mathrm{T}}C_j A) \leq P - C_j^{\mathrm{T}}C_j + \sum_{k=1}^{p} b_{jk}(C_k^{\mathrm{T}}C_k - C_j^{\mathrm{T}}C_j), \ \ j = 1, 2, \cdots, p. \tag{17}$$

The invariant set L_V can be maximized with respect to a certain shape reference set X_R such that $\eta X_R \subset L_V$ for the maximal η. The set inclusion condition $\eta X_R \subset L_V$ can be stated as LMIs if X_R is a polytope or an ellipsoid. For example, consider $X_R = \mathrm{co}\{x_i : i = 1, 2, \cdots, K\}$. Then $\eta X_R \subset L_V$ if and only if

$$x_i^{\mathrm{T}}Px_i \leq 1/\eta^2, \ \ x_i^{\mathrm{T}}C_j^{\mathrm{T}}C_j x_i \leq 1/\eta^2, \quad \forall\, i = 1, \cdots, K, j = 1, \cdots, J. \tag{18}$$

An optimization problem can be formulated to maximize η satisfying (16), (17) and (18). Note that all the conditions in (16) are LMIs and the condition (17) is LMI for a fixed α_0.

The method can be easily extended to linear differential inclusions by duplicating the matrix inequalities for each vertex matrix A_k, with respective coefficients $a_{ik}, b_{ijk}, \alpha_{0k}, \alpha_{jk}$. This is because L_V is a convex set. It is invariant for the linear differential inclusion if and only if it is invariant for each vertex system.

Example 4: Consider a third-order system in [14]:

$$\dot{x} = (A - BK)x,$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & -1 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and $K = [0.360 \ \ -0.053 \ \ -0.671]$. The state constraint set is

$$X_c = \{x \in \mathbb{R}^3 : |x_i| \le 1, i = 1, 2, 3, |Kx| \le 1\}.$$

The truncated ellipsoid obtained in [14] is $S_1 = X_c \cap \{x : x^{\mathrm{T}} P x \le 1\}$, where

$$P = \begin{bmatrix} 0.130 & 0.127 & -0.030 \\ 0.127 & 0.228 & 0.198 \\ -0.030 & 0.198 & 1.007 \end{bmatrix}.$$

The surface of the set S_1 is plotted in Fig. 3.

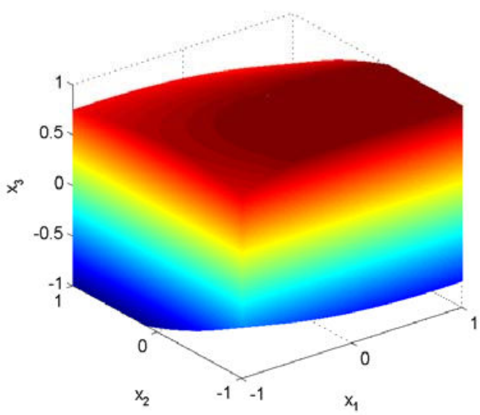

Fig. 3. Invariant set obtained in [14]

The truncated ellipsoid obtained in this paper is $S_2 = X_c \cap \{x : x^{\mathrm{T}} P_1 x \le 1\}$ where

$$P_1 = \begin{bmatrix} 0.0342 & 0.0679 & 0.0109 \\ 0.0679 & 0.1368 & 0.0638 \\ 0.0109 & 0.0638 & 0.8427 \end{bmatrix}.$$

The surface of the set S_2 is plotted in Fig. 4.

Using digital integration, the volume of S_1 and S_2 are obtained as, 7.3082 and 7.9252, respectively. The later is very close to the volume of the state constraint set X_c, which is 7.9847.

V. Conclusions

This paper uses truncated ellipsoids as simple tools for the analysis of several performance indices for linear systems subject to persistent disturbances. The performance indices include the maximal output under a set of initial conditions, the overshoot under a step input and the size of the maximal invariant set within state constraints. Condition for the invariance of intersections of ellipsoids is presented and used for different analysis purposes. The performance indices

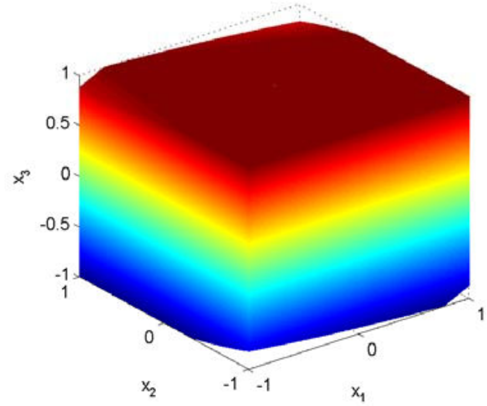

Fig. 4. Invariant set obtained with this paper's method

are estimated via optimization problems whose constraints become LMIs when a few scalar variables are fixed.

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