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Reliable H_∞ Dynamic Output Feedback Synthesis for Linear Systems

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***Abstract*— This paper considers the problem of reliable H_∞ control against actuator failures for linear continuous-time systems. The model including actuator partial failures and actuator lock-in-place failures is presented, and the faults of being locked in place are accurately described as zero-frequency disturbances and their attenuation performances are described by using small gain conditions based on finite frequency approach. Then a two-step LMI-based algorithm is developed for designing reliable H_∞ dynamic output feedback controllers. The resulting designs guarantee the stability and disturbance/fault attenuation performances of the closed-loop systems, not only when all control components are operational, but also in case of some actuator failures. Finally, a numerical example is given to illustrate the advantage of the proposed design method in comparison with the entire frequency approach by using bounded real lemma.**

I. Introduction

The research area of reliable control has recently attracted considerable attention. The reliability of control systems in the presence of system component failures is of great importance. The overall reliability is enhanced not by using more reliable components, but by managing them in a way that the reliability of the overall system is greater than the reliability of its parts. The ultimate goal is to preserve the stability and high-priority performances of the plant by a single controller which can tolerate a severe component failure. That is, the essential stability and performance requirements for the control systems remain achieved, not only when all control components are operational, but also in case of sensor or actuator failures. Models of control component failures can be classified as outages, partial degradations and lock-in-place failures (stuck-faults). When a failure modeled as outage occurs, the measured signal (in the case of sensors) or the control input (in the case of actuators) simply becomes zero. The partial failure is represented by a scaling factor with upper and lower bounds to the signal to be measured or to the control input.

A number of theoretical results as well as application examples have now been given in the literature (see, e.g., [1]-[6]). [1] presented a new methodology for the design of reliable centralized and decentralized control systems by using the algebraic Riccati equation approach. [2] employed the linear matrix inequality approach to study the reliable guaranteed cost control problem for discrete-time systems. Also, the frequency domain approach given in [3] has been proved to be another effective design technique for designing the linear reliable control systems. [5] addressed the problem of fault-tolerant flight tracking control against actuator lock-in-place failures based on the H_∞ and peak-to-peak gain performance indexes in a multi-objective optimization setting in terms of an iterative LMI algorithm. The previous results of reliable control in the presence of locked-in-place failures are developed in the framework of robust control theory, and the faulty signals of being locked in place were modeled as the entire frequency disturbances. However, the modeling method for the severe faults is not exact because the faulty signals of being locked in place correspond to zero frequency disturbances, and might be very conservative for some practical situations.

This paper is concerned with the reliable H_∞ control problem against actuator failures for linear continuous-time systems. The cases of actuator partial failures and actuator lock-in-place failures are simultaneously considered. In particular, the faults of being locked in place are modeled as zero-frequency disturbances and their rejection performances are described in terms of small gain conditions via the finite frequency approach given in [7]-[9]. The considered reliable H_∞ control problem essentially is a control design problem in mixed frequency domains for systems with multiple modes, which has not been addressed in the literature. In this study, a new method is developed to approach the reliable H_∞ control problem, and a two-step procedure is presented for designing reliable H_∞ dynamic output feedback controllers. The resulting designs guarantee the stability and disturbance/fault attenuation performances of the closed-loop systems, not only when all control components are operational, but also in case of some actuator failures. Compared with the entire frequency approach which is devoted to the H_∞ framework (see, e.g., [10]-[12]), the new proposed method is potentially less conservative due to the exact modeling of the faults of being locked in place. A comparison is given via a numerical example.

The paper is organized as follows. Section 2 presents the considered problem. The new method of designing reliable H_∞ dynamic output feedback controllers are given in Section 3. The entire frequency approach for solving the reliable control problem is described in Section 4. In Section 5, an example is provided to illustrate the new proposed design procedure and their effectiveness compared with the entire frequency approach. Some concluding remarks are given in Section 6.

Notation: For a matrix A, A^* denote its complex conjugate transpose. The Hermitian part of a square matrix A is denoted by $\mathbf{He}(A) := A + A^*$. The symbol $\mathbf{H}_n$ stands for the set of $n \times n$ Hermitian matrices. The symbol $*$ within a matrix represents the the symmetric entries. I denotes the identity matrix with an appropriate dimension. For a transfer function matrix G, its H_∞ norm is defined by

$$\|G(j\omega)\|_\infty := \sup_\omega \overline{\sigma}\{G(j\omega)\}$$

where $\overline{\sigma}(G) = \{\lambda_{\max}(G^*G)\}^{\frac{1}{2}}$ represents the maximum singular value of G, $\lambda_{\max}$ represents maximum eigenvalue.

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II. Problem Statement

Consider a linear time-invariant plant described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_1\varpi(t) \\ z(t) &= Cx(t) + Du(t) \\ y(t) &= C_1x(t) + D_1\varpi(t) \end{aligned} \tag{1}$$

with a dynamic output feedback controller of the following form:

$$\begin{aligned} \dot{\xi}(t) &= A_K\xi(t) + B_Ky(t) \\ u(t) &= C_K\xi(t) \end{aligned} \tag{2}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the control input, $\varpi(t) \in \mathbf{R}^r$ is the disturbance input and $z(t) \in \mathbf{R}^q$ is the regulated output, $y(t) \in \mathbf{R}^p$ is the measured output and $\xi(t) \in \mathbf{R}^{n_c}$ is the state of the controller, respectively. A, B, B_1, C, D, C_1 and D_1 are known constant matrices of appropriate dimensions. A_K, B_K and C_K are the dynamic output feedback controller parameter matrices.

For the control input u, let u^F denote the signal vector in the case of some actuator failures. The actuator partial failure model is as follows:

$$u^F = \alpha u, \tag{3}$$

where

$$\alpha := diag\begin{bmatrix}\alpha_1 & \alpha_2 & \ldots & \alpha_m\end{bmatrix} \tag{4}$$

with α_i satisfies

$$0 \le \underline{\alpha}_i \le \alpha_i \le \overline{\alpha}_i, \qquad (i = 1,2,\ldots,m). \tag{5}$$

Denote

$$\underline{\alpha} = diag\begin{bmatrix}\underline{\alpha}_1 & \underline{\alpha}_2 & \cdots & \underline{\alpha}_m\end{bmatrix}, \tag{6}$$

$$\overline{\alpha} = diag\begin{bmatrix}\overline{\alpha}_1 & \overline{\alpha}_2 & \cdots & \overline{\alpha}_m\end{bmatrix}. \tag{7}$$

While the actuator lock-in-place failure model is:

$$u^F = F_ju + (I - F_j)\beta_j, \qquad (j = 1,2,\ldots,L) \tag{8}$$

where

$$\begin{aligned} F_j &= diag\{f_{j1}, f_{j2}, \ldots, f_{jm}\}, \\ \beta_j &= \begin{bmatrix}\beta_{j1} & \beta_{j2} & \ldots & \beta_{jm}\end{bmatrix}^T \end{aligned} \tag{9}$$

with

$$f_{ji} = \begin{cases} 1 & \text{the } i\text{th actuator is operational} \\ 0 & \text{the } i\text{th actuator is locked in place} \end{cases} \tag{10}$$

where $i = 1,2,\ldots,m$. Here, the index j denotes the jth failure mode and L is the total failure modes. And β_{ji} is an unknown constant. Actually, the actuator failure mode adopted in this paper is:

$$u^F = F_j\alpha u + \overline{F}_j\beta_j, \qquad (j = 1,2,\ldots,L) \tag{11}$$

where $\overline{F}_j := I - F_j$.

Remark 1: It considers the case of actuator partial failure and actuator locking in place simultaneously. When $f_{ji} = 1$ and $\underline{\alpha}_i = \overline{\alpha}_i = 0$, it covers the case of outage of the ith actuator u_i in the jth failure mode. When $f_{ji} = 1$ and $\underline{\alpha}_i > 0$, it corresponds to the case of partial failure of the ith actuator u_i in the jth failure mode. When $f_{ji} = 0$, the ith actuator u_i is locked in place in the jth failure mode. Without loss of generality, we assume that $F_0 = I$. Note that, when $F_0 = I$ and $\underline{\alpha} = \overline{\alpha} = I$, it corresponds to the normal control input vector $u^F(t) = u(t)$.

Then the resulting closed-loop system in the event of the actuator failures described by (11) is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BF_j\alpha C_K\xi(t) + B_1\varpi(t) + B\overline{F}_j\beta_j \\ \dot{\xi}(t) &= B_KC_1x(t) + A_K\xi(t) + B_KD_1\varpi(t) \\ z(t) &= Cx(t) + DF_j\alpha C_K\xi(t). \end{aligned} \tag{12}$$

Remark 2: Here, $z(t)$ is defined as the regulated output. Since the inputs of being locked in place are uncontrollable, we do not consider the lock-in-place inputs in $z(t)$.

In the closed-loop system, if β_j in the failed actuator is regarded as a zero-frequency disturbance, then the transfer function matrices $G_i(s)(i = 1,2)$ from $\varpi(t)$ and β_j to $z(t)$ are respectively denoted by

$$G_i(s) = \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B}_i + \mathbf{D}_i, \tag{13}$$

where state space realizations $(\mathbf{A}, \mathbf{B}_i, \mathbf{C}, \mathbf{D}_i)$ of $G_i(s)$ are correspondingly given by

$$\left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C} & \mathbf{D}_1 & \mathbf{D}_2 \end{array}\right] = \left[\begin{array}{cc|cc} A & BF_j\alpha C_K & B_1 & B\overline{F}_j \\ B_KC_1 & A_K & B_KD_1 & 0 \\ \hline C & DF_j\alpha C_K & 0 & 0 \end{array}\right]. \tag{14}$$

The control synthesis problem under consideration is to find a dynamic output feedback controller (2) such that the resulting closed-loop system is asymptotically stable and the following constraints

$$\left\|G_1(j\omega)\right\|_\infty < \gamma_1 \quad \text{for all} \quad \omega \in \mathbf{R} \cup \{\infty\}, \tag{15}$$

$$\overline{\sigma}(G_2(j\omega)) < \gamma_2 \quad \text{for} \quad \omega = 0 \tag{16}$$

hold not only when all control components are operational, but also in the case of some actuator failures by (11).

The above reliable control problem essentially is a control design problem in mixed frequency domains for systems with multiple modes, which is non-convex one [13] and has not been addressed in the literature. In next section, a two-step algorithm will be given for designing reliable H_∞ dynamic output feedback controllers.

III. Reliable H_∞ Control via Dynamic Output Feedback

In this section, a new method of designing reliable H_∞ dynamic output feedback controllers is given by using a two-step algorithm. The following theorem presents a sufficient condition for the solvability of the reliable control problem.

Theorem 1: Consider the linear time-invariant system (1). Let C_K be given. If there exist scalars $\mu > 0$, $\nu > 0$, symmetric matrices $\begin{bmatrix}P_{11} & P_{12} \\ P_{12}^* & P_{22}\end{bmatrix}$, $\begin{bmatrix}Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22}\end{bmatrix} > 0$, $Y > N > 0$ and matrices G, H such that, for all $j = 1,2,\ldots,L$, the following inequalities

$$\begin{bmatrix} \mathbf{He}(YA - HC_1) & YBF_jaC_K - G - A^*N + C_1^*H^* & \mu C^* & YB_1 - HD_1 \\ * & \mathbf{He}(-NBF_jaC_K + G) & \mu C_K^*aF_jD^* & HD_1 - NB_1 \\ * & * & -\mu\gamma_1^2 I & 0 \\ * & * & * & -\mu I \end{bmatrix} < 0, \tag{17}$$

$$\begin{bmatrix} -Q_{11} & -Q_{12} & P_{11} - Y & P_{12} + N & 0 & 0 \\ * & -Q_{22} & P_{12}^* + N & P_{22} - N & 0 & 0 \\ * & * & \mathbf{He}(YA - HC_1) & YBF_jaC_K - G - A^*N + C_1^*H^* & \nu C^* & YB\overline{F}_j \\ * & * & * & \mathbf{He}(-NBF_jaC_K + G) & \nu C_K^*aF_jD^* & -NB\overline{F}_j \\ * & * & * & * & -\nu\gamma_2^2 I & 0 \\ * & * & * & * & * & -\nu I \end{bmatrix} < 0 \tag{18}$$

hold with $a \in \{\underline{\alpha}, \overline{\alpha}\}$, then the resulting closed-loop system (12) with

$$A_K := N^{-1}G, \quad B_K := N^{-1}H \tag{19}$$

is asymptotically stable and meets constraints (15) and (16) not only when all control components are operational, but also in the case of some actuator failures by (11).

Proof. (15) and (16) are equivalent to the following inequalities with positive scalars θ and τ, respectively, for $i = 1,2$

$$J^* \begin{bmatrix} \mathbf{B}_1^* & \mathbf{D}_1^* \\ 0 & I \end{bmatrix}^* \theta \Pi_1 \begin{bmatrix} \mathbf{B}_1^* & \mathbf{D}_1^* \\ 0 & I \end{bmatrix} J < 0, \quad \omega \in \mathbf{R} \cup \{\infty\}, \tag{20}$$

$$J^* \begin{bmatrix} \mathbf{B}_2^* & \mathbf{D}_2^* \\ 0 & I \end{bmatrix}^* \tau \Pi_2 \begin{bmatrix} \mathbf{B}_2^* & \mathbf{D}_2^* \\ 0 & I \end{bmatrix} J < 0, \quad \omega = 0, \tag{21}$$

where $J := \begin{bmatrix} (j\omega I - \mathbf{A}^*)^{-1}\mathbf{C}^* \\ I \end{bmatrix}$, $\Pi_i := \begin{bmatrix} I & 0 \\ 0 & -\gamma_i^2 I \end{bmatrix}$. Then we can get the following from (21) by using Theorem 1 in [7]

$$\begin{bmatrix} \mathbf{A} & I \\ \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} -Q & P \\ P & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A} & I \\ \mathbf{C} & 0 \end{bmatrix}^* + \begin{bmatrix} \mathbf{B}_2 & 0 \\ \mathbf{D}_2 & I \end{bmatrix} \tau \Pi_2 \begin{bmatrix} \mathbf{B}_2 & 0 \\ \mathbf{D}_2 & I \end{bmatrix}^* < 0 \tag{22}$$

where $P = P^*$ and $Q = Q^* > 0$. In view of Lemma 1 in [7] with $R = \begin{bmatrix} 0 & I & 0 \end{bmatrix}$, it follows that

$$\begin{bmatrix} -Q & P & 0 \\ P & \tau \mathbf{B}_2 \mathbf{B}_2^* & \tau \mathbf{B}_2 \mathbf{D}_2^* \\ 0 & \tau \mathbf{D}_2 \mathbf{B}_2^* & \tau \mathbf{D}_2 \mathbf{D}_2^* - \tau \gamma_2^2 I \end{bmatrix} < \mathbf{He} \begin{bmatrix} 0 & W & 0 \\ 0 & -\mathbf{A}W & 0 \\ 0 & -\mathbf{C}W & 0 \end{bmatrix} \tag{23}$$

where $W = W^* > 0$. By using Schur complement lemma, it is equivalent to

$$\begin{bmatrix} -Q & P - W & 0 & 0 \\ P - W & \mathbf{A}W + W\mathbf{A}^* & W\mathbf{C}^* & \mathbf{B}_2 \\ 0 & \mathbf{C}W & -\tau\gamma_2^2 I & \mathbf{D}_2 \\ 0 & \mathbf{B}_2^* & \mathbf{D}_2^* & -\tau^{-1} I \end{bmatrix} < 0. \tag{24}$$

Define $\nu := \tau^{-1}$, $X := W^{-1}$, $\mathscr{P} := XPX$ and $\mathscr{Q} := XQX$. Multiplying (24) by $diag\{X, X, \nu I, I\}$ on the left and right, respectively, we obtain

$$\begin{bmatrix} -\mathscr{Q} & \mathscr{P} - X & 0 & 0 \\ \mathscr{P} - X & X\mathbf{A} + \mathbf{A}^* X & \nu \mathbf{C}^* & X\mathbf{B}_2 \\ 0 & \nu \mathbf{C} & -\nu\gamma_2^2 I & \nu \mathbf{D}_2 \\ 0 & \mathbf{B}_2^* X & \nu \mathbf{D}_2^* & -\nu I \end{bmatrix} < 0. \tag{25}$$

By virtue of Lemma 1 in Appendix, X can be given to be

$$X = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix} \tag{26}$$

where $Y > N > 0$. Defining the new variables

$$\mathscr{P} := \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}, \qquad \mathscr{Q} := \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix},$$
$$G := NA_K, \qquad H := NB_K, \tag{27}$$

(25) holds if (18) hold with a given C_K in view of the fact that (25) is convex.

Then in view of dual bounded real lemma, we can get the following inequality from (20)

$$\begin{bmatrix} \mathbf{A} & I \\ \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} 0 & P_1 \\ P_1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A} & I \\ \mathbf{C} & 0 \end{bmatrix}^* + \begin{bmatrix} \mathbf{B}_1 & 0 \\ \mathbf{D}_1 & I \end{bmatrix} \theta \Pi_1 \begin{bmatrix} \mathbf{B}_1 & 0 \\ \mathbf{D}_1 & I \end{bmatrix}^* < 0 \tag{28}$$

where $P_1 = P_1^* > 0$. By using the Schur complement lemma, we can get

$$\begin{bmatrix} W\mathbf{A}^* + \mathbf{A}W & W\mathbf{C}^* & \mathbf{B}_1 \\ \mathbf{C}W & -\theta\gamma_1^2 I & \mathbf{D}_1 \\ \mathbf{B}_1^* & \mathbf{D}_1^* & -\theta^{-1} I \end{bmatrix} < 0. \tag{29}$$

Define $P_1 := W$ and $\mu := \theta^{-1}$. Multiplying (29) by $diag\{X, \mu I, I\}$ on the left and right, respectively, it follows that

$$\begin{bmatrix} \mathbf{A}^* X + X\mathbf{A} & \mu \mathbf{C}^* & X\mathbf{B}_1 \\ \mu \mathbf{C} & -\mu\gamma_1^2 I & \mu \mathbf{D}_1 \\ \mathbf{B}_1^* X & \mu \mathbf{D}_1^* & -\mu I \end{bmatrix} < 0. \tag{30}$$

If (30) holds, it follows that $\mathbf{A}^* X + X\mathbf{A} < 0$. That is to say, the closed-loop system (12) is asymptotically stable. (30) is convex. Then (30) holds if (17) hold. Thus, the proof is complete.

Remark 3: In Theorem 1, the entire frequency H_∞ conditions (17) correspond to the requirements of exogenous disturbance rejection performances, and the zero-frequency small gain conditions (18) correspond to the performance requirements for lock-in-place signal rejection. It should be pointed out that if C_K is not fixed, then the conditions (17) and (18) are non-convex. When C_K is given, the problem of designing reliable H_∞ dynamic output feedback controllers based on Theorem 1 can be reduced to a generalized eigenvalue problem (GEVP). In fact, let $\gamma_2 > 0$ be given. Define a scalar $\zeta > 0$, and replace $\mu\gamma_1^2$ with ζ in (17). The GEVP is to minimize γ_1^2 subject to the LMI constraints (17) and (18) for all $j = 1, 2, \ldots, L$ and $\nu > 0$, $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} > 0$, $Y > 0$, $N > 0$, $Y - N > 0$, $\zeta > 0$, $\mu > 0$, $\zeta < \mu\gamma_1^2$. As a matter of fact, the GEVP is a quasi-convex optimization problem since the constraint is convex and the objective is quasi-convex. It can be solved by using Toolbox of MATLAB.

We now turn our attention to the choice of C_K. The state feedback case for system (1) is equivalent to taking $C_1 = I$, $D_1 = 0$, so that $y = x$. Furthermore, we consider the state feedback controller $u = C_K x$, which results in the closed-loop transfer function matrices

$$\overline{G}_1(s) = (C + DF_j\alpha C_K)(sI - (A + BF_j\alpha C_K))^{-1} B_1,$$
$$\overline{G}_2(s) = (C + DF_j\alpha C_K)(sI - (A + BF_j\alpha C_K))^{-1} (B\overline{F}_j), \tag{31}$$

in case of actuator failures by (11) with no lock-in-place inputs in the regulated output z. The following theorem provides a solution to the corresponding reliable H_∞ state feedback control design problem for continuous-time systems.

Theorem 2: Consider system (1) and assume that $C_1 = I$, $D_1 = 0$. Consider the controller $u = C_K x$ resulting in the closed-loop transfer function matrices $\overline{G}_i (i = 1,2)$ defined by (31). Assume that the conditions of Theorem 1 hold, then there exist scalars $\theta > 0$, $\tau > 0$, symmetric matrices P, $Q > 0$, $V > 0$ and a matrix M such that, for all $j = 1, 2, \ldots, L$, the following inequalities

$$\begin{bmatrix} \mathbf{He}(AV + BF_j aM) & VC^* + M^* aF_j D^* & \theta B_1 \\ * & -\theta\gamma_1^2 I & 0 \\ * & * & -\theta I \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} -Q & P - V & 0 & 0 \\ * & \mathbf{He}(AV + BF_j aM) & VC^* + M^* aF_j D^* & \tau B\overline{F}_j \\ * & * & -\tau\gamma_2^2 I & 0 \\ * & * & * & -\tau I \end{bmatrix} < 0 \tag{33}$$

hold with $a \in \{\underline{\alpha}, \overline{\alpha}\}$. Furthermore, the constraints

$$\left\|\overline{G}_1(j\omega)\right\|_\infty < \gamma_1 \quad \text{for all} \quad \omega \in \mathbf{R} \cup \{\infty\}, \tag{34}$$

$$\overline{\sigma}(\overline{G}_2(j\omega)) < \gamma_2 \quad \text{for} \quad \omega = 0 \tag{35}$$

are satisfied for the closed-loop system with the stabilizing state feedback gain matrix

$$C_K := MV^{-1} \tag{36}$$

even in the presence of some actuator failures by (11).

Proof. Suppose that the conditions of Theorem 1 hold, then inequalities (17) and (18) hold. Defining $\rho := \begin{bmatrix} I & I & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \theta I & 0 \\ 0 & 0 & 0 & \theta I \end{bmatrix}$, then pre- and post-multiplying (17) by ρ and ρ^*, respectively, and deleting the second row and column, we can get

$$\begin{bmatrix} \mathbf{He}[(Y-N)A+(Y-N)BF_jaC_K] & C^*+C_K^*aF_jD^* & \theta(Y-N)B_1 \\ * & -\theta\gamma_1^2 I & 0 \\ * & * & -\theta I \end{bmatrix} < 0. \tag{37}$$

Subsequently, defining $V := (Y-N)^{-1}$ and $M := C_K V$, then pre- and post-multiplying (37) by $diag\{V, I, I\}$, respectively, they are equivalent to (32).

Similarly, defining $\eta := \begin{bmatrix} I & I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau I & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau I \end{bmatrix}$, then pre- and post-multiplying (18) by η and η^*, respectively, and deleting the second and the fourth rows and columns, it follows that

$$\begin{bmatrix} -Q_{11}-\mathbf{He}(Q_{12})-Q_{22} & P_{11}+\mathbf{He}(P_{12})+P_{22}-(Y-N) & 0 & 0 \\ * & \mathbf{He}[(Y-N)A+(Y-N)BF_jaC_K] & C^*+C_K^*aF_jD^* & \tau(Y-N)B\overline{F}_j \\ * & * & -\tau\gamma_2^2 I & 0 \\ * & * & * & -\tau I \end{bmatrix} < 0. \tag{38}$$

Then pre- and post-multiplying (38) by $diag\{V, V, I, I\}$, respectively, and defining $P := V[P_{11}+\mathbf{He}(P_{12})+P_{22}]V$ and $Q := V[Q_{11}+\mathbf{He}(Q_{12})+Q_{22}]V$, we can obtain (33). We can conclude that conditions (32) and (33) hold because (17) and (18) hold.

In view of the proof of Theorem 1, the specifications (34) and (35) for the closed-loop system with the state feedback controller $u = C_K x$ are equivalent to the followings, respectively

$$\begin{bmatrix} \mathbf{He}(AV+BF_j\alpha M) & VC^*+M^*\alpha F_jD^* & \theta B_1 \\ * & -\theta\gamma_1^2 I & 0 \\ * & * & -\theta I \end{bmatrix} < 0, \tag{39}$$

$$\begin{bmatrix} -Q & P-V & 0 & 0 \\ * & \mathbf{He}(AV+BF_j\alpha M) & VC^*+M^*\alpha F_jD^* & \tau B\overline{F}_j \\ * & * & -\tau\gamma_2^2 I & 0 \\ * & * & * & -\tau I \end{bmatrix} < 0. \tag{40}$$

It is known that (32) and (33) hold. So (39) and (40) hold accordingly due to the convexity of them. As a result, C_K can be obtained by $C_K = MV^{-1}$. That is to say, the closed-loop system with the state feedback controller $u = C_K x$ is asymptotically stable and captures the specifications (34) and (35). Thus, the proof is complete.

Remark 4: Theorem 2 gives a sufficient condition for the solvability of the reliable H_∞ state feedback control problem which captures the specifications (34) and (35). As a matter of fact, Theorem 2 is a necessary condition of Theorem 1, which provides a method of determining state feedback gain C_K.

Based on Theorems 1 and 2, a two-step algorithm is presented for designing reliable H_∞ dynamic output feedback controllers as follows:

Algorithm 1: Let $\gamma_2 > 0$ be given.

Step 1. Define a scalar $\varsigma > 0$. Use ς instead of $\theta\gamma_1^2$ in (32). Minimize γ_1^2 subject to the LMI constraints (32) and (33) for all $j = 1, 2, \ldots, L$ and $\tau > 0$, $\varsigma > 0$, $Q > 0$, $V > 0$, $\theta > 0$, $\varsigma < \theta\gamma_1^2$. Denoting the optimal solutions as $\hat{M}$, $\hat{V}$ and $\hat{\gamma}_s$, let $\hat{C_K} = \hat{M}\hat{V}^{-1}$.

Step 2. Plug $\hat{C_K}$ into (17) and (18). Define a scalar $\zeta > 0$, and replace $\mu\gamma_1^2$ with ζ in (17). Minimize γ_1^2 subject to the LMI constraints (17) and (18) for all $j = 1, 2, \ldots, L$ and $\nu > 0$, $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} > 0$, $Y > 0$, $N > 0$, $Y - N > 0$, $\zeta > 0$, $\mu > 0$, $\zeta < \mu\gamma_1^2$. The optimal solutions are denoted by $\hat{N}$, $\hat{G}$, $\hat{H}$ and $\hat{\gamma}_d$, then let $\hat{A_K} = \hat{N}^{-1}\hat{G}$, $\hat{B_K} = \hat{N}^{-1}\hat{H}$ and $\hat{\gamma}_1 = \sqrt{\hat{\gamma}_d}$.

Remark 5: In Algorithm 1, Step 1 corresponds to an GEVP for obtaining a reliable H_∞ state feedback gain matrix C_K, which provides an initial state feedback gain for the optimization in Step 2. Step 2 is for searching a solution to the multi-objective control problem via dynamic output feedback defined in (15) and (16) by solving the corresponding GEVP in terms of the remaining unknown controller matrices A_K and B_K.

IV. Entire frequency approach

In this section, we can derive a similar result in the case where (16) is replaced by an H_∞-norm constraint. By virtue of [13], an algorithm corresponding to the entire frequency approach for solving the reliable H_∞ dynamic output feedback control problem can be summarized as follows:

Algorithm 2: Let $\gamma_2 > 0$ be given.

Step 1. Minimize γ_1 subject to the LMI constraints

$$\begin{bmatrix} \mathbf{He}(AV+BF_jaM) & VC^*+M^*aF_jD^* & B_1 \\ * & -\gamma_1 I & 0 \\ * & * & -\gamma_1 I \end{bmatrix} < 0, \tag{41}$$

$$\begin{bmatrix} \mathbf{He}(AV+BF_jaM) & VC^*+M^*aF_jD^* & B\overline{F}_j \\ * & -\gamma_2 I & 0 \\ * & * & -\gamma_2 I \end{bmatrix} < 0 \tag{42}$$

for all $j = 1, 2, \ldots, L$ and $V > 0$. Obtain the optimal solutions $\hat{M}$, $\hat{V}$ and $\hat{\gamma}_s$, then let $\hat{C_K} = \hat{M}\hat{V}^{-1}$.

Step 2. Minimize γ_1 subject to the LMI constraints

$$\begin{bmatrix} \mathbf{He}(YA-HC_1) & YBF_jaC_K-G-A^*N+C_1^*H^* & C^* & YB_1-HD_1 \\ * & \mathbf{He}(-NBF_jaC_K+G) & C_K^*aF_jD^* & HD_1-NB_1 \\ * & * & -\gamma_1 I & 0 \\ * & * & * & -\gamma_1 I \end{bmatrix} < 0, \quad a \in \{\underline{\alpha}, \overline{\alpha}\}, \tag{43}$$

$$\begin{bmatrix} \mathbf{He}(YA-HC_1) & YBF_jaC_K-G-A^*N+C_1^*H^* & C^* & YB\overline{F}_j \\ * & \mathbf{He}(-NBF_jaC_K+G) & C_K^*aF_jD^* & -NB\overline{F}_j \\ * & * & -\gamma_2 I & 0 \\ * & * & * & -\gamma_2 I \end{bmatrix} < 0, \quad a \in \{\underline{\alpha}, \overline{\alpha}\} \tag{44}$$

with obtained $\hat{C_K}$ for all $j = 1, 2, \ldots, L$ and $Y > 0$, $N > 0$, $Y - N > 0$. The optimal solutions are denoted by $\hat{N}$, $\hat{G}$, $\hat{H}$ and $\hat{\gamma}_d$, then the controller gains are given by $\hat{C_K}$, $\hat{A_K} = \hat{N}^{-1}\hat{G}$, and $\hat{B_K} = \hat{N}^{-1}\hat{H}$, and $\hat{\gamma}_1 = \hat{\gamma}_d$.

Remark 6: Algorithm 2 is based on that the inputs being locked in place are modeled as the entire frequency disturbances and their rejection performances are devoted to the robust H_∞ framework [13]. Compared with Algorithm 1, the entire frequency approach given by Algorithm 2 is not exact enough.

V. Example

Consider a linear time-invariant system of the form (1) with

$$A=\begin{bmatrix}-6.8319 & -2.8352\\ 1.8152 & -5.2208\end{bmatrix},\qquad B_1=\begin{bmatrix}-8.1410\\ -5.8689\end{bmatrix},$$

$$B=\begin{bmatrix}3.4857 & -0.9688 & -4.4319\\ 2.4201 & -1.8905 & -9.2010\end{bmatrix},$$

$$C=\begin{bmatrix}-2.0771 & 0.8754\end{bmatrix},\quad C_1=\begin{bmatrix}2.1344 & 7.2740\end{bmatrix},$$

$$D=\begin{bmatrix}1.1472 & -6.2044 & 3.5000\end{bmatrix},\qquad D_1=-2.5509.$$

Our objective is to design a stabilizing dynamic output feedback controller (2) such that the resulting closed-loop system is asymptotically stable and the H_∞-norm bound constraints (15) and (16) hold not only when all control components are operational, but also in the case of some actuators being locked in place. The value of γ_2 is fixed as $\gamma_2=2$ and optimize γ_1. In this example we consider the following six sorts of actuator failure modes:
(1) $u_2(t)$ and $u_3(t)$ are in normal case, while $u_1(t)$ is locked in place, which is modeled as $u_1(t)=\beta_{11}$ for $t\geq 20$ sec. Correspondingly, we have that $\underline{\alpha}=\overline{\alpha}=I$ and $F_1=diag\{0,1,1\}$;
(2) $u_1(t)$ and $u_3(t)$ are in normal case, while $u_2(t)=\beta_{22}$ for $t\geq 20$ sec. As a result, $\underline{\alpha}=\overline{\alpha}=I$ and $F_2=diag\{1,0,1\}$;
(3) $u_1(t)$ and $u_2(t)$ are in normal case, while $u_3(t)=\beta_{33}$ for $t\geq 20$ sec. For this reason, $\underline{\alpha}=\overline{\alpha}=I$ and $F_3=diag\{1,1,0\}$;
(4) $u_1(t)$ is in normal case, while $u_2(t)=\beta_{42}$ and $u_3(t)=\beta_{43}$ for $t\geq 20$ sec. Because of this, $\underline{\alpha}=\overline{\alpha}=I$ and $F_4=diag\{1,0,0\}$;
(5) $u_2(t)$ is in normal case, while $u_1(t)=\beta_{51}$ and $u_3(t)=\beta_{53}$ for $t\geq 20$ sec. Consequently, $\underline{\alpha}=\overline{\alpha}=I$ and $F_5=diag\{0,1,0\}$;
(6) $u_3(t)$ is in normal case, while $u_1(t)=\beta_{61}$ and $u_2(t)=\beta_{62}$ for $t\geq 20$ sec. That is, $\underline{\alpha}=\overline{\alpha}=I$ and $F_6=diag\{0,0,1\}$.
i) By using Algorithm 1, the optimal value of γ_1 and the corresponding controller gains are obtained as follows:

$$\gamma_{1min}=0.6778,\quad A_K=\begin{bmatrix}-16.7354 & -27.6802\\ -5.4896 & -21.9653\end{bmatrix},$$

$$B_K=\begin{bmatrix}3.5522\\ 2.4872\end{bmatrix},\quad C_K=\begin{bmatrix}0.9800 & -0.3781\\ -0.1411 & 0.0525\\ 0.2725 & -0.1270\end{bmatrix}.$$

ii) In contrast, by using Algorithm 2, the optimal value of γ_1 and the corresponding controller gains are obtained as follows:

$$\gamma_{1min}=1.5826,\quad A_K=\begin{bmatrix}-15.2789 & -33.8413\\ -9.0370 & -39.4504\end{bmatrix},$$

$$B_K=\begin{bmatrix}4.2849\\ 4.8707\end{bmatrix},\quad C_K=\begin{bmatrix}0.9840 & -0.3829\\ -0.1440 & 0.0535\\ 0.2733 & -0.1309\end{bmatrix}.$$

The actual achieved values of γ_1 and γ_2 in some cases are in Table 1. The corresponding closed-loop frequency responses are respectively shown in Fig.1-Fig.4 (on the last page).
Based on the output responses and the actual achieved values of γ_1 and γ_2, it is easy to see that the new proposed method obtains better performance (comparison between the values of γ_1 and 10s-20s in Fig.1-Fig.4) and is less conservative (comparison between the values of γ_2 and 20s-30s in Fig.2-Fig.4) than the method given by bounded real lemma in most conditions for the example. So, compared with the entire frequency approach in terms of bounded real lemma, the new proposed method can be a good alternative for designing reliable H_∞ dynamic output feedback controllers with actuators locked in place.

Table 1 The actual achieved values of γ_1 and γ_2

F_i	$diag\{1,1,1\}$		$diag\{1,1,0\}$		$diag\{1,0,0\}$		$diag\{0,1,0\}$	
Actual values	γ_1	γ_2	γ_1	γ_2	γ_1	γ_2	γ_1	γ_2
Our design	0.4083	-	0.0989	1.7615	0.5353	1.7432	0.5130	1.7228
BRL	1.2793	-	0.8782	0.9967	0.7660	1.3259	0.6946	1.3349

VI. Conclusion

In this paper, the problem of reliable H_∞ control via dynamic output feedback for linear continuous-time systems against actuator failures has been investigated. A two-step LMI-based method for synthesizing dynamic output feedback controllers is developed to achieve the desired disturbance/fault attenuation performances not only when the system is operating properly, but also in the event of some actuator failures. The advantage of the new proposed design method in comparison with the entire frequency approach by using bounded real lemma has been illustrated via a numerical example.

Appendix

Before presenting the proof for Theorem 1, some preliminaries are required.
Lemma 1: Consider the system described by (12). Let $(\mathbf{A},\mathbf{B}_i,\mathbf{C},\mathbf{D}_i)(i=1,2)$ in (14) be given. Then the following statements are equivalent:
(i)There exist symmetric matrices $X>0$, $\mathscr{P}$, $\mathscr{Q}>0$ and a dynamic output feedback controller $\mathbf{K}$ described by (2) such that

$$\begin{bmatrix}X\mathbf{A}+\mathbf{A}^*X & \mathbf{C}^* & X\mathbf{B}_1\\ \mathbf{C} & -\gamma_1^2I & \mathbf{D}_1\\ \mathbf{B}_1^*X & \mathbf{D}_1^* & -I\end{bmatrix}<0 \tag{45}$$

holds for $\omega\in\mathbf{R}\cup\{\infty\}$, and

$$\begin{bmatrix}-\mathscr{Q} & \mathscr{P}-X & 0 & 0\\ \mathscr{P}-X & X\mathbf{A}+\mathbf{A}^*X & \mathbf{C}^* & X\mathbf{B}_2\\ 0 & \mathbf{C} & -\gamma_2^2I & \mathbf{D}_2\\ 0 & \mathbf{B}_2^*X & \mathbf{D}_2^* & -I\end{bmatrix}<0 \tag{46}$$

holds.
(ii)There exist symmetric matrices $X_a=\begin{bmatrix}Y & -N\\ -N & N\end{bmatrix}>0(Y>N>0)$, $\mathscr{P}_a$, $\mathscr{Q}_a>0$ and a dynamic output feedback controller described by (2) with $A_K=A_{Ka}$, $B_K=B_{Ka}$, $C_K=C_{Ka}$ such that

$$\begin{bmatrix}X_a\mathbf{A}_a+\mathbf{A}_a^*X_a & \mathbf{C}_a^* & X_a\mathbf{B}_{1a}\\ \mathbf{C}_a & -\gamma_1^2I & \mathbf{D}_{1a}\\ \mathbf{B}_{1a}^*X_a & \mathbf{D}_{1a}^* & -I\end{bmatrix}<0 \tag{47}$$

holds for $\omega\in\mathbf{R}\cup\{\infty\}$, and

$$\begin{bmatrix}-\mathscr{Q}_a & \mathscr{P}_a-X_a & 0 & 0\\ \mathscr{P}_a-X_a & X_a\mathbf{A}_a+\mathbf{A}_a^*X_a & \mathbf{C}_a^* & X_a\mathbf{B}_{2a}\\ 0 & \mathbf{C}_a & -\gamma_2^2I & \mathbf{D}_{2a}\\ 0 & \mathbf{B}_{2a}^*X_a & \mathbf{D}_{2a}^* & -I\end{bmatrix}<0 \tag{48}$$

holds, where

$$
\begin{aligned}
\mathbf{A}_a &= \begin{bmatrix} A & BF_j\alpha C_{Ka} \\ B_{Ka}C_1 & A_{Ka} \end{bmatrix}, \quad \mathbf{B}_{1a} = \begin{bmatrix} B_1 \\ B_{Ka}D_1 \end{bmatrix}, \\
\mathbf{B}_{2a} &= \begin{bmatrix} B\overline{F}_j \\ 0 \end{bmatrix}, \quad \mathbf{C}_a = \begin{bmatrix} C & DF_j\alpha C_{Ka} \end{bmatrix}, \\
\mathbf{D}_{1a} &= 0, \quad \mathbf{D}_{2a} = D\overline{F}_j.
\end{aligned} \tag{49}
$$

Proof. Let $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}$ where $X_{11} = X_{11}^* > 0$, $X_{22} = X_{22}^* > 0$ and X_{12} is nonsingular. Letting $Y = X_{11}$ and $N = X_{12}X_{22}^{-1}X_{12}^*$, it follows that

$$
\begin{aligned}
X_a = TXT^* &= \begin{bmatrix} I & 0 \\ 0 & -X_{12}X_{22}^{-1} \end{bmatrix} X \begin{bmatrix} I & 0 \\ 0 & -X_{12}X_{22}^{-1} \end{bmatrix}^* \\
&= \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}
\end{aligned} \tag{50}
$$

where $X > 0$ is equivalent to $Y > N > 0$. Set $\mathscr{P}_a = T\mathscr{P}T^*$ and $\mathscr{Q}_a = T\mathscr{Q}T^*$. Then pre- and post-multiplying (46) by $diag\{T,T,I,I\}$ and $diag\{T^*,T^*,I,I\}$ respectively, (46) is equivalent to (48) by letting $A_{Ka} = (X_{12}^{-1})^* X_{22} A_K X_{22}^{-1} X_{12}^*$, $B_{Ka} = -(X_{12}^{-1})^* X_{22} B_K$ and $C_{Ka} = -C_K X_{22}^{-1} X_{12}^*$.
Similarly, pre- and post-multiplying (45) by $diag\{T,I,I\}$ and $diag\{T^*,I,I\}$ respectively, (45) is equivalent to (47). Thus, the proof is complete.

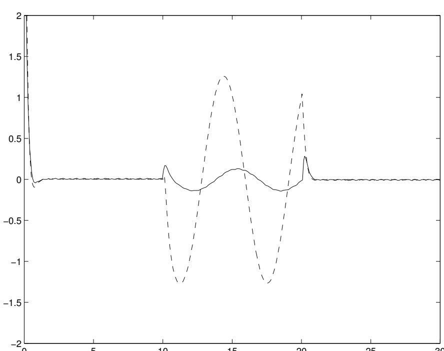

Fig. 1 Output responses comparison between solid by using our design procedure and dashed by using bounded real lemma in normal case.

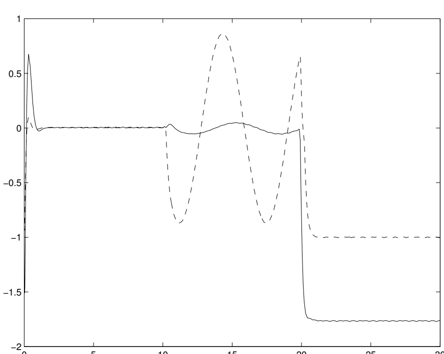

Fig. 2 Output responses comparison between solid by using our design procedure and dashed by using bounded real lemma for failure pattern (3).

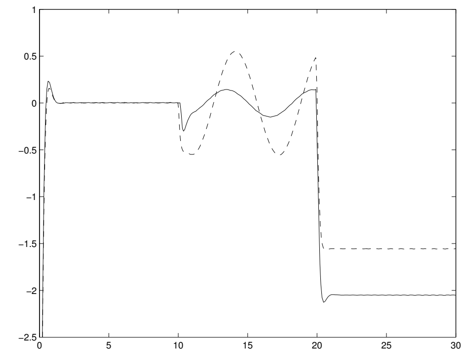

Fig. 3 Output responses comparison between solid by using our design procedure and dashed by using bounded real lemma for failure pattern (4).

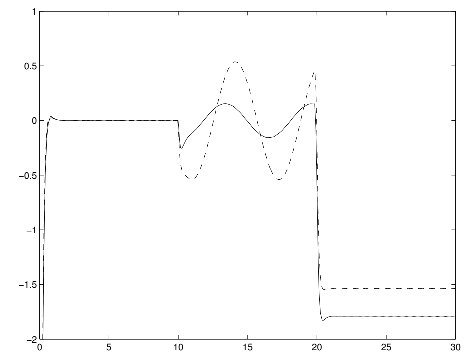

Fig. 4 Output responses comparison between solid by using our design procedure and dashed by using bounded real lemma for failure pattern (5).

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