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Observer Design for Untimed Continuous Petri Nets

C. Mahulea, C. Seatzu, M.P. Cabasino, L. Recalde and M. Silva

***Abstract*— Continuous Petri nets (conPNs) are an approximation of (discrete) Petri nets (PNs) introduced to cope with the state explosion problem typical of discrete event systems. We consider a free-labeled Petri net model and assume that certain transitions, including all that model faulty behaviors are unobservable, i.e., they are labeled with the empty word. We show how to design a marking observer that consists in a linear algebraic characterization of the set of markings that is consistent with a given observation. For some subclasses of nets the results can be extended to the discrete case where the computational complexity is much bigger than the one of continuous systems.**

I. Introduction

Reconstructing the state of a system from available measurements is a fundamental issue in several applications. State observation can be seen as a self-standing problem, but also as a pre-requisite for solving problems of different nature, such as stabilization, state-feedback control, diagnosis, filtering, and others.

This problem has been extensively investigated in time driven systems. On the contrary, despite the attention payed by several authors in the last years, there are relatively few works addressing this topic in discrete and hybrid systems, thus several related problems are still open.

In the case of discrete event systems modeled by PNs, different approaches for observability have been recently proposed. In [7] the problem was that of reconstructing the initial marking (assumed only partially known) from the observation of transition firings. In [9] this approach was extended to the observation and control of timed nets. In other works it was assumed that some of the transitions of the net are not observable [4] or undistinguishable [6], thus complicating the observation problem. Benasser [2] has studied the possibility of defining the set of markings reached firing a "partially specified" step of transitions using logical formulas, without having to enumerate this set. Ramirez *et al.* [17] have discussed the problem of estimating the marking of a Petri net using a mix of transition firings and place observations.

Recently, a particular hybrid model based on Petri nets has received some attention. This model is called contPNs [1], [20]. It can be seen as a relaxation of Petri nets where the constraints that markings and transitions firings are integer is removed. While for timed case the problem has been extensively studied for *infinite server semantics* [11], [15] and exist some works also for *finite* server semantics [14], in the untimed case the problem has been studied in [13] for a particular class of backward conflict free nets.

In this paper we focus on the problem of designing a state observer for general case of *untimed* contPNs. Thus the net behavior is asynchronous and sequential as in discrete net systems, and the only difference between the first and the latter model is the relaxation of the integer constraint.

We assume that the net structure is known, as well as the initial marking. The set of transitions is partitioned in two sets: observable and unobservable transitions. After an observable transition firing we observe its firing quantity, which is the continuous counterpart of the number of firings of each transition. Our goal is that of reconstructing the set of markings consistent with the given observation (namely, the set of markings in which the system may be given the actual observation).

We first prove that, under certain assumptions on the unobservable subnet, the set of consistent markings is convex. An iterative algorithm is also given to compute it if the net system is bounded. Some important remarks are made allowing to simplify the computational complexity of the algorithm, moving off-line the most burdensome part of the procedure.

We also investigate when it is possible to use such results when dealing with discrete PNs, namely if it is possible to study the state observation problem via relaxation. We show via simple numerical examples that this is in general not possible unless the unobservable subnet is either a marked graph or a state machine. This confirms what has already been proved in [3], i.e., the requirement of enumerating the set of consistent markings for general unobservable nets.

II. Background on Untimed ContPNs

In this section we provide the basic background on untimed contPNs.

Definition 1: A contPN system is a pair $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$, where:

- $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ is the net structure with two disjoint sets of places P and transitions T; pre and post incidence functions $\boldsymbol{Pre}, \boldsymbol{Post} : P \times T \to \mathbb{R}_{\geq 0}$, denote the weight of the arcs from places to transitions (respectively, transitions to places);
- $\boldsymbol{m}_0 : P \to \mathbb{R}_{\geq 0}$ is the initial marking. ■

We denote as $m = |P|$ and $n = |T|$ the cardinality of the set of places and transitions, respectively.

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C. Mahulea, L. Recalde and M. Silva are with the Aragón Institute for Engineering Research (I3A), University of Zaragoza, Maria de Luna 1, 50018 Zaragoza, Spain {cmahulea,lrecalde,silva@unizar.es}.

C. Seatzu and M.P. Cabasino are with the Department of Electrical and Electronic Engineering, University of Cagliari, Piazza D'Armi, 09123 Cagliari, Italy {seatzu,cabasino@diee.unica.it}.

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The input and output set of a node $x \in P \cup T$ is denoted by ${}^\bullet x$ and $x^\bullet$, respectively. The token load of a place p_i at the marking $\boldsymbol{m}$ is denoted by $\boldsymbol{m}(p_i)$ or simply by m_i.

A transition $t_j \in T$ is enabled at a marking $\boldsymbol{m}$ iff $\forall p_i \in {}^\bullet t_j$, $\boldsymbol{m}(p_i) \geq 0$ and the enabling degree of t_j at $\boldsymbol{m}$ is:

$$enab(t_j, \boldsymbol{m}) = \min_{p_i \in {}^\bullet t_j} \frac{m_i}{\boldsymbol{Pre}(p_i, t_j)}. \quad (1)$$

When a transition t_j is enabled at a marking $\boldsymbol{m}$ it can be fired. The main difference with respect to discrete PNs is that in the case of contPNs it can be fired in any real amount α, with $0 \leq \alpha \leq enab(t_j, \boldsymbol{m})$ and it is not limited only to a natural number. Such a firing yields to a new marking $\boldsymbol{m'} = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}[\cdot, t_j]$, where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the *token flow matrix* (or *incidence matrix*). This firing is also denoted $\boldsymbol{m}[t_j(\alpha)\rangle \boldsymbol{m'}$.

If a marking $\boldsymbol{m}$ is reachable from the initial marking through a firing sequence $\sigma = t_{r1}(\alpha_1) t_{r2}(\alpha_2) \cdots t_{rk}(\alpha_k)$, and we denote by $\boldsymbol{\sigma} : T \to \mathbb{R}_{\geq 0}$ the *firing count vector*, then we can write $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, which is called the *fundamental equation* or *state equation*.

The set of all fireable sequences in the net is $\mathcal{L}(\mathcal{N}, \boldsymbol{m}_0)$, while the set of all markings that are reachable with a finite firing sequence is denoted by $RS^{ut}(\mathcal{N}, \boldsymbol{m}_0)$. An interesting property of $RS^{ut}(\mathcal{N}, \boldsymbol{m}_0)$ is that it is a *convex set* [18]. That is, if two markings $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are reachable, then any marking $\boldsymbol{m}_3 = \alpha \cdot \boldsymbol{m}_1 + (1 - \alpha) \cdot \boldsymbol{m}_2$, $\forall \alpha \in [0, 1]$ is also a reachable marking.

The net $\mathcal{N}$ is called *consistent* iff $\exists \boldsymbol{x} > 0$ such that $\boldsymbol{C} \cdot \boldsymbol{x} = 0$. The support of a vector $\boldsymbol{v}$ is denoted by $||\boldsymbol{v}||$ and represents the indices of its not null components.

A Petri net $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ is a *marked graph* if $\forall p \in P$, $|{}^\bullet p| = |p^\bullet| \leq 1$ and $Pre(p, t), Post(p, t) \in \{0, 1\}$ for any $p \in P$ and any $t \in T$. It is a *state machine* if $\forall t \in T$, $|{}^\bullet t| = |t^\bullet| \leq 1$ and $Pre(p, t), Post(p, t) \in \{0, 1\}$ for any $p \in P$ and any $t \in T$.

Given a net $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$, and a subset $T' \subseteq T$ of its transitions, the T'*−induced subnet of* $\mathcal{N}$ is the new net $\mathcal{N}' = \langle P, T', \boldsymbol{Pre}', \boldsymbol{Post}' \rangle$ where $\boldsymbol{Pre}', \boldsymbol{Post}'$ are the restriction of $\boldsymbol{Pre}, \boldsymbol{Post}$ to T'. The net $\mathcal{N}'$ can be thought as obtained from $\mathcal{N}$ removing all transitions in $T \setminus T'$. We also write $\mathcal{N}' \prec_{T'} \mathcal{N}$.

Given a subset $T' \subseteq T$, the projection Π of a sequence $\sigma \in T^*$ over T' is defined as $\Pi : T^* \to T'^*$ such that: (i) $\Pi(\varepsilon) = \varepsilon$, where ε denotes the empty word; (ii) for all $\sigma \in T^*$ and $t \in T$, $\Pi(\sigma t) = \Pi(\sigma) t$ if $t \in T'$, and $\Pi(\sigma t) = \Pi(\sigma)$ otherwise. Here T^* denotes the set of all possible sequences obtainable combining elements in T, included the empty word.

Given a sequence $\sigma \in \mathcal{L}(\mathcal{N}, M_0)$, we denote $w = \Pi_o(\sigma)$ the corresponding *observed word*, where $\Pi_o(\sigma)$ is the projection of σ to T_o. In the following, with a little abuse of notation, we will write that $w \in T_o^*$.

III. PROBLEM STATEMENT

In this paper we propose a procedure to design an observer for contPNs based on the following three assumptions.

(A1) The initial marking of the net is known.

(A2) The set of transitions is partitioned as $T = T_o \cup T_u$ where T_o is the set of *observable* transitions and T_u is the set of *unobservable* transitions.

(A3) The T_u-induced net has no *spurious solution*.

A spurious marking is a marking satisfying the state equation but not reachable, i.e., there exists no firing sequence corresponding to the firing vector. Thus, the third assumption implies that all markings $\boldsymbol{m} \in \mathbb{R}^m_{\geq 0}$ such that $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, with $\boldsymbol{\sigma} \geq \boldsymbol{0}$, are reachable.

The following proposition provides two constructive criteria to establish the validity of assumption (A3).

Proposition 2: Let $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ be a contPN system. All markings $\boldsymbol{m} \in \mathbb{R}^m_{\geq 0} : \boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, with $\boldsymbol{\sigma} \geq \boldsymbol{0}$, are reachable, i.e., $\mathcal{N}$ has no spurious solution, if one of the following two conditions is satisfied:

- $\mathcal{N}$ is *acyclic* [16];
- $\mathcal{N}$ is *consistent* and all transitions are firable from $\boldsymbol{m}_0$ [18], or, equivalently, there exists no empty syphon at $\boldsymbol{m}_0$.

Now, given an observed word $w = t_{r1}(\alpha_1) t_{r2}(\alpha_2) \ldots t_{rk}(\alpha_k)$, our goal is that of characterizing the set of markings in which the system may be given the actual observation w. We denote this set $\mathcal{C}(w)$ and call it *set of consistent markings at* w.

Definition 3: Let $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ be a contPN system and $w \in T_o^*$ be a sequence of observable transitions. The set of *consistent markings at* w is

$$\mathcal{C}(w) = \{\boldsymbol{m} \in \mathbb{R}^m_{\geq 0} \mid \exists \sigma \in T^* : \boldsymbol{m}_0[\sigma\rangle \boldsymbol{m},\ \Pi_o(\sigma) = w\}. \quad (2)$$

■

IV. CHARACTERIZATION OF THE SET OF CONSISTENT MARKINGS

The following proposition claims that $\mathcal{C}(w)$ is convex. This is the key feature of our approach. In fact it enables us to characterize $\mathcal{C}(w)$ in linear algebraic terms.

Proposition 4: Let us consider a contPN system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ and an observed sequence w.

Under assumptions (A1) to (A3), the set of consistent markings $\mathcal{C}(w)$ is convex.

Proof: We prove this by induction.

(*Basic step*) For $w = \varepsilon$ the result obviously holds. In fact, $\mathcal{C}(\varepsilon)$ is the set of markings that can be reached from $\boldsymbol{m}_0$ firing sequences $\sigma_u \in T_u^*$, i.e., $\mathcal{C}(\varepsilon)$ coincides with the reachability set of the T_u-induced subnet. Being the reachability space of a contPN system convex [18], the set $\mathcal{C}(\varepsilon)$ is convex as well.

(*Inductive step.*) Assume the result holds for an observation v. We prove that it also holds for $w = vt(\alpha)$.

We prove this in two steps. The first step (a) consists in proving that the set of markings that are obtained from a marking in $\mathcal{C}(v)$ constitute a convex set. The second step (b) consists in proving that any marking in $\mathcal{C}(w)$ is obtained from a marking in $\mathcal{C}(v)$.

(a) Let us consider two markings that are consistent with the observation w:

$$\begin{aligned}\boldsymbol{m}' &= \boldsymbol{m}_0' + \boldsymbol{C}_u \cdot \boldsymbol{\sigma}_u' + \alpha \cdot \boldsymbol{C}(\cdot, t),\\ \boldsymbol{m}'' &= \boldsymbol{m}_0'' + \boldsymbol{C}_u \cdot \boldsymbol{\sigma}_u'' + \alpha \cdot \boldsymbol{C}(\cdot, t),\end{aligned}$$

where $\boldsymbol{m}_0', \boldsymbol{m}_0'' \in \mathcal{C}(v)$. Let $\tilde{\boldsymbol{m}}$ be the generic marking obtained by a convex combination of $\boldsymbol{m}'$ and $\boldsymbol{m}''$, i.e.,

$$\begin{aligned}\tilde{\boldsymbol{m}} &= \delta \cdot \boldsymbol{m}' + \beta \cdot \boldsymbol{m}'' = \delta \cdot \boldsymbol{m}_0' + \beta \cdot \boldsymbol{m}_0'' \\ &\quad + \boldsymbol{C}_u \cdot (\delta \cdot \boldsymbol{\sigma}_u' + \beta \cdot \boldsymbol{\sigma}_u'') + \alpha \cdot \boldsymbol{C}(\cdot, t)\end{aligned}$$

with $\delta, \beta \in [0, 1]$ and $\delta + \beta = 1$. Now, keeping into account that in a contPN if a sequence σ is firable at $\boldsymbol{m}$, then $\gamma \cdot \sigma$ is firable at $\gamma \cdot \boldsymbol{m}$, for any $\gamma \in \mathbb{R}_{\geq 0}$, under assumption (A3) we may conclude that $\tilde{\boldsymbol{m}} \in \mathcal{C}(w)$.

(b) Let $\boldsymbol{m}'$ be an arbitrary marking in $\mathcal{C}(vt(\alpha))$. According to equation (2), we can write:

$$\boldsymbol{m}' = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma} + \boldsymbol{C}_u \cdot \boldsymbol{\sigma}_u,$$

where $\boldsymbol{m}_0$ is the initial marking, $\sigma \in T^*$, $\Pi_o(\sigma) = vt(\alpha)$ and $\sigma_u \in T_u^*$.
Now, we can write σ as $\sigma = \sigma'\sigma''t(\alpha)$, where $\Pi_o(\sigma') = v$ and $\sigma'' \in T_u^*$. Thus,

$$\boldsymbol{m}' = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}' + \boldsymbol{C}_u \cdot \boldsymbol{\sigma}'' + \alpha \cdot \boldsymbol{C}(\cdot, t) + \boldsymbol{C}_u \cdot \boldsymbol{\sigma}_u. \quad (3)$$

However, by Definition 3,

$$\boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}' + \boldsymbol{C}_u \cdot \boldsymbol{\sigma}'' = \boldsymbol{m} \in \mathcal{C}(v), \quad (4)$$

where the inclusion relationship follows by assumption (A3). Thus the statement holds being $\boldsymbol{m}' = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}(\cdot, t) + \boldsymbol{C}_u \cdot \boldsymbol{\sigma}_u$. □

V. COMPUTATION OF THE SET OF CONSISTENT MARKINGS

Clearly, if the net system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ is *bounded*, the set $\mathcal{C}(w)$ is a closed convex set. In particular, we can characterize it as the projection in the $\mathbb{R}^m$-space of a polytope[1] in the $\mathbb{R}^{m+n_u}$ space. It is well-known that a closed convex set can be represented also by the convex hull of its finite set of vertices. Let us denote by $\mathcal{E}(w)$ the set of vertices of the polytope $\mathcal{C}(w)$. The following algorithm provides a procedure to compute it.

***Algorithm** 5 (Computation of $\mathcal{C}(w)$):*

1) Let $v = \varepsilon$.
2) Let
$$\mathcal{C}(v) = \{\boldsymbol{m} \in \mathbb{R}^m_{\geq 0} \mid \boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C}_u \boldsymbol{\sigma}_u,\ \boldsymbol{\sigma}_u \geq \boldsymbol{0}\}.$$
3) Let $t(\alpha)$ be a new observation and $w = vt(\alpha)$.
4) Let
$$\mathcal{C}'(v) = \{\boldsymbol{m} \in \mathbb{R}^m_{\geq 0} \mid \boldsymbol{m} \in \mathcal{C}(v),\ \boldsymbol{m} \geq \alpha \cdot Pre(\cdot, t)\}. \quad (5)$$
5) Compute the set of vertices $\mathcal{E}'(v)$ of $\mathcal{C}'(v)$.
6) Let $E = \emptyset$.
7) For all $\boldsymbol{e}_i = [\boldsymbol{m}_i^T;\ \boldsymbol{\sigma}_{u,i}^T] \in \mathcal{E}'(v)$:
   a) compute the set of vertices E_i of the polytope defined as
   $$\begin{cases} \boldsymbol{m} = \boldsymbol{m}_i + \boldsymbol{C}_u \cdot \boldsymbol{\sigma}_u + \alpha \cdot \boldsymbol{C}(\cdot, t) \\ \boldsymbol{m} \geq \boldsymbol{0} \\ \boldsymbol{\sigma}_u \geq \boldsymbol{0}; \end{cases} \quad (6)$$
   b) let $E = E \cup E_i$.
8) Let $\bar{\mathcal{C}}(w)$ be the convex hull of E and $\mathcal{C}(w)$ the projection of $\bar{\mathcal{C}}(w)$ in the $\mathbb{R}^m$ space[2].
9) Let $v = w$ and goto Step 3. ■

In simple words, Algorithm 5 computes first the set of markings that are consistent with the observation of the empty word ($v = \varepsilon$). Then, given an observation $t(\alpha)$, it computes the set of markings $\mathcal{C}'(v)$ that are consistent with the empty word and that also enable the firing of t for an amount α. At this point, for each vertex of $\mathcal{C}'(v)$, namely for each element of $\mathcal{E}'(v)$, it computes the vertices of the polytope obtained solving the system of inequalities (6). The vertices obtained considering all $\boldsymbol{m}_i$'s in $\mathcal{E}'(v)$ define the set E. The projection in the $\mathbb{R}^m$ space of the convex hull of E provides the set of consistent markings $\mathcal{C}(w)$. The algorithm iterates when a new observation $t(\alpha)$ is detected.

Note that the boundness assumption is essential when computing $\mathcal{C}(w)$ with Algorithm 5. In fact, if the net system is unbounded, the set of consistent markings may be unbounded as well, and the procedure should be modified to accept directions, not only vertices.

***Corollary** 6:* Let $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ be a *bounded* contPN system. Let assumptions (A1) to (A3) be satisfied and w be a given observation.

The set $\mathcal{C}(w)$ computed using Algorithm 5 is the set of markings consistent with w.

Proof: Let us preliminary observe that the set $\mathcal{C}'(v)$ is convex for any $v \in T_o^*$. In fact it is obtained by the set $\mathcal{C}(v)$, that is convex by Proposition 4, simply removing all markings $\boldsymbol{m}$ such that $\boldsymbol{m} < \alpha \cdot Pre(\cdot, t)$ (see Step 4 of Algorithm 5).

Now, since $\mathcal{C}(vt(\alpha))$ is computed from the vertices of $\mathcal{C}'(v) \subseteq \mathcal{C}(v)$ firing $t(\alpha)$ and a series of unobservable transitions, it is obvious that to any interior point of $\mathcal{C}'(v)$ we can associate at least one point in $\mathcal{C}(vt(\alpha))$.

Finally, we have to prove that any point of $\mathcal{C}(vt(\alpha))$ can be obtained from $\mathcal{C}'(v)$, so when the convex hull is computed *all* consistent markings are obtained. This can be proved using exactly the same arguments we used in the proof of case (b) of Proposition 4. □

A. A remark on the implementation of Algorithm 5

Let us make an important remark concerning the implementation of Algorithm 5. In particular, we want to show how its computational complexity can be drastically reduced

[1] A *bounded polyhedron* $\mathcal{P} \subset \mathbb{R}^n$, $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{B}\}$ is called a *polytope*.

[2] Note that we project the first m-entries of $\bar{\mathcal{C}}(w)$ corresponding to the marking, while the remaining n_u entries correspond to the unobservable firing vector.

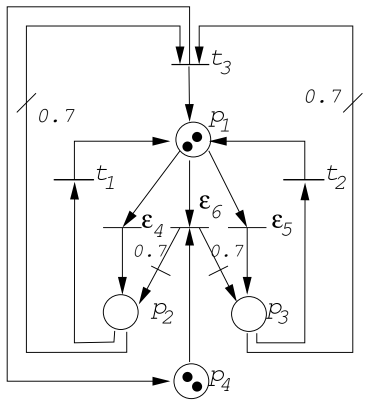

Fig. 1. The Petri net of Subsection V-B.

taking advantage from the structure of the constraints in equation (6).

Step 7 of Algorithm 5 requires the enumeration of all the vertices of the polytope defined by equation (6). As it is well known such enumeration cannot be done in general, in polynomial time [19]. However, we can avoid this taking into account that each vertex can be identified by a different *basis*. To show this, let us rewrite equation (6) as:

$$\begin{cases} \left[\begin{array}{cc} \boldsymbol{I} & -\boldsymbol{C}_u \end{array}\right] \cdot \left[\begin{array}{c} \boldsymbol{m} \\ \boldsymbol{\sigma}_u \end{array}\right] = \boldsymbol{m}_i + \alpha \cdot \boldsymbol{C}(\cdot, t) \\ \boldsymbol{m} \geq \boldsymbol{0} \\ \boldsymbol{\sigma}_u \geq \boldsymbol{0} \end{cases} \tag{7}$$

Let $\boldsymbol{B} \in \mathbb{R}^{m \times m}$ be a full rank matrix obtained extracting from $\left[\begin{array}{cc} \boldsymbol{I} & -\boldsymbol{C}_u \end{array}\right]$ a number m of independent linear columns. If we denote as $\boldsymbol{b}$ the right hand side term of equation (7), namely $\boldsymbol{b} = \boldsymbol{m}_i + \alpha \cdot \boldsymbol{C}(\cdot, t)$, then $\boldsymbol{x_B} = \boldsymbol{B}^{-1} \cdot \boldsymbol{b}$ is a *basic solution* of (6). Moreover, if $\boldsymbol{x_B} \geq \boldsymbol{0}$, the positiveness constrains are satisfied and $\boldsymbol{x_B}$ is a *basic feasible solution* of (6).

Now, let $\mathcal{I}_B = \{i_1, i_2, \ldots, i_m\}$ be the indices of the m columns of matrix $\left[\begin{array}{cc} \boldsymbol{I} & -\boldsymbol{C}_u \end{array}\right]$ generating $\boldsymbol{B}$. We define a vector $\boldsymbol{y}$ such that $\boldsymbol{y}(i_j) = \boldsymbol{x_B}(j)$ if $i_j \in \mathcal{I}_B$, and 0 otherwise. It is well known that, for bounded polyedra, $\boldsymbol{y}$ is a basic feasible solutions iff it is an *extreme point* [19].

Since the matrix $\left[\begin{array}{cc} \boldsymbol{I} & -\boldsymbol{C}_u \end{array}\right]$ is the same for all $\boldsymbol{e}_i \in \mathcal{E}'(v)$ and for all $v \in T_o^*$ (obviously the unobservable subnet is not changing) all its basis can be computed only once and off-line.

Whenever a new observation is detected, we update the value of $\boldsymbol{b} = \boldsymbol{m}_i + \alpha \cdot \boldsymbol{C}(\cdot, t)$, we compute the basis solutions by simple matrix manipulation and evaluate their feasibility. The set of basic feasible solutions coincides with the set of extreme points [19].

The same reasoning applies for the computation of the set of vertices of $\mathcal{C}'(v)$ (see equation (5)).

B. A numerical example

Let us consider the net system in Fig. 1.

By Step 2 of Algorithm 5, we know that the set of markings consistent with the empty word $\mathcal{C}(\varepsilon)$ can be defined:

$$\begin{cases} [\boldsymbol{I} \;\; -\boldsymbol{C}_u] \cdot \left[\begin{array}{c} \boldsymbol{m} \\ \boldsymbol{\sigma}_u \end{array}\right] = \boldsymbol{m}_0 \\ \boldsymbol{m} \geq \boldsymbol{0} \\ \boldsymbol{\sigma}_u \geq \boldsymbol{0} \end{cases} \tag{8}$$

where

$$\boldsymbol{C}_u = \left[\begin{array}{rrr} -1 & -1 & -1 \\ 1 & 0 & 0.7 \\ 0 & 1 & 0.7 \\ 0 & 0 & -1 \end{array}\right].$$

The number of basis of this system is equal to 26. Now, if we denote as $\boldsymbol{b}_i$ the generic i-th column of matrix $\boldsymbol{B} = [\boldsymbol{I} \;\; -\boldsymbol{C}_u]$, some basis are $\boldsymbol{B}_1 = [\boldsymbol{b}_1 \; \boldsymbol{b}_2 \; \boldsymbol{b}_3 \; \boldsymbol{b}_4]$; $\boldsymbol{B}_2 = [\boldsymbol{b}_1 \; \boldsymbol{b}_2 \; \boldsymbol{b}_3 \; \boldsymbol{b}_7]$; $\boldsymbol{B}_3 = [\boldsymbol{b}_2 \; \boldsymbol{b}_3 \; \boldsymbol{b}_4 \; \boldsymbol{b}_5]$; $\boldsymbol{B}_4 = [\boldsymbol{b}_2 \; \boldsymbol{b}_3 \; \boldsymbol{b}_4 \; \boldsymbol{b}_6]$; etc. All basis and their inverse can be computed initially off-line.

Now, for clarity of presentation, we also compute the extreme points of the set $\mathcal{C}(\varepsilon)$, thus we compute all products of the form $\boldsymbol{B}_i^{-1} \cdot \boldsymbol{m}_0$ and only consider the positive vectors. In this case we obtain the following basic solutions (extreme points), respectively:

$$\begin{array}{rcl} \boldsymbol{e}_1 & = & [2 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0]^T, \\ \boldsymbol{e}_2 & = & [0 \quad 1.4 \quad 1.4 \quad 0 \quad 0 \quad 0 \quad 2]^T, \\ \boldsymbol{e}_3 & = & [0 \quad 2 \quad 0 \quad 2 \quad 2 \quad 0 \quad 0]^T, \\ \boldsymbol{e}_4 & = & [0 \quad 0 \quad 2 \quad 2 \quad 0 \quad 2 \quad 0]^T. \end{array}$$

This means that $\boldsymbol{m}_1 = [2\ 0\ 0\ 2]^T$ is the extreme marking corresponding to $\boldsymbol{\sigma}_1 = [0\ 0\ 0]^T$, i.e., the marking obtained from the initial one firing no unobservable transition; $\boldsymbol{m}_2 = [0\ 1.4\ 1.4\ 0]^T$ corresponds to $\boldsymbol{\sigma}_2 = [0\ 0\ 2]^T$, i.e., firing $\varepsilon_6(2)$, and so on.

Note that, when making the products $\boldsymbol{B}_i^{-1} \cdot \boldsymbol{m}_0$, the same solution can be obtained more than once. As an example, in this case 24 basis provide basic feasible solutions, but 20 of them are duplications of the above four ones.

Thus, $\mathcal{E}(\varepsilon) = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ and the convex hull of $\mathcal{E}(\varepsilon)$ is the polytope

$$\bar{\mathcal{C}}(\varepsilon) = \left\{ \left[\begin{array}{c} \boldsymbol{m} \\ \boldsymbol{\sigma}_u \end{array}\right] \in \mathbb{R}_{\geq 0}^{m+n_u} \mid \boldsymbol{A}_\varepsilon \cdot \left[\begin{array}{c} \boldsymbol{m} \\ \boldsymbol{\sigma}_u \end{array}\right] \leq \boldsymbol{b}_\varepsilon \right\}$$

where

$$\boldsymbol{A}_\varepsilon = \left[\begin{array}{rrrrrrr} 2.33 & 2.33 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 2.33 & 1 & 2.33 & 0 & 0 & 0 & 0 \\ -2.5 & -2.5 & -2.5 & -1 & 0 & 0 & 0 \\ -1.75 & -0.75 & -1.75 & 0 & -1 & 0 & 0 \\ -1.75 & -1.75 & -0.75 & 0 & 0 & -1 & 0 \\ 2.5 & 2.5 & 2.5 & 0 & 0 & 0 & -1 \end{array}\right],$$

$$\boldsymbol{b}_\varepsilon = \left[\begin{array}{cccccccc} 4.67 & 0 & -2 & 4.67 & -7 & -3.5 & -3.5 & 5 \end{array}\right]^T$$

have been computed using cdd [5].

Now, assume that $t_1(0.7)$ is observed.

We first have to compute the set $\mathcal{C}'(\varepsilon) \subseteq \mathcal{C}(\varepsilon)$, that coincides with the set of markings that are consistent with the empty word and enable the actual observation $t_1(0.7)$.

To do this, we need to restrict $\mathcal{C}(\varepsilon)$ adding the constraint $m_2 \geq 0.7$, i.e.,

$$\mathcal{C}'(\varepsilon) = \{\boldsymbol{m} \mid \boldsymbol{m} \in \mathcal{C}(\varepsilon),\ m_2 \geq 0.7\}.$$

Note that $\mathcal{C}'(\varepsilon) \subseteq \mathcal{C}(\varepsilon)$ thus there can exist points that belong to $\mathcal{C}(\varepsilon)$ but not to $\mathcal{C}'(\varepsilon)$. As an example, $\boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathcal{C}(\varepsilon)$ but $\boldsymbol{e}_1, \boldsymbol{e}_4 \notin \mathcal{C}'(\varepsilon)$ since $\boldsymbol{e}_1(p_2), \boldsymbol{e}_4(p_2) = 0 < 0.7$.

Using the above characterization of $\mathcal{C}(\varepsilon)$ in terms of $\boldsymbol{A}_\varepsilon$ and $\boldsymbol{b}_\varepsilon$, plus the constraint

$$[0\ 1\ 0\ \dots\ 0] \cdot \begin{bmatrix} \boldsymbol{m} \\ \boldsymbol{\sigma}_u \end{bmatrix} \geq 0.7,$$

we obtain $\mathcal{E}'(\varepsilon) = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6\}$ where

$$\begin{array}{rcl} \boldsymbol{e}_1 &=& [\ 1.3\ \ 0.7\ \ 0\ \ 2\ \ 0.7\ \ 0\ \ 0]^T, \\ \boldsymbol{e}_2 &=& [\ 0\ \ 2\ \ 0\ \ 2\ \ 2\ \ 0\ \ 0]^T, \\ \boldsymbol{e}_3 &=& [\ 0\ \ 0.7\ \ 1.3\ \ 2\ \ 0.7\ \ 1.3\ \ 0]^T, \\ \boldsymbol{e}_4 &=& [\ 0\ \ 0.7\ \ 1.7\ \ 1\ \ 0\ \ 1\ \ 1]^T, \\ \boldsymbol{e}_5 &=& [\ 1\ \ 0.7\ \ 0.7\ \ 1\ \ 0\ \ 0\ \ 1]^T, \\ \boldsymbol{e}_6 &=& [\ 0\ \ 1.4\ \ 1.4\ \ 0\ \ 0\ \ 0\ \ 2]^T. \end{array}$$

Now, to each vertex $\boldsymbol{e}_i = [\boldsymbol{m}_i^T;\ \boldsymbol{\sigma}_{u,i}^T] \in \mathcal{E}'(\varepsilon)$ we associate a polytope defined as

$$\begin{cases} [\boldsymbol{I}\ \ -\boldsymbol{C}_u] \cdot \begin{bmatrix} \boldsymbol{m} \\ \boldsymbol{\sigma}_u \end{bmatrix} = \boldsymbol{m}_i^T + 0.7 \cdot \boldsymbol{C}(\cdot, t_1) \\ \boldsymbol{m}, \boldsymbol{\sigma}_u \geq 0 \end{cases} \tag{9}$$

and compute its vertices E_i.

Note that the vertices of such polytopes can be computed using the same basis we used to characterize $\bar{\mathcal{C}}(\varepsilon)$, that have been determined off-line.

The convex hull of $E = \cup_i E_i$ provides the set $\bar{\mathcal{C}}(t_1(0.7))$ defined by matrix $\boldsymbol{A}_{t_1(0.7)}$ and vector $\boldsymbol{b}_{t_1(0.7)}$ where

$$\boldsymbol{A}_{t_1(0.7)} =$$

$$\begin{bmatrix} 2.33 & 1 & 2.33 & 0 & 1.33 & 0 & 0 \\ 6.58 & 6.58 & 7.58 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.39 & 4.25 & 3.25 & 0 & -2.86 & -1.86 & -1 \\ -1.43 & 0 & 0 & 0 & -1.43 & -1.43 & -1.43 \\ -2.5 & -2.5 & -2.5 & 0 & 0 & 0 & 1 \\ 0 & -1.43 & 0 & 0 & 1.43 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 2.33 & 2.33 & 1 & 0 & 0 & 1.33 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -2.5 & -2.5 & -2.5 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{b}_{t_1(0.7)} = [5.6\ 19.83\ 0\ 8.5\ -1\ -5\ 0\ 0\ 4.67\ 0\ 0\ -7]^T.$$

VI. Extending these results to Discrete PNs

Some of us have studied marking estimation of discrete PNs using the notions of *basis marking* and *minimal explanation* (see in particular [3]).

Basis markings are those markings that are reached firing the observed sequences of transitions and the minimal explanations, namely minimal sequences of unobservable transitions that enable them. The set of basis markings is a subset of the set of reachable markings, and in most of the cases it is strictly contained in the set of reachable markings. However, it still requires enumeration, thus making the procedure computationally demanding in certain cases.

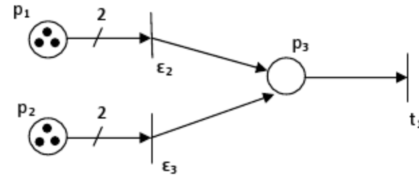

Fig. 2. The Petri net of Example 7, where ε_2 and ε_3 are not observable.

In this section we want to investigate under which conditions the linear algebraic characterization we proposed in Section V for untimed contPNs can also be useful for discrete PNs. In particular, we wonder under which conditions on the net structure the problem of marking estimation of a discrete PN can be studied within the continuous framework, simply relaxing the discrete net to a continuous one. The following example shows that unfortunately this is not possible in general cases for two main reasons.

(a) There can exist integer markings consistent with an observation in the continuous net that are not consistent with the same observation in the discrete net. An explanation of this lies in the fact that in the continuous case we can fire a transition in a real amount, while in the discrete case each transition may only be fired in a discrete amount.

(b) The set of consistent markings of a discrete PN, in general, is not an integer polytope so it cannot be characterized by a set of linear inequalities.

***Example** 7:* Let us consider the PN in Fig. 2.

If the net is assumed to be discrete (D), the set of markings that are consistent with the empty word ε is

$$\mathcal{C}^D(\varepsilon) = \{[3\ 3\ 0]^T, [1\ 3\ 1]^T, [3\ 1\ 1]^T, [1\ 1\ 2]^T\}.$$

In fact, both ε_2 and ε_3 can fire at most once.

If the net is assumed to be continuous (C), the set of markings that are consistent with ε coincides with the polytope whose set of vertices is

$$\mathcal{E}^C(\varepsilon) = \{[3\ 3\ 0]^T, [0\ 3\ 1.5]^T, [3\ 0\ 1.5]^T, [0\ 0\ 3]^T\}.$$

Indeed in such a case both p_1 and p_2 can be emptied since ε_2 and ε_3 can both fire in an amount 1.5.

The above issues (a) and (b) apply to this example.

(a) Marking $\boldsymbol{m} = [0\ 0\ 3]^T \in \mathcal{C}^C(\varepsilon)$ but $\boldsymbol{m} \notin \mathcal{C}^D(\varepsilon)$ even if it has integer entries. In particular, we may observe that this marking is reachable in the continuous net firing the sequence $\sigma = \varepsilon_2(1.5)\varepsilon_3(1.5)$, namely firing transitions ε_2 and ε_3 in non integer amounts.

(b) Let us consider markings $\boldsymbol{m}_1 = [1\ 3\ 1]^T$ and $\boldsymbol{m}_2 = [3\ 1\ 1]^T$, where $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{C}^D(w)$. Marking

$$\begin{aligned} \boldsymbol{m} &= 0.5 \cdot \boldsymbol{m}_1 + 0.5 \cdot \boldsymbol{m}_2 \\ &= [.5\ 1.5\ .5]^T + [1.5\ .5\ .5]^T = [2\ 2\ 1]^T \end{aligned}$$

is integer and is a convex combination of them but $\boldsymbol{m} \notin \mathcal{C}^D(\varepsilon)$. This shows that the set of consistent markings of the discrete net is not an integer polytope. ■

The above example confirms that Proposition 4 cannot be extended to discrete PNs with arbitrary structures, thus enumerating the set of consistent markings is in general a requirement, as already discussed in [3].

However, there exist particular classes of PNs to which Proposition 4 applies. In particular, as shown in the following, this is true if the unobservable subnet is a *marked graph* or a *state machine*.

The following definition and propositions are necessary to prove the main result of this section.

***Definition** 8:* A matrix $\boldsymbol{A}$ is *totally unimodular* if each of its minors is equal to -1, 0 or 1. ■

***Proposition** 9:* [10] A matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is *totally unimodular* iff the set $\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\cdot\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ is an integer polytope $\forall\ \boldsymbol{b} \in \mathbb{Z}^m$, i.e., iff all vertices of $\mathcal{S}$ are integers $\forall\ \boldsymbol{b} \in \mathbb{Z}^m$.

***Proposition** 10:* [10] A matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ satisfying the following conditions is *totally unimodular*.

- Every column of $\boldsymbol{A}$ contains at most two non-zero entries.
- All entries of $\boldsymbol{A}$ are in $\{-1, 0, +1\}$.
- The columns of $\boldsymbol{A}$ can be partitioned in two sets $\boldsymbol{B}$ and $\boldsymbol{C}$ that satisfy the following two conditions.
  – If two non-zero entries in a column of $\boldsymbol{A}$ have the same sign, then the row of one of them is in $\boldsymbol{B}$, and the other one is in $\boldsymbol{C}$.
  – If two non-zero entries in a column of $\boldsymbol{A}$ have opposite sign, then the rows of both of them are either in $\boldsymbol{B}$ or in $\boldsymbol{C}$.

The above proposition enables us to prove:

***Corollary** 11:* Let $\langle \mathcal{N}, \boldsymbol{m}_0\rangle$ be a *bounded discrete* PN system satisfying assumptions (A1) to (A3). Assume that the unobservable subnet is a *state machine* or a *marked graph*.

The set $\mathcal{C}(w)$ of markings consistent with any observation $w \in T_o^*$ is an integer convex polytope and can be computed using Algorithm 5 by relaxation (namely relaxing the discrete net to a continuous net, and then only considering integer solutions).

Proof: Let us first consider the case of an unobservable subnet that is a state machine. In such a case the j-th column of $\boldsymbol{C}_u$ contains at most one element equal to 1 (corresponding to the output place of t_j) and at most one element equal to -1 (corresponding to the input place of t_j). Thus, by Proposition 10, the matrix $[\boldsymbol{I}\ \ -\boldsymbol{C}_u]$ is totally unimodular. Moreover, by Proposition 9 the vertices of all the sets defined by equation (6) are integers for any observation $w = t_{r_1}(\alpha_1)t_{r_2}(\alpha_2)\dots t_{r_k}(\alpha_k)$ being $\alpha_1, \dots, \alpha_k$ integers.

Finally, any other integer point inside the convex hull of such vertices is a consistent marking as well. In fact, the unobservable subnet is a state machine thus the firing of any unobservable transition keeps the marking integer whenever it is fired in an integer amount.

Exactly the same reasoning applies if the unobservable subnet is a marked graph: in such a case the transpose of the incidence matrix of the unobservable subnet satisfies Proposition 10, i.e., it is a totally unimodular matrix. □

VII. Conclusions

In this paper we dealt with the problem of designing a marking observer for untimed contPNs. We first shown that the set of markings consistent with a given observation is a convex set; later, an algorithm to compute it is given. For some subclasses of nets, it is proved that the results here can be used for the design of observers in the discrete case (it reduces the computation complexity in this case drastically). The problem of characterizing how the number of vertices of the polytope changes with the observed word will be addressed in a future work.

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