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Error Tolerant Arrangements of Sensors For Sampling Fields

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***Abstract*— We introduce and address the *sensor arrangement problem*. We ask when and where it is best to sample an unknown scalar field (say of temperatures or chemical concentrations) in order to estimate it to within a certain error tolerance. This question is necessary to decide where to place sensors in sensing phenomena which can be described as fields over large areas. We assume that the field is modeled as a linear combination of a set of basis functions and that the sensor measurements are noisy. Based on linear estimation theory, estimation error can be shown to be a function of sampling errors and of the geometric arrangement of sampling locations. We refer to the latter as the *sensor arrangement*. Our approach is to characterize different classes of sensor arrangements and to understand the circumstances under which the reconstructed fields for arrangements from these classes satisfy an error tolerance limit. We refer to these classes as Error Tolerant Arrangement Classes or ETAC's. With a knowledge of the nature of ETAC's we will have articulated constraints for placing sensors in time and space; furthermore, by identifying possible sampling locations *in advance*, we will also have simplified the planning of the motion of mobile sensors for that field. In this paper we discuss three types of ETAC's for fields that are modeled as 2D trigonometric polynomials: uniform sensor arrangements, Δ-dense sensor arrangements and incrementally constructed sensor arrangements.**

I. INTRODUCTION

In this paper we consider the *sensor arrangement problem*. The question we ask is when and where to sample an unknown spatio-temporal field in order to obtain a *good* estimate of the field. We refer to the geometric arrangement of sampling locations as a *sensor arrangement*. The forward problem of estimating a field given the noisy sample values for a given sensor arrangement is widely addressed in many areas of science and engineering. However the inverse problem, *i.e.*, understanding where to take samples, is seldom addressed. We argue that the sensor arrangement problem is of fundamental importance in the nascent area of sensor networks.

Sensor networks have shown great promise in understanding phenomena by providing measurements at spatial and temporal scales that were not possible before. Many of the phenomena that sensor networks can sense involve physical quantities such as temperature, light intensity, chemical concentration and density, and can be represented as spatio-temporal fields. For instance, in large ecological systems such as lakes, rivers and oceans, one is interested in monitoring temperature, pH levels, chemical levels, algae concentrations, salinity, etc. in certain regions. In addition, in the case of potential disasters such as chemical spills and the leakage of hazardous gases into the environment, one is interested in monitoring the evolution of concentration levels. The spatio-temporal mapping of such physical quantities are often continuous and can be viewed as scalar fields.

Depending on the accuracy required, estimating a spatio-temporal field may require a large number of measurements. Some arrangements of sampling sensors may be disadvantageous in that they may require a larger number of sensors to reconstruct the field with the same accuracy as other geometric arrangements of sensors, and the question of where to place static sensors is crucial. Sometimes using just static sensors alone may not be feasible or sufficient, and a few additional mobile sensors may be necessary. Mobility has the effect of *multiplying* the number of sensors in the field. Theoretically, a sensor which can move infinitely fast can be at many places at one time instance. A sensor moving at some finite velocity effectively enables that single sensor to act like some finite number of sensors in time-space, enabling what we refer to as *multiplicity*. In our view, sensor mobility can be categorized into two types, *incidental* and *intentional*. We define incidental mobility as a situation in which a sensor does not have control over its motion. In these situations a sensor moves passively under the influence of the environment (*e.g.*, sensors moving with water currents and nodes mounted on animals). We define intentional mobility as a situation in which a sensor has control over its motion and can actively move to a desired location (*e.g.*, nodes mounted on mobile robots). However, a clear understanding does not exist for how the sensors, static or mobile, must be arranged to ensure *good* sampling. [1]

In this paper we define the sensor arrangement problem and propose an approach to determining feasible solution spaces to the question of which sensor arrangements ensure that the estimation error does not exceed some pre-specified threshold. We assume that an unknown field to be estimated is represented as a linear combination of a set of basis functions. We further assume that the measurement model involves random additive noise. In this setting, according to linear estimation theory, [1], [2], the estimation error is a function of the sensor arrangement. Any solution to the sensor arrangement problem belongs to the space of sensor arrangements each of which guarantees that the estimation error is less than the error tolerance. However characterizing this space is difficult. Instead, our approach consists of considering different classes of sensor arrangements such that for each class, any of its sensor arrangement ensures that the

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[1]In this paper, we exclusively focus on the *sensing* part of 'sensor' networks and leave the *networking* part for future work.

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estimation error is always less than the error tolerance. We refer to these classes as Error Tolerant Arrangement Classes or ETAC's. We say that each instance of an arrangement from an ETAC is error tolerant. ETAC's are suitably characterized feasible solution spaces to the sensor arrangement problem. They are also convenient when we study mobile sensor networks. In the case of incidentally mobile sensors, ETAC's can be used as a tool to analyze the feasibility of a particular mobility model for sampling. In the case of intentionally mobile sensors, ETAC's may simplify and characterize the motion planning problem.

Here we consider 2D fields that are modeled as trigonometric polynomials. We assume Additive White Gaussian Noise (AWGN) model for measurements. We discuss three types of ETAC's - uniform sensor arrangements, Δ-dense sensor arrangements and incrementally constructed arrangements. The first two classes are characterized by a single parameter whereas as the name suggests the third class involves an explicit construction method.

The remainder of the paper is organized as follows. In Section II, we summarize the related work. In Section III, we formulate the sensor arrangement problem for general cases as well 2D trigonometric polynomial fields and state our approach. We discuss three types of ETAC's for trigonometric polynomial fields in Section IV in detail and conclude in Section V.

II. RELATED WORK

In almost every signal processing application, the uniform sampling arrangement is the most widely used sampling arrangement. According to the Shannon sampling theorem, the uniform sampling density must be at least the Nyquist rate (twice the highest frequency) in a band-limited signal in order to be able to reconstruct the signal perfectly [3]. Yen proposed a method to reconstruct a band-limited signal from non-uniform samples when the average sampling density is at least the Nyquist rate [4]. However, this method is numerically unstable. Duffin and Schaeffer first proposed conditions on the sampling pattern under which stable reconstruction of a band-limited signal is possible [5]. Such a sampling pattern is referred to as a *frame*. In [6], [7], [8], theoretical and numerical aspects of the reconstruction of a special type of band-limited signal, a trigonometric polynomial, from non-uniform samples have been studied. In this case the stable reconstruction is related to 'conditioning' of a certain matrix. In [7], Gröchenig shows that for Δ-dense sampling arrangements, this matrix is well-conditioned and stable reconstruction is possible. Our choice of Δ-dense arrangements as a class of error tolerant arrangements for trigonometric polynomial fields is inspired from the work in [7], [8].

In the area of sensor networks, the importance of dealing with inevitable non-uniform sensor arrangements is emphasized recently [9]. In [10], [11], [12], [13], [14] the authors deal with the sensor arrangement problem. The sensor arrangement problem is closely related to optimal experiment design and is also known as the *sampling design* problem. In [10], the authors consider the near optimal sensor arrangement for Gaussian processes. In [11], Zhang and Sukhatme propose an adaptive sampling approach for a single mobile sensor mounted on a boat in combination with a few static sensors. The approach is based on the optimal experiment design work. In [12], the authors consider elliptical motion paths for mobile underwater vehicles and consider the sensor arrangement problem restricted to these paths. In [13], [14], the authors consider the sensor arrangement problem for trigonometric polynomials in the Bayesian estimation framework. They consider the probabilistically generated sampling arrangements and use the asymptotic analysis techniques to obtain error tolerance values. In addition to the above work, in [15], the authors use active learning methods for mobile sensors for adaptive sampling.

III. PROBLEM FORMULATION

Let $f(\mathbf{x})$ indicate an unknown scalar field on the bounded domain $[0,1]^d$ in the d-dimensional space. We assume that $f(\mathbf{x})$ has the following form,

$$f(\mathbf{x}) = \sum_{k=1}^{m} a_k \phi_k(\mathbf{x}) \tag{1}$$

where ϕ_k's form a set of m known *orthonormal* basis functions. Orthonormality is defined as:

$$\int_D \phi_k(\mathbf{x})\phi_l(\mathbf{x}) d\mathbf{x} = \delta_{kl}, \quad k,l = 1,2,...,m \tag{2}$$

where $D = [0,1]^d$ and δ_{kl} indicates the Kronecker Delta function. Thus, a_k's constitute a set of m unknown *fixed* parameters of the field. We assume that sensors take measurements at a point. Typically, these measurements are noisy and we use an additive noise model to capture these imperfections. Thus, we assume that measurement $y(\mathbf{x})$ has the following form.

$$y(\mathbf{x}) = f(\mathbf{x}) + \varepsilon(\mathbf{x}) \tag{3}$$

where $\varepsilon(\mathbf{x})$ denotes noise. We refer to $\mathbf{x}$ as a *sensor location* or *sampling location*. We measure field values at n different sampling locations, where $n \geq m$. Let y_i denote the field value at location $\mathbf{x}_i$, where $i = 1, 2, \cdots, n$. Under these settings, the problem of estimating $f(\mathbf{x})$ is now reduced to estimating a_k's as a function of n data pairs $(y_i, \mathbf{x}_i)$. We use the following vector and matrix notation:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} \varepsilon(\mathbf{x}_1) \\ \varepsilon(\mathbf{x}_2) \\ \vdots \\ \varepsilon(\mathbf{x}_n) \end{pmatrix}, \quad \mathbf{z}_k = \begin{pmatrix} \phi_k(\mathbf{x}_1) \\ \phi_k(\mathbf{x}_2) \\ \vdots \\ \phi_k(\mathbf{x}_n) \end{pmatrix},$$

$$\mathbf{v}(\mathbf{x}) = \begin{pmatrix} \phi_1(\mathbf{x}) \\ \phi_2(\mathbf{x}) \\ \vdots \\ \phi_m(\mathbf{x}) \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} \mathbf{v^T}(\mathbf{x}_1) \\ \mathbf{v^T}(\mathbf{x}_2) \\ \vdots \\ \mathbf{v^T}(\mathbf{x}_n) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_1 & \mathbf{z}_2 & \cdots & \mathbf{z}_m \end{pmatrix}.$$

Further, let $X = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ be the set of n sampling locations. We refer to X as a *sensor arrangement*. $\mathbf{y}$ is referred to as an *observation vector* and $\mathbf{V}$ as an *observation matrix*. Using the above notation, we can express $f(\mathbf{x})$ and the measurement data as:

$$f(\mathbf{x}) = \mathbf{v}^T(\mathbf{x}) \cdot \mathbf{a} \tag{4}$$

$$\mathbf{y} = \mathbf{V}\mathbf{a} + \mathbf{e} \tag{5}$$

We assume that the noise vector $\mathbf{e}$ has zero mean, and its covariance matrix $\mathbf{Q} = \mathrm{E}[\mathbf{e}^T\mathbf{e}]$ is known and is positive definite. Given the setting in (5), the problem of finding a *good* estimate $\hat{\mathbf{a}}$ of $\mathbf{a}$ has been studied extensively in the literature on linear estimation [1],[2]. A variety of error metrics that capture the difference between $\hat{\mathbf{a}}$ and $\mathbf{a}$ and the estimators that minimize these metrics have been discussed. In this paper, we consider the *minimum variance unbiased estimator* (MVUE). MVUE is based on zero *bias*, *i.e.*, $\mathrm{E}\left[\hat{\mathbf{a}} - \mathbf{a}\right] = 0$, and minimizes the variance of the error $\hat{\mathbf{a}} - \mathbf{a}$.

$$\mathrm{E}\left[\|\hat{\mathbf{a}} - \mathbf{a}\|^2\right] = \sum_{k=1}^{m} \mathrm{E}\left[(\hat{a}_k - a_k)^2\right] \tag{6}$$

According to the Gauss-Markov theorem, the optimal estimate $\hat{\mathbf{a}}$ is given by:

$$\hat{\mathbf{a}} = (\mathbf{V}^*\mathbf{Q}^{-1}\mathbf{V})^{-1}\mathbf{V}^*\mathbf{Q}^{-1}\mathbf{y} \tag{7}$$

and the error covariance is given by:

$$\mathrm{E}\left[(\hat{\mathbf{a}} - \mathbf{a})(\hat{\mathbf{a}} - \mathbf{a})^T\right] = (\mathbf{V}^*\mathbf{Q}^{-1}\mathbf{V})^{-1} \tag{8}$$

Here $\mathbf{V}^*$ denotes the transpose of the complex conjugate of $\mathbf{V}$. Thus, the value of the minimum error variance is given by:

$$\min \mathrm{E}\left[\|\hat{\mathbf{a}} - \mathbf{a}\|^2\right] = \mathrm{trace}(\mathbf{V}^*\mathbf{Q}^{-1}\mathbf{V})^{-1} \tag{9}$$

Given X and $\mathbf{y}$ in this setting, this is the best one can do [2].

A. Sensor Arrangement Problem

The error covariance and the optimal estimation error corresponding to the MVUE as in (8) and (9) are functions of the sensor arrangement X alone and are independent of the observation vector. This is important from the sampling perspective. Through rest of the paper, we denote the optimal estimation error corresponding to the MVUE as $\mathrm{Err}(X)$ to capture its dependence on the sensor arrangement X. Thus,

$$\mathrm{Err}(X) = \mathrm{trace}(\mathbf{V}^*\mathbf{Q}^{-1}\mathbf{V})^{-1} \tag{10}$$

We refer to $\mathrm{Err}(X)$ as the *error metric*. We say that the field estimate $\hat{\mathbf{a}}$ is *good* if the error metric $\mathrm{Err}(X)$ is less than a certain error tolerance value. We formally define the sensor arrangement problem as follows:

$$\text{Find } X, \text{ s. t. } \mathrm{Err}(X) \le \Theta \tag{11}$$

where Θ denotes a certain tolerance value and we refer to it as *error tolerance*. Thus, the sensor arrangement problem involves finding a sensor arrangement that guarantees that the error metric is less than the error tolerance. Note that the solution to the sensor arrangement problem need not be unique. It is also possible to define the optimal sensor arrangement problem as finding the sensor arrangement that yields the optimal value of the error metric for a given number of samples. We would like to note that the sensor arrangement problem can be defined in a similar way for other settings that involve different types of fields, measurement models and the corresponding estimators.

B. Our Approach – Error Tolerant Arrangement Classes (ETAC's)

Suppose $S(\Theta)$ denote the set of all feasible solutions to the sensor arrangement problem in (11). Ideally we would like to characterize the space $S(\Theta)$. This would allow us to formulate an optimization problem over the space $S(\Theta)$. For instance, in case of intentionally mobile sensors, we can imagine a motion planning problem that involves *touring* sampling locations corresponding to a sensor arrangement in $S(\Theta)$ such that the energy spent in motion is minimal. On the other hand, in the incidental motion of sensors, we can pose a problem to verify whether a particular mobility model guarantees motion paths that conform with a solution in $S(\Theta)$. Unfortunately characterizing $S(\Theta)$ is complicated by the highly non-linear nature of the error metric. In order to deal with this problem, we follow a reverse approach. We define so called the *Error Tolerant Arrangement Classes* (ETAC's) for sampling. We say that a class of sensor arrangements where each arrangement satisfies certain properties is an ETAC if every sensor arrangement in that class guarantees that the corresponding error metric value is less than the error tolerance. More formally,

Definition: Let $\mathbb{X}_{\mathcal{A}}$ denote a set of sensor arrangements characterized by certain properties $\mathcal{A}$. We say that $\mathbb{X}_{\mathcal{A}}$ is an *Error Tolerant Arrangement Class* or an ETAC, if $\forall\ X \in \mathbb{X}_{\mathcal{A}},\ \mathrm{Err}(X) \le \Theta$, where Θ is a certain error tolerance.

Note that it is likely to be easier to analyze motion planning problem over the ETAC space because it is more explicitly defined than $S(\Theta)$. This approach is fairly general because it allows us to deal with different kinds of classes. For instance, in the case of intentionally mobile sensors, it is easier to plan motion that is characterized by certain properties. This will allow us to define an appropriate $\mathbb{X}_{\mathcal{A}}$ and identify conditions under which it is an ETAC. Similarly, incidental motion itself may lead to a certain $\mathbb{X}_{\mathcal{A}}$ and we can analyze conditions under which it is an ETAC. Through the rest of the paper we consider a few specific ETAC's for fields that are modeled as trigonometric polynomials.

C. Trigonometric polynomials

2D spatial fields can be modeled via number of ways. Here we limit our analysis to fields that are modeled as trigonometric polynomials; and we set up the estimation framework for this class of fields.

$$f(x,y) = \sum_{k=-M}^{+M} \sum_{l=-M}^{+M} a(k,l) e^{2\pi j(kx+ly)} \quad (12)$$

where $f(x,y)$ indicates the field on domain $[0,1] \times [0,1]$ and each basis function is of the form $e^{2\pi j(kx+ly)}$. Note that the field is real. Also, a trigonometric polynomial is nothing but a truncation of the Fourier series representation up to a certain number of finite terms. In this case we have $(2M+1)^2$ complex exponential basis functions and $(2M+1)^2$ unknown complex coefficients, $a(k,l)$. As before, we assume that the measurement model involves additive random noise. In this paper, we exclusively deal with Additive White Gaussian Noise (AWGN). Thus our measurement model is:

$$z(x,y) = f(x,y) + \varepsilon_{(x,y)} \quad (13)$$

where $z(x,y)$ is a field measurement value at location (x,y), and $\varepsilon_{(x,y)}$ is white gaussian noise with $E[\varepsilon_{(x,y)}] = 0$ and $E[\varepsilon^2_{(x,y)}] = \sigma^2$.

Suppose we take N samples at points $(x_1,y_1),(x_2,y_2),\cdots,(x_N,y_N)$. Let X denote the sensor arrangement of these sampling locations. We denote field values at the corresponding locations by $z_1,z_2,\cdots,z_N$ and we denote the observation vector by $\mathbf{z}$. We arrange unknown coefficients $a(k,l)$'s as a vector $\mathbf{a}$. Using the vector and matrix notation that we defined earlier, we can represent N samples in terms of the following system of linear equations.

$$\mathbf{z} = \mathbf{V}\mathbf{a} + \mathbf{e} \quad (14)$$

Note that each column of $\mathbf{V}$ can be indexed by the pair (k,l) corresponding to the unknown coefficient $a(k,l)$. Let $\mathbf{V}_{(k,l)}$ denote this column. Thus,

$$\mathbf{V}_{(k,l)} = \begin{pmatrix} e^{2\pi j(kx_1+ly_1)} \\ e^{2\pi j(kx_2+ly_2)} \\ \vdots \\ e^{2\pi j(kx_N+ly_N)} \end{pmatrix} \quad (15)$$

We further assume that noise values at any two locations are independent and identically distributed random variables. Thus $E[\mathbf{e}] = 0$ and $E[\mathbf{e}\mathbf{e}^T] = \sigma^2\mathbf{I}$, where $\mathbf{I}$ is the $N \times N$ identity matrix. Let $\hat{\mathbf{a}}$ indicate the estimate of $\mathbf{a}$ obtained using the MVUE. The Gauss-Markov theorem implies the following.

$$\hat{\mathbf{a}} = (\mathbf{V}^*\mathbf{V})^{-1}\mathbf{V}^*\mathbf{z} \quad (16)$$

$$\mathrm{E}\left[(\hat{\mathbf{a}}-\mathbf{a})(\hat{\mathbf{a}}-\mathbf{a})^T\right] = \sigma^2(\mathbf{V}^*\mathbf{V})^{-1} \quad (17)$$

$$\mathrm{Err}(X) = \mathrm{E}\left[\|\hat{\mathbf{a}}-\mathbf{a}\|^2\right] = \sigma^2 \mathrm{trace}\{(\mathbf{V}^*\mathbf{V})^{-1}\} \quad (18)$$

Let $\mathbf{T} = \mathbf{V}^T\mathbf{V}$. Thus,

$$\mathrm{Err}(X) = \sigma^2 \mathrm{trace}\{\mathbf{T}^{-1}\} \quad (19)$$

$\mathbf{T}$ is a matrix of size $(2M+1)^2 \times (2M+1)^2$. Note that each element of $\mathbf{T}$ can be indexed by indices of two columns of $\mathbf{V}$ that yield it. Thus,

$$\mathbf{T}_{kl,mn} = \mathbf{V}^*_{(k,l)} \cdot \mathbf{V}_{(m,n)} = \sum_{i=1}^{N} e^{-2\pi j[(k-m)x_i+(l-n)y_i]}$$
$$\text{where } k,l,m,n = -M,-M+1,\cdots,0,1,\cdots,+M \quad (20)$$

Note that $\mathbf{T}$ has a special structure. Each element of $\mathbf{T}$ just depends on k,l,m,n. Such a matrix is called as a *block Toeplitz* matrix.

Lemma: The error metric in (18) is invariant to the translation of the sensor arrangement. Suppose we translate each point in X by $s = (\Delta x, \Delta y)$ along x and y-axis and let $X+s$ denote the new arrangement after mapping all the points to $[0,1] \times [0,1]$. Note that a trigonometric polynomial $f(x,y)$ is a periodic function and the sampling domain $[0,1]\times[0,1]$ corresponds to one period. Thus for any point (x,y) outside this domain, it is always possible to find an equivalent point in $[0,1] \times [0,1]$. Then,

$$\mathrm{Err}(X+s) = \mathrm{Err}(X) \quad (21)$$

This invariance of the error metric to the translation of a sensor arrangement allows us to rearrange the sensor arrangement such that one of the sampling locations is always at $(0,0)$.

IV. ETAC's for Trigonometric Polynomials

A. Uniform Sensor Arrangements

Uniform sampling is undoubtedly the simplest sensor arrangement to specify and analyze. The only information that we need to specify a uniform arrangement is the number of samples. Suppose we take samples at points of a $N\times N$ 2D Cartesian grid with grid spacing $\frac{1}{N}$. We assume that the origin of the lattice is at $(0,0)$ because the error metric is translation invariant. We now prove that a uniform arrangement yields the optimal value of the error metric for a given number of samples.

Theorem: Let $f(x,y)$ be a 2D trigonometric polynomial as in (12). Suppose we take sample values at $N\times N$ sampling locations to form the sensor arrangement X. Let the measurement model be as in (13) that gives a system of linear equations (14). Let $N^2 \geq (2M+1)^2$. Then we have the following result.

$$\min_X \ \mathrm{Err}(X) = \frac{\sigma^2(2M+1)^2}{N^2} \quad (22)$$

We prove that the uniform arrangement of $N\times N$ yields this optimal error.

Proof: Note that each diagonal element of $\mathbf{T}$ is just N^2.

$$\mathbf{T}_{kl,kl} = \mathbf{V}^*_{(k,l)}\mathbf{V}_{(k,l)} = N^2$$

$\mathbf{T}$ is a Hermitian matrix. Hence all its eigenvalues are real [16]. In addition, $\mathbf{T}$ is a block Toeplitz matrix and hence it is positive definite [8]. Let $\lambda_1,\lambda_2,\cdots,\lambda_{(2M+1)^2}$ denote the eigenvalues of $\mathbf{T}$. Note that $\lambda_i>0$. We have the following relations.

$$\text{trace}(\mathbf{T}) = \sum_{i=1}^{(2M+1)^2} \lambda_i = N^2(2M+1)^2$$

$$\text{Err}(X) = \sigma^2 \text{trace}(\mathbf{T}^{-1}) = \sigma^2 \sum_{i=1}^{(2M+1)^2} \frac{1}{\lambda_i}$$

Since all λ_i's are positive, we use the Arithmetic Mean - Harmonic Mean inequality to obtain the following result:

$$\frac{\sum_{i=1}^{(2M+1)^2} \lambda_i}{(2M+1)^2} \geq \frac{(2M+1)^2}{\sum_{i=1}^{(2M+1)^2} \frac{1}{\lambda_i}}$$

Using the above inequality and previous two equations,

$$\sigma^2 \sum_{i=1}^{(2M+1)^2} \frac{1}{\lambda_i} \geq \frac{(2M+1)^2}{N^2}$$

$$\text{Err}(X) \geq \frac{\sigma^2(2M+1)^2}{N^2}$$

We obtain a lower bound on the value of error metric for any sensor arrangement X of size N^2. Next we prove that this lower bound is tight by showing that uniform arrangement indeed achieves it. Consider the uniform arrangement of $N \times N$ sampling locations denoted by X_U. The grid spacing is $\frac{1}{N}$. Let $\mathbf{T}^U$ denote the corresponding block Toeplitz matrix.

$$\mathbf{T}^U_{kl,mn} = \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1} e^{-2\pi j[(k-m)\frac{i_1}{N} + (l-n)\frac{i_2}{N}]},\ k \neq m, l \neq n$$

$$= \sum_{i_1=0}^{N-1} e^{-2\pi j(k-m)\frac{i_1}{N}} \cdot \sum_{i_2=0}^{N-1} e^{-2\pi j(l-n)\frac{i_2}{N}}$$

Using geometric series summation formula for each of the above expressions, we can easily show:

$$\mathbf{T}^U_{kl,mn} = 0,\ k \neq m, l \neq n$$

Thus, $\mathbf{T} = N^2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of size $(2M+1)^2 \times (2M+1)^2$. Therefore,

$$\text{Err}(X_U) = \text{trace}((\mathbf{T}^U)^{-1}) = \frac{\sigma^2(2M+1)^2}{N^2}$$

Thus, uniform arrangement achieves this lower bound. Hence,

$$\min_X \text{ Err}(X) = \frac{\sigma^2(2M+1)^2}{N^2}$$

■

With the above result, it is easy to find conditions under which the class of uniform sensor arrangements is an ETAC. Let Θ denote the error tolerance. We know that $\text{Err}(X_U) = \frac{\sigma^2(2M+1)^2}{N^2}$. We find the smallest N_o such that the inequality $\frac{\sigma^2(2M+1)^2}{N_o^2} \leq \Theta$ holds. Then, for any $N > N_o$, the error is always less that Θ. Thus, for any given Θ we can easily find conditions under which the uniform arrangements form an ETAC.

B. Δ-dense Sensor Arrangement

Δ-dense sensor arrangements have been studied in the context of stable reconstruction of band limited signals using non-uniform sampling and numerical issues involved therein [7], [8].

Definition: Let X denote a sensor arrangement in $[0,1] \times [0,1]$. We say that X is *Δ-dense*, if for any point $(x,y) \in [0,1] \times [0,1]$, there exists some $(x_i, y_i) \in X$ such that $\max\{|x - x_i|, |y - y_i|\} \leq \Delta$, *i.e.*, the L_∞ distance between (x,y) and (x_i, y_i) is at most Δ. Intuitively, within a square of size 2Δ, *i.e.*, an L_∞-disc of radius Δ, placed anywhere in $[0,1] \times [0,1]$, there is at least one sampling location. We refer to a collection of all Δ-dense sensor arrangements as the class of Δ-dense sensor arrangements.

Note that if X is Δ-dense, then it is also Δ'-dense for any $\Delta' > \Delta$. However a Δ'-dense arrangement need not be Δ-dense. Note that the L_∞ distance between any two nearest neighbor sampling locations from a Δ-dense sensor arrangement is at most 2Δ. Intuitively, a Δ-dense arrangement does not contain a square *hole* of size larger than 2Δ. Based on this geometric interpretation, we can conclude that any Δ-dense arrangement contains at least $\frac{1}{(2\Delta)^2}$ sampling points.

We find the conditions on Δ under which the Δ-dense arrangements form an ETAC. Let Θ be the error tolerance. In the previous section, we found the conditions under which uniform arrangements form an ETAC. Suppose we place $N \times N$ samples uniformly to form the sensor arrangement X_U as in the previous section. Let N be the smallest integer such that $\text{Err}(X_U) \leq \Theta$. Note that X_U is in fact $\frac{1}{2N}$-dense and $\text{Err}(X_U) = \frac{\sigma^2(2M+1)^2}{N^2}$. Loosely speaking, X_U is the *tightest* among all the $\frac{1}{2N}$-dense arrangements since it needs the smallest number of samples than any other arrangement in this class. Based on several simulation runs, we make the following claim.

Claim: Let X be a $\frac{1}{2N}$-dense sensor arrangement, where N is the smallest integer such that $\frac{\sigma^2(2M+1)^2}{N^2} \leq \Theta$. Then $\text{Err}(X) \leq \Theta$. Thus, the class of $\frac{1}{2N}$-dense sensor arrangements is an ETAC with respect to the error tolerance Θ.

Thus, the uniform arrangement with $N \times N$ samples is also the *tightest* in terms of the error because it represents the upper bound on the estimation error for any $\frac{1}{2N}$-dense arrangement. We observed this over several simulation runs, however we have not been able to prove the claim.

The class of Δ-dense sensor arrangements has a connection with the geometric *coverage problem* of covering the domain with L_∞-discs (square discs) of radius Δ. Note that any Δ-dense sensor arrangement is a valid solution to the *coverage* problem because if we locate a square of size 2Δ centered at each sampling location of a Δ-dense arrangement, then by the definition the entire domain $[0,1] \times [0,1]$ is guaranteed to be *covered* by these L_∞ discs. Furthermore, if we imagine a few mobile square shaped robots of size 2Δ *tour* through the sites of a Δ-dense arrangement, then the entire domain is *swept* by these robots. Note that the tour involves only translational motion of the robots (no rotations) according to the definition of a Δ-dense arrangement.

C. Incrementally Constructed Sensor Arrangements

Independent of the previous two ETAC's, we consider another type of ETAC inspired from an *active learning* method. As the name suggests we incrementally add chosen sampling sites to the already existing sensor arrangement one by one such that the estimation error is reduced at every step until the error is less than the error tolerance value. At this point the sensor arrangement is error tolerant. We propose a heuristic to construct such an arrangement. The class of arrangements obtained from various initial sensor arrangements is particularly useful for the case of intentionally mobile sensors. Suppose $(2M+1)^2$ points are already chosen in the sensor arrangement. These might be obtained by placing a few static sensors or measurements available from a few mobile sensors. Suppose we need to make a few additional measurements to guarantee that the error metric is within the error tolerance. In this case, the class of incrementally constructed sensor arrangements allows us to find a set of additional measurements.

Let X_n denote a sensor arrangement of n points. Suppose we wish to add a sampling location to X_n such that the estimation error is further reduced. Let $\mathbf{T}_n$ denote Toeplitz matrix corresponding to X_n. $\mathbf{T}_n = \mathbf{V}_n^* \mathbf{V}_n$. Suppose we add the sampling location (x_{n+1}, y_{n+1}) to obtain X_{n+1} and let $\mathbf{T}_{n+1}$ be the new Toeplitz matrix. We add a new observation to the already existing system of linear equations as in (14), we add a new row to $\mathbf{V}_n$. Let $\mathbf{v}$ denote the column vector corresponding to all the basis function values at (x_{n+1}, y_{n+1}). Thus,

$$\mathbf{T}_{n+1} = \mathbf{T}_n + \bar{\mathbf{v}}\mathbf{v}^T \tag{23}$$

where $\bar{\mathbf{v}}$ denotes the complex conjugate of $\mathbf{v}$. $\mathbf{T}_{n+1}^{-1}$ can be expressed in the closed form as follows [17]:

$$\mathbf{T}_{n+1}^{-1} = \mathbf{T}_n^{-1} - \frac{1}{1 + \mathbf{v}^T \mathbf{T}_n^{-1} \bar{\mathbf{v}}} \{\mathbf{T}_n^{-1}\} \bar{\mathbf{v}} \mathbf{v}^T \{\mathbf{T}_n^{-1}\} \tag{24}$$

Hence,

$$\text{trace}(\mathbf{T}_{n+1}^{-1}) = \text{trace}(\mathbf{T}_n^{-1}) - \frac{\mathbf{v}^T \mathbf{T}_n^{-1} \mathbf{T}_n^{-1} \bar{\mathbf{v}}}{1 + \mathbf{v}^T \mathbf{T}_n^{-1} \bar{\mathbf{v}}} \tag{25}$$

Therefore,

$$\text{Err}(X_{n+1}) = \text{Err}(X_n) - \frac{\mathbf{v}^T \mathbf{T}_n^{-1} \mathbf{T}_n^{-1} \bar{\mathbf{v}}}{1 + \mathbf{v}^T \mathbf{T}_n^{-1} \bar{\mathbf{v}}} \tag{26}$$

Since $\mathbf{T}_n$ is positive definite and Hermitian, the second term in the above equation is always positive and this shows that any extra sample reduces the estimation error. We consider the following optimization problem of finding (x_{n+1}, y_{n+1}) that reduces the estimation error the most.

$$(x_{n+1}^*, y_{n+1}^*) = \arg \max_{(x_{n+1}, y_{n+1})} \frac{\mathbf{v}^T \mathbf{T}_n^{-1} \mathbf{T}_n^{-1} \bar{\mathbf{v}}}{1 + \mathbf{v}^T \mathbf{T}_n^{-1} \bar{\mathbf{v}}} \quad \text{s.t.} \quad (x_{n+1}, y_{n+1}) \in [0,1] \times [0,1] \tag{27}$$

The above constrained optimization problem is highly nonlinear and has many local maxima. Figure 1 shows an arrangement of uniformly placed 8×8 points except one sample in the center is missing. Clearly, if we add the missing sample, we achieve the global optima. We numerically evaluated the value of $\text{Err}(X)$ as a function of location that is added to the current arrangement. Figure 2 shows $\text{Err}(X)$ as a function of (x, y). Note that there are many local optima. Moreover, the reduction in the estimation error for the missing sample location from the uniform arrangement is significantly higher than that for any other location. In our numerical simulations, we observed that various initial choices lead to local optima when we solve the above constrained optimization problem. Thus the choice of initial solution to the optimization solver is very crucial.

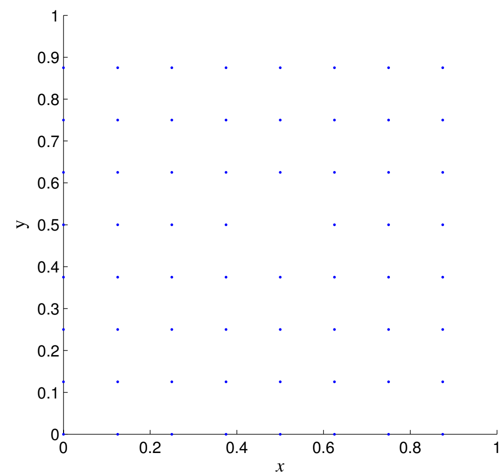

Fig. 1. $M = 3$, $\sigma^2 = 1$; samples are uniformly placed at points of a regular 8×8 grid, except one sample is missing.

In order to get around these local effects, we propose an approach to calculate initial solution to the optimization solver using a heuristic based on Voronoi diagrams of sampling locations. The underlying idea is that we find the Voronoi cell that has the maximum area and choose the farthest vertex of that cell from the source point as the initial solution. Intuitively a large Voronoi cell is present due to a large gap among the points and in this approach the optimization solver searches for the next sample in the vicinity of this large gap. As mentioned before, we assume toroidal topology of the domain $[0, 1] \times [0, 1]$. Hence we consider Voronoi diagrams on a torus rather than on a plane. We summarize our approach in the form of an algorithm as follows.

Once we find an initial guess at any step, we find the next sample and add it to the existing arrangement. At each step the estimation error is reduced and we continue till it is less than the tolerance error. At this point the arrangement is error tolerant.

Numerical example: Let $M = 3$ and $\sigma^2 = 1$. We chose 64 sampling sites randomly. They are shown in Figure 3.

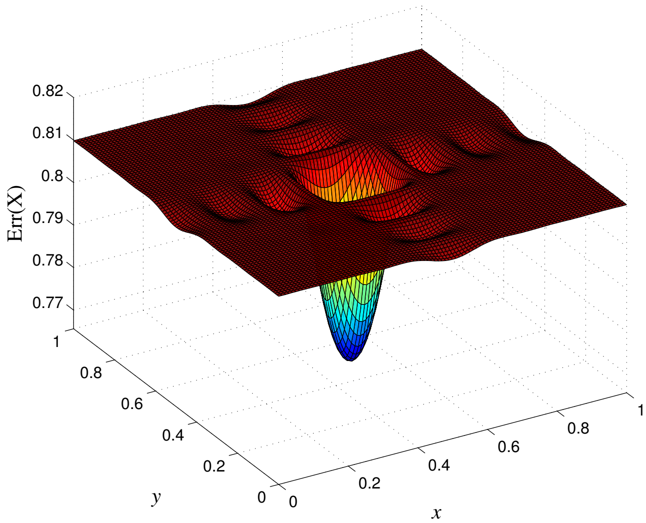

Fig. 2. For the arrangement shown in Figure 1, Err(X) is shown as a function of (x, y) where (x, y) is added to the already existing arrangement of samples. There are many local optima. The estimation error is reduced the most when the sample is placed at the missing sample site.

Algorithm 1 A heuristic based on Voronoi diagrams to find an initial guess for the optimization problem in (27)

Input: X, a sensor arrangement
Output: Initial guess (x_o, y_o)
Draw Voronoi diagram of points in X
Find the Voronoi cell P of the maximum area and its source (x, y)
Choose the vertex of P which is farthest from (x, y) as (x_o, y_o)

We use the nonlinear constrained optimization module of MATLAB to solve the optimization problem at each stage. We compare three schemes of adaptive sampling by adding 10 samples to the arrangement in each case. In the first scheme, we follow the brute force search method to find the global optima that yields the lowest estimation error. In the second scheme, we use our heuristic in Algorithm 1 to find an initial guess at each stage. In the third scheme, we choose initial guess randomly. In Figure 4, we show how Err(X) behaves as a function of any point (x, y) in the domain. We observe that for the first extra sample, our heuristic gives global optima. The figure also indicates that there are many local optima and usually the larger reduction in the estimation error occurs when a new sample is placed in the region of large gaps. In Figure 5, we compare the estimation error for 10 additional samples for these different schemes. The brute force search method yields the lowest estimation error at each stage. However it is extremely time consuming. Our heuristic approach yielded the estimation error always less than the random guess at each stage. Moreover it is easy to implement and the error is close to the brute force search method.

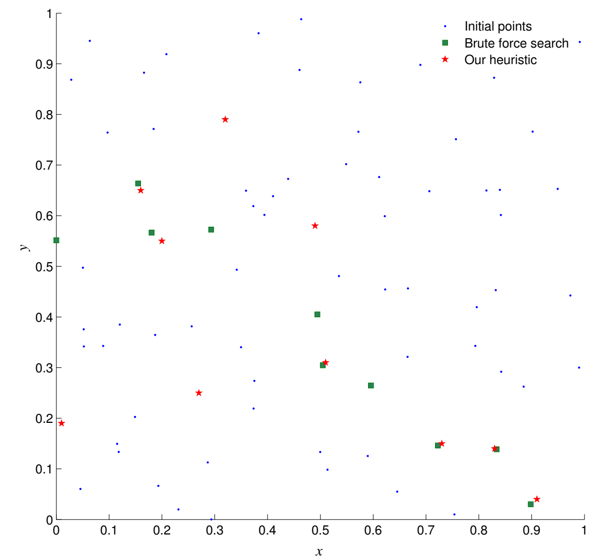

Fig. 3. 64 sample sites are randomly chosen. 10 additional sample points obtained using the brute force search method and our heuristic are shown along with the initial randomly chosen sites.

V. CONCLUSIONS AND FUTURE WORKS

In this paper, we defined the sensor arrangement problem – when and where to sample in order to obtain a good estimate of a field. We proposed the concept of ETAC's as a means to obtain suitably characterized feasible solution spaces to the sensor arrangement problem. Specifically, we discussed three types of ETAC's for the sensor arrangement problem of fields that are modeled as 2D trigonometric polynomials – uniform arrangements, Δ-dense arrangements and incrementally constructed arrangements. The first two classes are characterized in terms of a single parameter whereas the third class involved an explicit construction method. We showed that for a fixed number of samples that is a perfect square, the uniform arrangement yields the least estimation error for trigonometric polynomials under the AWGN measurement model. We discussed the relation between the class of Δ-dense arrangements and the geometric coverage problem. We proposed a Voronoi diagram based heuristic to select an initial guess at each step for adaptively constructing ETAC's. For future research, we would like to study ETAC's for other types of field models, implications of ETAC's on the motion planning problems for intentionally mobile sensors and ETAC as a tool to analyze sampling for incidentally mobile sensors.

REFERENCES

[1] D. Luenberger, *Optimization by Vector Space Methods*. John Wiley and Sons Inc., 1969.

[2] T. Kailath, A. h. Sayed, and B. Hassibi, *Linear Estimation*. Prentice Hall, 2000.

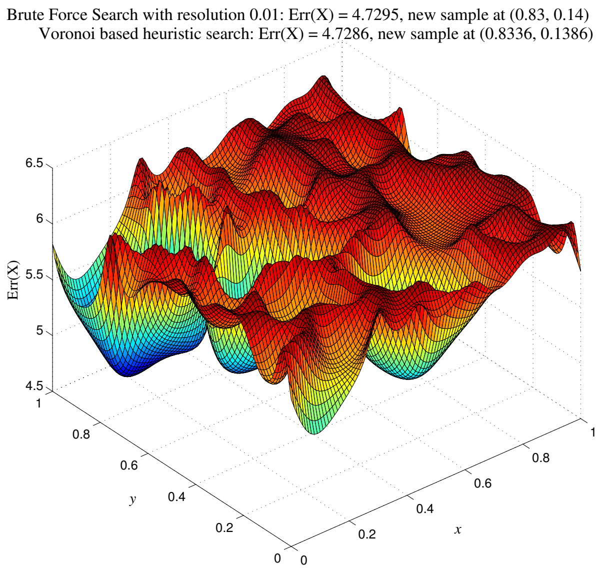

Fig. 4. For the arrangement shown in Figure 3, Err(X) is shown as a function of (x, y) where (x, y) is added to the already existing sensor arrangement. The resolution between consecutive points is 0.01. There are many local optima.

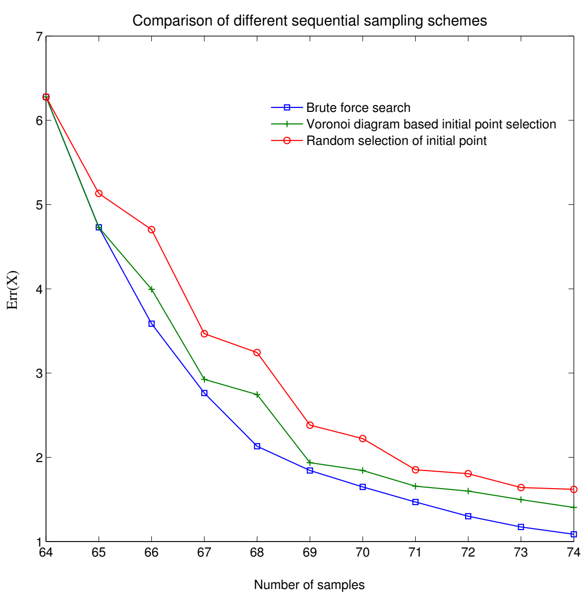

Fig. 5. Comparison of the estimation error values for three different schemes of incremental sampling: (1) the brute force search method for global minima at each step carried at resolution of 0.01 (2) our heuristic based on Voronoi diagrams to choose initial point for optimum search at each step (3) random selection of an initial point for optimum search at each step.

[3] C. E. Shannon, "Communication in the presence of noise," *Proceedings of the IEEE*, vol. 86, no. 2, pp. 447–457, 1998.

[4] J. L. Yen, "On nonuniform sampling of bandwidth-limited signals," *IRE Trans. on Circuit Theory*, vol. CT-3, pp. 251–257, December 1956.

[5] R. Duffin and A. Schaeffer, "A class of nonharmonic fourier series," *Transactions of American Mathematical Society*, vol. 72, no. 2, pp. 341–366, 1952.

[6] K. Gröchenig, "Reconstruction algorithms in irregular sampling," *Mathematics of Computation*, vol. 59, no. 199, pp. 181–194, July 1992.

[7] ——, "A discrete theory of irregular sampling," *Linear Algebra Applications*, vol. 193, no. 1, pp. 129–150, 1993.

[8] K. Gröchenig and T. Strohmer, *Theory and Practice of Nonuniform Sampling*. Kluwer Academic/Plenum Publishers, 2000, ch. Numerical and theoretical aspects of non-uniform sampling of band-limited images, pp. 283–322.

[9] D. Ganesan, S. Ratnasamy, H. Wang, and D. Estrin, "Coping with irregular spatio-temporal sampling in sensor networks," *SIGCOMM Comput. Commun. Rev.*, vol. 34, no. 1, pp. 125–130, 2004.

[10] C. Guestrin, A. Krause, and A. P. Singh, "Near-optimal sensor placements in gaussian processes," in *ICML*, 2005, pp. 265–272.

[11] B. Zhang and G. S. Sukhatme, "Adaptive sampling for estimating a scalar field using a robotic boat and a sensor network," in *IEEE International Conference on Robotics and Automation*, 2007, pp. 3673–3680.

[12] N. E. Leonard, D. A. Paley, F. Lekien, R. Sepulchre, D. M. Fratantoni, and R. E. Davis, "Collectivemotion, sensor networks, and ocean sampling," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 48–74, 2007.

[13] A. Nordio, C.-F. Chiasserini, and E. Viterbo, "The impact of quasi-equally spaced sensor layouts on field reconstruction," in *IPSN*, 2007, pp. 274–282.

[14] ——, "Quality of field reconstruction in sensor networks," in *INFOCOM*, 2007, pp. 2406–2410.

[15] A. Singh, R. Nowak, and P. Ramanathan, "Active learning for adaptive mobile sensing networks," in *IPSN '06: Proceedings of the fifth international conference on Information processing in sensor networks*. New York, NY, USA: ACM Press, 2006, pp. 60–68.

[16] G. Golub and C. V. Loan, *Matrix Computations*. Johns Hopkins University Press, Baltimore, MD, USA, 1996.

[17] T. Jaakkola, "6.867 machine learning class notes," http://courses.csail.mit.edu/6.867, 2007.