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Distributed Sliding Mode Control Design for a Class of Positive Compartmental Systems

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***Abstract*— We concern ourselves with flow control of a class of positive compartmental systems, which represent interconnected networks of reservoirs and the flow of material between these reservoirs. Using a sliding mode control approach, we design controllers which, using only knowledge of the amount of material in their own section and the flow out of the network, match a desired throughput profile between the inlet and outlet ports. One of the advantages of this control strategy is that, since it is not required that each section have knowledge of the states of all other sections, it requires only a limited amount of communication between sections. Exploiting the particular compartmental structure of the system, we give proofs of asymptotic stability of the control scheme, along with an upper bound on the time needed for the tracking error to fall below a prescribed level. We also analyze the role of the positivity constraints on the state and control variables of the system on closed-loop performance.**

I. INTRODUCTION

We are concerned with the control of positive compartmental systems. These types of systems represent the flow of material through a network of interconnected reservoirs and their dynamics is dictated by mass conservation laws and the underlying structure of the interconnection network [3]. These types of models can be used to describe a variety of different systems including automobile or aircraft traffic flow, job-balancing in computer clusters [5], or any system of connected reservoirs with natural constraints, such as irrigation networks [4].

In this paper, we show how to design distributed control policies for such systems in which controllers in charge of different sections can locally direct flow so as to match a desired network outflow. By "distributed" we mean that controllers need only share a small amount of information regarding the states of their sections with each other. Our design strategy builds on techniques from the theory of sliding mode control. In particular, and as is typical of this field, we can guarantee that this control law produces a tracking error between the true and desired outflow which converges to zero in finite time, provided some assumptions on the system's parameters hold. This is in contrast with some of our earlier results presented in [2], in which the tracking error could only be guaranteed to have bounded L_2-norm. For the present result to hold, however, we need to know the inflow rate function exactly.

In order to demonstrate the applicability of our method, we use it to design control laws for a very simple network of air traffic flow described by an Eulerian flow model, results of which are presented in Section IV. Eulerian models have recently become popular in the control-oriented modeling of air traffic in the National Airspace System. This type of model describes the dynamics of groups of aircraft instead of focusing on individual vehicles in the limit of dense traffic (see [8] for a survey of available Eulerian framework and a comparison of their predictive abilities).

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II. PROBLEM DESCRIPTION

We focus on networks consisting of one inlet and one outlet port. The route between the inlet and outlet port is represented by a directed graph connecting different sections. The flow out of a given section can diverge and enter multiple subsequent sections, including itself, and similarly, flow from multiple sections can converge and enter one subsequent section.

For our later modeling assumptions to make sense, we require that the graph is such that (i) every vertex is connected to the inlet and outlet ports, (ii) given a specific vertex (including the outlet port), every path between the inlet port and this vertex has the same length. In that case, we can structure the graph into levels by saying that a section i belongs to level L if all paths from the inlet to section i have length L. We will use $\mathcal{S}_L$ to represent the set of sections in level L, that is, if section i is in level L, then $i \in \mathcal{S}_L$. We assume that there are $m+2$ levels overall in the graph, namely $L \in \{\mathcal{I}\} \cup \{1, ..., m\} \cup \{\mathcal{F}\}$, where the symbols $\mathcal{I}$ and $\mathcal{F}$ refer to the 'initial' and 'final' levels, respectively. We make the convention that material flows from sections in level L to sections in level $L+1$. Material flowing into the network at the inlet port enters sections in the initial level and material flowing out of sections in the final

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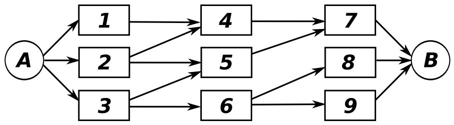

Fig. 1. Example of a network of flow between inlet port, A, and outlet port, B. Sections 1, 2 and 3 are in level $\mathcal{I}$, sections 4, 5, 6 are in level 1 and sections 7, 8, 9 are in level $\mathcal{F}$.

level exits the network at the outlet port. An example of this type of network structure is given in Fig. 1.

The state of a section $i \in \mathcal{S}_L$, which represents the amount of material in that section, is denoted by $x_i(t)$. Since we are dealing with positive systems, $x_i(t)$ must be a nonnegative value for all t. Material in each section is assumed to be traveling at the maximum allowable speed, corresponding to a section traversal time of τ_i. The control input, u_i, can be thought of as a recirculation rate, that is, the control input is modeled as taking part of the natural outflow of the section and recirculating it back into that same section, thus effectively reducing the outflow of the section. With parameters defined in this way, we can define the outflow rate of section i as

$$f_i(t) = \frac{x_i(t)}{\tau_i} - u_i(t).$$

The terms β_{ij}, $0 \leq \beta_{ij} \leq 1$, denote the fraction of the outflow rate of section i that flows into section j. Clearly, if $i \in \mathcal{S}_L$ and $L \neq \mathcal{F}$, $\sum_{j \in \mathcal{S}_{L+1}} \beta_{ij} = 1$, since all material leaving a section in level L must enter a section in level $L+1$ by definition of the levels. We consider the fractions β_{ij} to be given as a part of the network model, which is only appropriate when considering operations around a given regime with routing control performed at a higher level. Incorporating flow routing as a control input is the subject of future work. The time-varying rate at which material enters the system from port A is denoted by d. Finally, $0 \leq \gamma_i \leq 1$ denotes the fraction of the inflow rate from port A that enters into section i. Our assumption on the structure of the network requires that $\gamma_i \neq 0$ if and only if $i \in \mathcal{S}_{\mathcal{I}}$.

The dynamics of section $i \in \mathcal{S}_L$ can now be given by

$$\dot{x}_i = -\frac{x_i}{\tau_i} + u_i + \sum_{j \in \mathcal{S}_{L-1}} \beta_{ji} \left(\frac{x_j}{\tau_j} - u_j \right) + \gamma_i d, \quad (1)$$

which is a continuous time analog of the Eulerian model of air traffic flow introduced in [7].

The global outflow from sections in the final level, namely $z = \sum_{i \in \mathcal{S}_{\mathcal{F}}} f_i$, is used as a performance output. The control design problem is then to determine feedback control policies $u = (u_1, ..., u_n)^T$ such that the system is internally stable in closed-loop and the output z behaves satisfactorily. More precisely, we measure the performance of the closed-loop system by comparing z to a desired output z_d generated by a given target system

$$\begin{aligned} \dot{\xi} &= \tilde{A}\xi + \tilde{B}d \\ z_d &= \tilde{C}\xi, \end{aligned} \quad (2)$$

with $\xi(t) \in \mathbb{R}^{\tilde{n}}$ for all $t \geq 0$. The parameters of (2) must be chosen to generate a desired output that is meaningful in relation to the system being controlled. The faster the error $e = z - z_d$ approaches zero (or falls below a prescribed level ϵ), i.e., the smaller T_ϵ defined as follows is, the better the performance:

$$T_\epsilon := \min\{t \mid |e(s)| \leq \epsilon, \ \forall \ s \geq t\}.$$

In other words, we desire to achieve asymptotic tracking of z_d in closed-loop. In addition to these closed-loop stability and performance requirements, the following constraints are imposed on the control input u:

- *Positivity:* Since the control input, $u_i(t)$, acts only to recirculate a fraction of the nominal outflow of section i and thus can only decrease the outflow, it must satisfy $0 \leq u_i(t) \leq \frac{x_i(t)}{\tau_i}$.
- *Decentralization:* In order to limit the amount of inter-section communication, $u_i(t)$ should not depend on all the components of the state vector $x(t) = (x_1(t), ..., x_n(t))^T$ but, preferably, only on $x_i(t)$ and a small number of other components.

III. Control Strategy and Closed-Loop Properties

In order to achieve our control design objective while respecting the positivity and decentralization constraints, we propose to use a control law of the form

$$u_i(t) = \alpha(t) \, \frac{x_i(t)}{\tau_i} \text{ for all } i = 1, ..., n, \quad (3)$$

for some function α to be determined later. The rationale for the structure of (3) is that (i) it agrees with the interpretation that u_i is a fraction of the outflow and positivity of the controller is easily recognized, (ii) if we could make α constant, this structure would give a fully decentralized controller. Substituting this expression into equation (1), we see that the closed-loop dynamics of each section is

$$\dot{x}_i = -(1-\alpha(t))\frac{x_i}{\tau_i} + \sum_{j \in \mathcal{S}_{L-1}} \beta_{ji}(1-\alpha(t))\frac{x_j}{\tau_j} + \gamma_i d, \quad (4)$$

and that, if $0 \leq \alpha(t) \leq 1$ for all t,

$$\left. \begin{array}{r} x_i(0) \in \mathbb{R}_+ \text{ for all } i \\ d(t) \in \mathbb{R}_+ \text{ for all } t \geq 0 \end{array} \right\} \Rightarrow x_i(t) \in \mathbb{R}_+, \ \forall \ i.$$

To simplify later developments, we need to introduce some new notation. First, we combine the dynamics of all individual sections specified by (1) in the form

$$\begin{aligned} \dot{x} &= Ax + Bu + B_d\, d \\ z &= Cx + Du, \end{aligned} \tag{5}$$

where matrices A, B, B_d, C, and D are derived from the interconnection of subsystems (1). We also rewrite control law (3) as $u = Kx$ where $K = \alpha T^{-1}$, and $T^{-1} = \mathrm{diag}(\frac{1}{\tau_1}, \frac{1}{\tau_2}, \ldots, \frac{1}{\tau_n})$. With this representation of K, α is the only design parameter.

With this notation, the closed-loop system is given by

$$\begin{aligned} \dot{x} &= \left(A + \alpha B T^{-1}\right) x + B_d\, d \\ z &= \left(C + \alpha D T^{-1}\right) x. \end{aligned} \tag{6}$$

Due to the specific structure of the interconnected systems, illustrated in Fig. 1, the output of the system is the controlled outflow of the system at port B and depends only on the material in level $\mathcal{F}$. Thus, we see that $C = DT^{-1}$, both of which will be denoted by w^T where $w_i = \frac{1}{\tau_i}$ for $i \in \mathcal{S}_{\mathcal{F}}$ and $w_i = 0$ otherwise. Thus, we can express the output as

$$z = (1-\alpha)\, w^T x. \tag{7}$$

A. Closed-loop Stability

We start by focusing on closed-loop stability. We show that, as long as the function α in control law (3) is chosen so that $0 \leq \alpha(t) < 1$ for all $t \geq 0$, the closed-loop system (6) exhibits global asymptotic stability. This is mainly due to the fact that, because of its compartmental structure, system (6) admits a fixed Lyapunov function, which is independent of the time-varying function α.

It should be noted, however, that traditional Lyapunov stability results, as given, e.g., in [6], cannot be readily applied in the present problem, since our candidate Lyapunov function will only be decreasing over $\mathbb{R}^n_+$, which does not contain the origin of state space in its interior. We thus need to provide a variant of these results suited to positive systems. This is the content of the following theorem.

Theorem 1: Let system (5) be a positive system (i.e., $x(t) \in \mathbb{R}^n_+$ for all t) and $x = 0$ be an equilibrium point of system (5). Assume there exists a continuously differentiable function $V : \mathbb{R}^n_+ \to \mathbb{R}$ such that

$$\begin{gathered} V(0) = 0 \text{ and } V(x) > 0,\ \forall\, x \in\ \mathbb{R}^n_+ - \{0\} \\ \lim_{\|x\|\to\infty,\ x\in\mathbb{R}^n_+} V(x) = +\infty \\ \dot{V}(x) < 0,\ \forall\, x \in\ \mathbb{R}^n_+ - \{0\} \end{gathered}$$

Then, $x = 0$ is globally asymptotically stable.

The proof is presented in [1] and follows similar steps as the proof of Lyapunov's stability theorem given in [6].

Using Theorem 1, we can prove the stability of system (5) under state-feedback control with varying α. With $K = \alpha(t)T^{-1}$ we will show that the closed-loop system is stable for any $\alpha(t)$ satisfying $0 \leq \alpha < 1$.

First we must show that the closed-loop system is positive. In closed-loop, recalling the expression for $\dot{x}_i$ in equation (4) with $d = 0$, $0 \leq \alpha < 1$, $x_i \geq 0,\ \forall\ i$,

$$(1-\alpha)\sum_{j=1}^{n} \frac{\beta_{ji}}{\tau_j} x_j \geq 0 \tag{8}$$

$$-(1-\alpha)\frac{1}{\tau_i} x_i \leq 0. \tag{9}$$

We see that whenever $x_i = 0$, the only possibly negative component of $\dot{x}_i$, namely (9), is zero. Thus, if $x(0) \geq 0$,

$$\begin{aligned} x_i = 0 &\Rightarrow \dot{x}_i \geq 0 \\ &\Rightarrow x_i(t) \geq 0,\ \forall\ t \geq 0. \end{aligned}$$

Now we can use Theorem 1 to show that the closed-loop system is stable. Define a function $V(x) = l^T x$ where l is a column vector of length n defined such that $l_1 > l_2 > \cdots > l_n > 0$. Since we are dealing with positive systems, it is clear that $V(x) \geq 0$ for all $t \geq 0$. Also note that, on $\mathbb{R}^n_+$, $V(x)$ is radially unbounded, $V(0) = 0$, and $V(x) > 0$ if $x \neq 0$. Taking the time derivative of V and after some algebra we have

$$\dot{V}(x) = -(1-\alpha)\sum_{i=1}^{n} \left\{ \left[l_i - \sum_{j\in\mathcal{S}_{L+1}} l_j \beta_{ij} \right] \frac{x_i}{\tau_i} \right\}.$$

Let $\bar{l}_i = \max_{j\in\mathcal{S}_{L+1}} l_j < l_i$, thus we have

$$\dot{V}(x) < -(1-\alpha)\sum_{i=1}^{n} \left\{ \left[l_i - \bar{l}_i \right] \frac{x_i}{\tau_i} \right\}.$$

Since $0 \leq \alpha < 1$, and $x \geq 0$, we see that $\dot{V}(x) \leq 0$ along trajectories of the system. Further, $\dot{V}(x) < 0,\ \forall\ x > 0$. Thus, using Theorem 1 we conclude that the closed-loop system is globally asymptotically stable for $0 \leq \alpha(t) < 1$ and $x(0) \in \mathbb{R}^n_+$.

B. Development of Sliding Mode Controller

In this section, we make a specific choice of function α in (3), so as to solve the tracking problem outlined in Section II. From here on, we assume that the input d to both system (5) and target system (2) is a known step function. In other words, we are interested in designing a controller which achieves good tracking performance in closed-loop *for a given input*.

More precisely, we use sliding mode control to bring the closed-loop system to the manifold defined in state-space by the equation $e = z - z_d$. The sliding mode controller will drive the system to $e = 0$ and then slide along the $e = 0$ manifold, which is precisely how we would like our system to behave. Using the system output specified in (7) and the reference output specified in (2) we define the output error as

$$e = (1-\alpha)\, w^T x - \tilde{C}\xi.$$

The general method of developing a sliding mode controller involves choosing a Lyapunov candidate function, V, and choosing dynamics for the control parameter such that $\dot{V} \leq -c_0|e|$, for some constant $c_0 > 0$. Here, our only control parameter is the value $\alpha(t)$ and thus our goal is to use the theory of sliding mode control to specify the dynamics of α, that is, find an expression for $\dot{\alpha}$.

We will start with the Lyapunov candidate function $V(x,\xi) = \frac{1}{2}e^2$. Computing $\dot{V}$ along trajectories of the system, we have

$$\dot{V} = e\left[(1-\alpha)\, w^T \dot{x} - \tilde{C}\dot{\xi}\right] - e\dot{\alpha}w^T x.$$

We want to find $\dot{\alpha}$ such that $\dot{V} \leq -c_0|e|$. Given this expression for $\dot{V}$, we must find $\rho(x,\xi) \geq 0$ satisfying

$$\rho(x,\xi) \geq \left|\frac{(1-\alpha)\, w^T \dot{x} - \tilde{C}\dot{\xi}}{w^T x}\right| \qquad (10)$$

Once such a bound is found, we will have

$$\dot{V} \leq |e|\rho(x,\xi) w^T x - e\dot{\alpha}w^T x.$$

With the following choice of $\dot{\alpha}$

$$\dot{\alpha} = \eta(x,\xi)\mathrm{sgn}(e), \qquad (11)$$

where

$$\eta(x,\xi) = \eta_0 + \frac{c_0}{w^T x} + \rho(x,\xi) \geq 0, \qquad (12)$$

for some constant $\eta_0 \geq 0$, we obtain

$$\dot{V} \leq -c_0|e| - \eta_0\left(w^T x\right) \leq -c_0|e|. \qquad (13)$$

Note that because α depends on the state of all sections in level $\mathcal{F}$, the proposed control law is not *fully* decentralized, in spite of the structure (3): the control input applied to each section not only depends on its local state, but also on the global outflow of the system.

In order to construct such a controller, we must find a function $\rho(x,\xi)$ that satisfies (10). We can split up the absolute value expression on the right hand side of (10) and find upper bounds on the terms

$$\left|\frac{(1-\alpha)\, w^T \dot{x}}{w^T x}\right| \text{ and } \left|\frac{\tilde{C}\dot{\xi}}{w^T x}\right|. \qquad (14)$$

Since $\gamma_i = 0,\ \forall\ i \in \mathcal{S}_{\mathcal{F}}$, and recalling that, due to the interconnection structure of the system, all material entering sections in $\mathcal{S}_{\mathcal{F}}$ must come from sections in $\mathcal{S}_m$, and all material leaving a section in $\mathcal{S}_m$ must enter a section in $\mathcal{S}_{\mathcal{F}}$ and that we require α to satisfy $0 \leq \alpha \leq 1$ we find the following bound on the first term in (14)

$$\left|\frac{(1-\alpha)\, w^T \dot{x}}{w^T x}\right| \leq \frac{1}{\tau_{min}}\left(\frac{\sum_{j\in\mathcal{S}_m}\frac{x_j}{\tau_j}}{w^T x}\right)$$

where $\tau_{min} = \min_{i\in\mathcal{S}_{\mathcal{F}}}\{\tau_i\}$.

In order to find a bound for the second term in (14) with the dynamics of ξ defined in (2), we need to make the following assumptions: $\xi(0) = 0$, $d(t)$ is step or square wave function with amplitude $d_0 > 0$ and (2) is a stable first order linear system with $\tilde{A} < 0$ and $\tilde{B} > 0$. We can then solve for $\xi(t)$ and differentiate to find the maximum of $\dot{\xi}(t)$ and thus arrive at the following bound

$$\left|\frac{\tilde{C}\dot{\xi}}{w^T x}\right| \leq \frac{\tilde{B}d_0}{w^T x}.$$

The details of these derivations can be found in [1].

Thus we arrive at an expression for $\rho(x,\xi)$ which satisfies (10),

$$\rho(x,\xi) = \frac{1}{\tau_{min}}\left(\frac{\sum_{j\in\mathcal{S}_m}\frac{x_j}{\tau_j}}{w^T x}\right) + \frac{\tilde{B}d_0}{w^T x}.$$

With this choice of $\dot{\alpha}$ given in (11), $V = \frac{1}{2}e^2$ is always decreasing along trajectories of the system, thus, $|e|$ decreases until it reaches zero and $e = 0$ for all t after this point.

In practice (and even in simulation), there will be delay between the time that e changes sign and the time that $\dot{\alpha}$ switches from $\eta(x,\xi)$ to $-\eta(x,\xi)$, or *vice versa*. This delay will cause e to become nonzero for some amount of time before being brought back to the manifold $e = 0$. This will occur each time the system approaches the manifold, thus resulting in chattering of the controller. A common resolution of this problem is to implement the control law

$$\dot{\alpha} = \eta(x,\xi)\mathrm{sat}\left(\frac{e}{\epsilon}\right) \qquad (15)$$

for some chosen $\epsilon > 0$ where

$$\mathrm{sat}(y) = \begin{cases} y, & \text{if } |y| \leq 1 \\ \mathrm{sgn}(y), & \text{if } |y| > 1 \end{cases}.$$

If implementing this control law on a physical system, e will *not* remain fixed at $e = 0$ once the trajectory first reaches the manifold, instead, $|e| \leq \epsilon$ will be satisfied once e comes within ϵ of zero. The value of ϵ is a design parameter and must be chosen as an acceptable level of error.

C. Performance Measure

With the proposed dynamics of α given by (15), the natural performance measure is the time required for the system to reach the boundary of $|e| \leq \epsilon$, which, as mentioned in Section II, we will denote as T_ϵ.

Define $W = \sqrt{2V} = |e|$ and notice that $D^+W \leq -c_0$, where D^+ indicates the upper right-hand derivative. Now we can use the comparison lemma (see [6]) along with the upper bound on $\dot{V}$ given in (13) to state

$$W\left(e\left(t\right)\right) \leq W\left(e\left(0\right)\right) - c_0 t. \tag{16}$$

Use (16) to find a value of T_ϵ which satisfies

$$|e(T_\epsilon)| \leq \epsilon. \tag{17}$$

By solving $|e(0)| - c_0 T_\epsilon \leq \epsilon$ we find that the smallest value of T_ϵ which satisfies (17) is $T_\epsilon = \frac{|e(0)|-\epsilon}{c_0}$. For all $t \geq T_\epsilon$, we know that $|e(t)| \leq \epsilon$.

D. Constraints on Controller

Throughout the derivation of the dynamics of α we have assumed that $0 \leq \alpha \leq 1$. Looking closely at the final expression for $\dot{\alpha}$ given in (11), we see that α will remain bounded from above by 1, provided that the state of the target system remains positive. To see this, note that

$$\alpha(t) = 1 \Rightarrow e(t) = -\tilde{C}\xi < 0.$$

Hence, according to control law (11),

$$\alpha(t) = 1 \Rightarrow \dot{\alpha}(t) < 0,$$

and we can then conclude that $\alpha(t) < 1$ for all time $t > 0$, if $\alpha(0) < 1$. As a result, the state of closed-loop system (4) remains strictly positive for all time under the sliding mode control law (11).

Unfortunately, this control law does not ensure that α remain positive. If $(1 - \alpha(t))w^T x(t) < \tilde{C}\xi(t)$ over some time interval (i.e., if $e < 0$ and α is decreasing over an interval), it is possible for α to be driven negative. This means that the input $u(t) = \alpha(t)T^{-1}x(t)$ may be negative for some time t.

In order to ensure that the control input is always physically meaningful, as captured by the *Positivity* requirement, we can simply set α to zero whenever relation (11) fails to naturally impose this constraint. As we have proved earlier, setting α to zero in such a way does not affect closed-loop stability. However, this may affect tracking performance, prevent the error e to monotonically decrease, or even converge to zero.

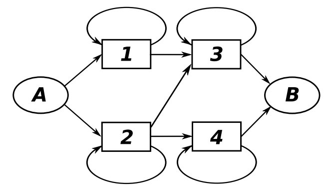

Fig. 2. Airspace between airports A and B divided into sections of 1-D flow, arrows indicate direction of flow.

Extensive numerical simulations for systems consisting of two levels of sections, with parameters and network input similar to those used in the example presented in Section IV, were performed. The results seem to indicate that for a certain class of parameters of the model and target systems, initial conditions and network input, control law (11) results in a strictly positive function α. This leads us to conjecture that, for every system of the form (5), there exists a class of target systems for which control law (11) leads to finite-time convergence of the tracking error e, while satisfying both the *Positivity* and *Decentralization* requirements. Characterizing this class of target systems rigorously is the subject of our current research.

IV. Application Example

In this section, we focus on the network presented in Fig. 2 representing the flow of aircraft between two airports and apply the sliding mode controller developed in Section III-B. Compartments correspond to sections of airspace and the state and dynamics of each section represent aggregate quantities, see [1] or [2] for more details. In this network, all aircraft take off from airport A and land at airport B. The state-space representation of this model is given by

$$\dot{x} = \begin{bmatrix} -\frac{1}{\tau_1} & 0 & 0 & 0 \\ 0 & -\frac{1}{\tau_2} & 0 & 0 \\ \frac{1}{\tau_1} & \frac{\beta_{23}}{\tau_2} & -\frac{1}{\tau_3} & 0 \\ 0 & \frac{\beta_{24}}{\tau_2} & 0 & -\frac{1}{\tau_4} \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -\beta_{23} & 1 & 0 \\ 0 & -\beta_{24} & 0 & 1 \end{bmatrix} u + \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ 0 \\ 0 \end{bmatrix} d$$

$$z = \begin{bmatrix} 0 & 0 & \frac{1}{\tau_3} & \frac{1}{\tau_4} \end{bmatrix} x + \begin{bmatrix} 0 & 0 & -1 & -1 \end{bmatrix} u,$$

where $x = [x_1, \ldots, x_4]^T$ is a vector of state variables and $u = [u_1, \ldots, u_4]^T$ is the control input.

The parameter values used in the simulation are

$$\tau_1 = 0.625,\ \tau_2 = 0.938,\ \tau_3 = 0.208,\ \tau_4 = 0.250,$$

$$\beta_{23} = 0.7,\ \beta_{24} = 0.3,\ \gamma_1 = 0.5,\ \gamma_2 = 0.5.$$

The traversal times, τ_i, have units of hours and were chosen to be comparable to the parameter values used in [7]. We desire to match a first-order system with time-constant 0.5. The system and model values used here

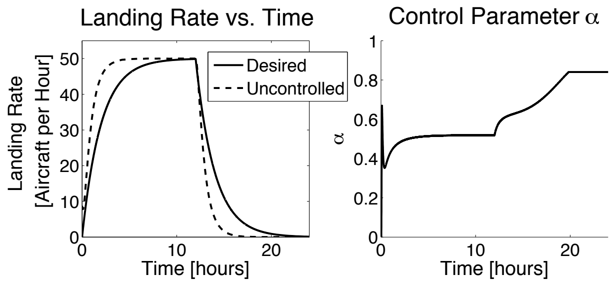

Fig. 3. **Left:** Desired landing rate and uncontrolled landing rate. **Right:** Control parameter α as a function of time.

are the same as those used in the example in [2]. The constant values in (12) were chosen to be $\eta_0 = 0,\ c_0 = 10$. We chose $\epsilon = 1$, indicating that an acceptable error in the landing rate is 1 aircraft per hour.

The initial conditions used are $x_0 = [1,1,1,1]^T$, $\xi_0 = 0$. With these initial conditions, the initial error is approximately 8.8 aircraft per hour. The input, $d(t)$, is a square wave function oscillating between 0 and 50 aircraft per hour, with a period of 24 hours and pulse width of 50%. With this input, a total of 600 aircraft enter the system. The uncontrolled system output is plotted along with the desired landing rate in Fig. 3. At this scale, the controlled landing rate is indistinguishable from the desired landing rate for most of the time history and is therefore not included in the plot.

A plot of α over the course of the simulation is presented in Fig. 3. Notice that α does not go below zero at any time during the simulation. Thus, we can conclude that $|e(t)| \leq \epsilon$ will be satisfied for all $t \geq T_\epsilon$. Also note that α remains fixed for all t greater than about 20 hours. This is because at this point, both the controlled outflow and the desired landing rate become less than ϵ and the adaptive controller is turned off.

Under the proposed control, the landing rate error is guaranteed to be within ϵ of zero by $T_\epsilon = 0.78$ hours. In simulation, the error actually enters this region before about 0.1 hours. Over the course of the 24 hours of the simulation, the uncontrolled system accumulates a total error of 104 aircraft, while the controlled system accumulates an error of only 2 aircraft.

V. Conclusion & future work

We developed a sliding mode control technique for network flow control with the goal of tracking a desired outflow. We provide a performance measure in the form of an upper bound on the time required for the outflow error, e, to be within some specified value, ϵ, after which $|e| \leq \epsilon$ will be satisfied. This performance measure is valid only for certain combinations of network, inflow rate, desired outflow rate and initial conditions, which result in nonnegative control input, i.e., $\alpha \geq 0$, under the proposed control law. We will continue to search for a mathematical description of the conditions under which the control input is nonnegative and thus the performance measure holds.

The controller developed here is not fully decentralized since it depends on the transfer of information from final sections to all other sections. Alternatively, we will look into methods that only involve the exchange of information between nearest neighbors. Such a method will likely result in the use of different values of α, which may evolve independently, for each section.

Focusing on air traffic management, large networks and multiple take-off and landing airports should be incorporated to better model realistic air traffic flow scenarios. One of the challenges associated with modeling multiple airports is that sections are not naturally layered into levels, in contrast to our assumptions of Section II. We have investigated the application of this control scheme to larger single inlet/outlet networks with some results and challenges detailed in [1].

VI. Acknowledgments

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