

# Local Stability Analysis For Uncertain Nonlinear Systems Using A Branch-and-Bound Algorithm

Ufuk Topcu, Andrew Packard, Peter Seiler, and Gary Balas

**Abstract**—We propose a method to compute invariant subsets of the region-of-attraction for the asymptotically stable equilibrium points of polynomial dynamical systems with bounded parametric uncertainty. Parameter-independent Lyapunov functions are used to characterize invariant subsets of the robust region-of-attraction. A branch-and-bound type refinement procedure is implemented to reduce the conservatism. We demonstrate the method on a two-state example from the literature and five-state controlled short period aircraft dynamics with and without time delay in the input to the plant.

## I. INTRODUCTION

We consider the problem of computing invariant subsets of the region-of-attraction (ROA) for uncertain systems with polynomial vector fields. Since computing the exact ROA, even for systems with known dynamics, is hard, researchers have focused on finding Lyapunov functions whose sublevel sets provide invariant subsets of the ROA [8], [9], [18]. Recent advances in polynomial optimization based on sum-of-squares (SOS) relaxations [11] are utilized to determine invariant subsets of the ROA for systems with known polynomial and/or rational dynamics solving optimization problems with matrix inequality constraints [14], [7], [13], [16]. Ref. [4] provides a generalization of Zubov's method to uncertain systems and [10] investigates robustness of the ROA under time-varying perturbations and proposes an iterative algorithm that asymptotically gives the robust ROA. Parametric uncertainties are considered in [5], [12], [17]. The focus in [5] is on computing the largest sublevel set of a *given* Lyapunov function that can be certified to be an invariant subset of the ROA. [12], [17] propose parameter-dependent Lyapunov functions which lead to potentially less conservative results (compared to parameter-independent Lyapunov functions) at the expense of increased computational complexity.

In this paper, we use a characterization for invariant subsets of the robust ROA based on parameter-independent Lyapunov functions, i.e., a common Lyapunov function is to certify the local stability of systems over the entire parameter uncertainty set. Similar to quadratic stability analysis [3], where a single quadratic Lyapunov function proves the

stability of an entire family of uncertain linear systems, this characterization is conservative, i.e., it leads to conservative estimates for the robust ROA. As a remedy for this conservativeness Lyapunov functions with polynomial parameter dependence are proposed in [5], [12]. Although SOS optimization can be used with parameter dependent Lyapunov functions theoretically, the ensuing optimization problem is computationally harder than that for parameter-independent Lyapunov functions. Uncertain parameters increase the size of the optimization problem in two ways: (i) uncertain parameters are treated as new variables in addition to state variables, and (ii) SOS constraints in the resulting problem are supposed to be satisfied for certain sets in the parameter space introducing extra set containment constraints, and, consequently, extra decision variables in corresponding multipliers. Moreover, choosing a polynomial basis for parameter-dependent Lyapunov functions is less intuitive than the parameter-independent case since the dependence on uncertain parameters in the optimization problem is different than that on state variables. Motivated by these difficulties, we restrict our attention to parameter-independent Lyapunov functions and propose a branch-and-bound based refinement procedure where the uncertainty set is partitioned and a different parameter-independent Lyapunov function is computed for each cell of the partition. This iterative procedure computes lower and upper bounds on a measure of the size of the computed invariant subset of the robust ROA (detailed in section II) at each iteration. These bounds approach each other as the partition gets finer and localizes the optimal value of this measure (this optimal value would be achieved if a different Lyapunov function could be computed for every singleton in the uncertainty set). This procedure is potentially more flexible than using parameter-dependent Lyapunov functions since it does not require a parametrization of the dependence on the uncertainty and reduces the conservativeness by partitioning the uncertainty set. Moreover, computations at each iteration trivially parallelize yielding implementations more suitable for parallel computing.

The rest of the paper is organized as follows: We formulate the problem of computing invariant subsets of the robust ROA for system with affine parametric uncertainty as a bilinear SOS problem in section II and propose a branch-and-bound based refinement procedure in section III. After a generalization of the method to systems with polynomial parametric uncertainty in section IV, we discuss a computationally efficient implementation of the method in section V. Illustration of the method on three examples is followed

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by conclusions.

**Notation:**  $\mathcal{R}^n$  denotes  $n$ -dimensional real vector space and  $\mathbf{C}^1$  denotes the set of real valued continuously differentiable functions on  $\mathcal{R}^n$ . A function  $f : \mathcal{R}^n \rightarrow \mathcal{R}$  is called positive semidefinite (definite) if  $f(0) = 0$  and  $f(x) \geq 0$  ( $f(x) > 0$ ) for all  $x \in \mathcal{R}^n$  (nonzero  $x \in \mathcal{R}^n$ ).  $f$  is called negative (semi)definite if  $-f$  is positive (semi)definite. For  $x \in \mathcal{R}^n$ ,  $x \succeq 0$  means that  $x_k \geq 0$  for  $k = 1, \dots, n$ . For a symmetric matrix  $X \in \mathcal{R}^{n \times n}$ ,  $X \succeq 0$  ( $X \succ 0$ ) means that  $X$  is positive semidefinite (definite). For  $X \in \mathcal{R}^{n \times m}$ ,  $X^T \in \mathcal{R}^{m \times n}$  denotes the transpose of  $X$ .  $\mathbb{R}[x]$  represents the set of polynomials in  $x$  with real coefficients. The subset  $\Sigma[x] := \{\pi \in \mathbb{R}[x] : \pi = \pi_1^2 + \pi_2^2 + \dots + \pi_m^2, \pi_1, \dots, \pi_m \in \mathbb{R}[x]\}$  of  $\mathbb{R}[x]$  is the set of SOS polynomials. For  $\pi \in \mathbb{R}[x]$ ,  $\partial(\pi)$  denotes the degree of  $\pi$ .

## II. COMPUTATION OF ROBUSTLY INVARIANT SUBSETS OF THE ROA FOR AFFINE PARAMETRIC UNCERTAINTY

Consider the system governed by

$$\dot{x}(t) = f(x(t)), \quad (1)$$

where  $x(t) \in \mathcal{R}^n$  is the state vector and  $f : \mathcal{R}^n \rightarrow \mathcal{R}^n$  is such that  $f(0) = 0$ , i.e., the origin is an equilibrium point of (1) and  $f$  is locally Lipschitz on  $\mathcal{R}^n$ . Let  $\varphi(\xi, t)$  denote the solution to (1) with the initial condition  $\xi$ . If the origin is asymptotically stable but not globally attractive, one often wants to know which trajectories converge to the origin as time approaches  $\infty$ . This gives rise to the following definition of the *region-of-attraction*:

*Definition 2.1:* The region-of-attraction  $R_0$  of the origin for the system (1) is

$$R_0 := \left\{ \xi \in \mathcal{R}^n : \lim_{t \rightarrow \infty} \varphi(\xi, t) = 0 \right\}.$$

For  $\eta > 0$  and a function  $V : \mathcal{R}^n \rightarrow \mathcal{R}$ , define the  $\eta$ -sublevel set of  $V$  as

$$\Omega_{V,\eta} := \{ \xi \in \mathcal{R}^n : V(\xi) \leq \eta \}.$$

For simplicity,  $\Omega_{V,1}$  is denoted by  $\Omega_V$ . Lemma 2.1, which is a modification of similar results in [19] and derived in [12], provides a characterization of invariant subsets of the ROA in terms of sublevel sets of appropriate Lyapunov functions.

*Lemma 2.1:* If there exists a  $\mathbf{C}^1$  function  $V : \mathcal{R}^n \rightarrow \mathcal{R}$  such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \neq 0, \quad (2)$$

$$\Omega_V \text{ is bounded, and} \quad (3)$$

$$\Omega_V \setminus \{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x)f(x) < 0\}, \quad (4)$$

then for all  $\xi \in \Omega_V$ , the solution of (1) exists, satisfies  $\varphi(\xi, t) \in \Omega_V$  for all  $t \geq 0$ , and  $\lim_{t \rightarrow \infty} \varphi(\xi, t) = 0$ , i.e.,  $\Omega_V$  is an invariant subset of  $R_0$ .

Now, consider the system governed by

$$\dot{x}(t) = f(x(t), \delta), \quad (5)$$

where  $x \in \mathcal{R}^n$  is the state vector,  $\delta \in \mathcal{R}^m$  is the parameter vector which satisfies  $\delta \in \Delta \subset \mathcal{R}^m$  for some polytope  $\Delta$  in  $\mathcal{R}^m$  and  $f(0, \delta) = 0$  for all  $\delta \in \Delta$ . The robust ROA is

the intersection of the ROAs for all systems governed by (5) and formally defined as

*Definition 2.2:* The robust ROA  $R_0^r$  of the origin for systems governed by (5) is

$$R_0^r := \bigcap_{\delta \in \Delta} \{ \xi \in \mathcal{R}^n : \lim_{t \rightarrow \infty} \varphi(\xi, t; \delta) = 0 \},$$

where  $\varphi(\xi, t; \delta)$  denotes the solution of (5) with the initial condition  $\xi$ .

We focus on computing invariant subsets of the robust ROA characterized by sublevel sets of appropriate Lyapunov functions. To this end, we modify Lemma 2.1 such that condition (4) holds for (5) for all  $\delta \in \Delta$ .

*Proposition 2.1:* If there exists a  $\mathbf{C}^1$  function  $V : \mathcal{R}^n \rightarrow \mathcal{R}$  such that, for all  $\delta \in \Delta$ , conditions (2)-(3), and

$$\Omega_V \setminus \{0\} \subset \{ \xi \in \mathcal{R}^n : \nabla V(\xi)f(\xi, \delta) < 0 \}, \quad (6)$$

hold, then for all  $\xi \in \Omega_V$  and for all  $\delta \in \Delta$ , the solution of (5) exists, satisfies  $\varphi(\xi, t; \delta) \in \Omega_V$  for all  $t \geq 0$ , and  $\lim_{t \rightarrow \infty} \varphi(\xi, t; \delta) = 0$ , i.e.,  $\Omega_V$  is an invariant subset of  $R_0^r$ .

We restrict our attention on a special case, where the dependence of  $f$  on  $\delta$  is affine, to obtain conditions that are equivalent to (6) for this special case and more suitable for numerical verification. A more general case, where polynomial dependence on the parameter  $\delta$  is allowed, will be discussed later. Now, assume that the vector field in (5) is

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^m \delta_i f_i(x), \quad (7)$$

where  $f_0, f_1, \dots, f_m : \mathcal{R}^n \rightarrow \mathcal{R}^n$ . Further, denote the set of vertices (extreme points) of  $\Delta$  by  $\mathcal{E}_\Delta$ . Note that, if

$$\Omega_V \setminus \{0\} \subseteq \left\{ \xi \in \mathcal{R}^n : \nabla V(\xi)(f_0(\xi) + \sum_{i=1}^m \delta_i f_i(\xi)) < 0 \right\} \quad (8)$$

holds for all  $\delta \in \mathcal{E}_\Delta$ , then it holds for all  $\delta \in \Delta$ . A proof for this straightforward argument is given in [15].

In order to enlarge the computed invariant subset of the robust ROA, we define a variable sized region  $\mathcal{P}_\beta := \{ \xi \in \mathcal{R}^n : p(\xi) \leq \beta \}$ , where  $p \in \mathbb{R}[x]$  is a fixed, positive definite, convex polynomial, and maximize  $\beta$  while imposing constraints (2)-(3), (8), and  $\mathcal{P}_\beta \subseteq \Omega_V$ . This can be written as an optimization problem.

$$\max_{V \in \mathcal{V}, \beta > 0} \beta \text{ subject to} \quad (9)$$

(2) – (3), (8),  $\mathcal{P}_\beta \subseteq \Omega_V$ .

In order to make the problem in (9) amenable to numerical optimization (specifically SOS programming), we require  $f_0, f_1, \dots, f_m$  to be polynomial and restrict  $V$  to be a polynomial in  $x$  of fixed degree. We use the well-known sufficient condition for polynomial positivity [11]: for any  $\pi \in \mathbb{R}[x]$ , if  $\pi \in \Sigma[x]$ , then  $\pi$  is positive semidefinite. Using the generalized S-procedure [13], we obtain sufficient conditions for set containment constraints. Specifically, let  $l_1$  and  $l_2$  be a positive definite polynomials (typically  $\epsilon x^T x$

with some (small) real number  $\epsilon$ ). Then, since  $l_1$  is radially unbounded, the constraint

$$V - l_1 \in \Sigma[x] \quad (10)$$

and  $V(0) = 0$  are sufficient conditions for (2) and (3). If  $s_1 \in \Sigma[x]$ , then

$$-[(\beta - p)s_1 + (V - 1)] \in \Sigma[x] \quad (11)$$

implies  $\mathcal{P}_\beta \subseteq \Omega_V$ . If for each  $\delta \in \mathcal{E}_\Delta$  there exist  $s_{2\delta}, s_{3\delta} \in \Sigma[x]$  such that

$$-\left[ (1 - V)s_{2\delta} + (\nabla V(f_0 + \sum_{i=1}^m \delta_i f_i))s_{3\delta} + l_2 \right] \in \Sigma[x] \quad (12)$$

holds, then (8) holds.

For a polytopic subset  $D$  of  $\Delta$ , we now define the quantity  $\beta_D(\mathcal{V}_{poly}, \mathcal{S})$  (which is also going to be used in later sections) as

$$\begin{aligned} \beta_D(\mathcal{V}_{poly}, \mathcal{S}) := & \max_{V, \beta, s_1, s_{2\delta}, s_{3\delta}} \beta \text{ such that} \\ & \beta > 0, s_1 \in \mathcal{S}_1, s_{2\delta} \in \mathcal{S}_{2\delta}, s_{3\delta} \in \mathcal{S}_{3\delta}, \\ & V(0) = 0, V \in \mathcal{V}_{poly}, \text{ and} \\ & (10) - (12) \text{ hold } \forall \delta \in \mathcal{E}_\Delta. \end{aligned} \quad (13)$$

Here,  $\mathcal{V}_{poly}$  and  $\mathcal{S}$ 's are prescribed finite-dimensional subspaces of  $\mathbb{R}[x]$  and  $\Sigma[x]$ , respectively. Note the optimal value of  $\beta$  in problem (13) depends on  $\Delta$ ,  $\mathcal{V}_{poly}$ , and  $\mathcal{S}$ . For simplicity, we will omit the dependence on  $\mathcal{V}_{poly}$  and  $\mathcal{S}$  in notation hereafter. By the definition in (13), a lower bound on the optimal value of  $\beta$  in (9) is  $\beta_\Delta(\mathcal{V}_{poly}, \mathcal{S})$ .

The optimization problem in (13), when applied with  $D = \Delta$ , provides a method for computing invariant subsets of the robust ROA characterized by a single Lyapunov function. Therefore, results by (13) may be conservative: certified invariant subset may be too small relative the robust ROA. On the other hand, a less conservative estimate of the robust ROA can be obtained by computing a different Lyapunov function for each  $\delta \in \Delta$ . Then  $\mathcal{P}_{\beta_\Delta^*}$ , where, for a subset  $D \subseteq \delta$ ,  $\beta_D^*$  is defined as

$$\beta_D^* := \min_{\delta \in D} \beta_{\{\delta\}}, \quad (14)$$

is a subset of the robust ROA.<sup>1</sup> In fact,  $\mathcal{P}_{\beta_\Delta^*}$  is the largest sublevel set of  $p$  that can be proven to be a subset of the robust ROA using the optimization problem in (13). However, computing  $\beta_\Delta^*$  requires solving an optimization problem for each  $\delta \in \Delta$ . In the following, we propose a ‘‘branch-and-bound’’ type procedure for computing lower and upper bounds for  $\beta_\Delta^*$ , i.e., localizing the value of  $\beta_\Delta^*$ . The method is based on computing a different Lyapunov function for each cell of a finite partition of  $\Delta$ . Therefore, it potentially leads to less conservative estimates of the robust ROA compared to directly solving (13) with  $D = \Delta$  and more conservative estimates compared to  $\mathcal{P}_{\beta_\Delta^*}$ .

<sup>1</sup>Note that for a singleton  $\{\delta\}$ ,  $\mathcal{E}_{\{\delta\}} = \{\delta\}$ .

### III. BRANCH-AND-BOUND IN THE PARAMETER SPACE

Branch-and-bound is an algorithmic method for global optimization. The method is based on two steps: first the search region is covered by smaller subregions and then upper and lower bounds for the objective function restricted to each subregion are computed. These steps are repeated refining the partition of the search region until the gap between the upper and lower bounds gets smaller than a prescribed tolerance or a maximum number of iterations is reached.

Now, let  $\mathcal{D} := \{D_k\}_{k=1}^K$  be a partition of  $\Delta$  and  $\mathcal{C}$  be defined as  $\mathcal{C} := \{c_k\}_{k=1}^K$  where  $c_k \in D_k$  for  $k = 1, \dots, K$ . Further define

$$L_{\mathcal{D}} := \min_{D \in \mathcal{D}} \beta_D,$$

$$U_{\mathcal{C}} := \min_{c \in \mathcal{C}} \beta_{\{c\}}.$$

Then, the following set of inequalities hold

$$\beta_\Delta \leq L_{\mathcal{D}} \leq \beta_\Delta^* \leq U_{\mathcal{C}},$$

where the first inequality follows from the fact that  $D \subseteq \Delta$  for all  $D \in \mathcal{D}$ , the second inequality follows from the fact that for each  $\delta \in \Delta$  there exists  $D \in \mathcal{D}$  such that  $\delta \in D$ , and the last inequality is due to  $\mathcal{C} \subseteq \Delta$ . Using the lower and upper bounds  $L_{\mathcal{D}}$  and  $U_{\mathcal{C}}$  for  $\beta_\Delta^*$ , the following branch-and-bound algorithm adapted from [2] can be implemented for localizing  $\beta_\Delta^*$ .

**Branch-and-Bound Algorithm:** Given an initial partition  $\mathcal{D}^0$  of  $\Delta$ , corresponding  $\mathcal{C}^0$ , positive integer  $N_{iter}$ , and positive number  $\epsilon > 0$ ,

- $k \leftarrow 0$ ;
- compute  $L_{\mathcal{D}^k}$  and  $U_{\mathcal{C}^k}$ ;
- while  $k \leq N_{iter}$  and  $U_{\mathcal{C}^k} - L_{\mathcal{D}^k} > \epsilon$ 
  - $k \leftarrow k + 1$ ;
  - pick  $\tilde{D} \in \mathcal{D}^{k-1}$  such that  $\beta_{\tilde{D}} = L_{\mathcal{D}^{k-1}}$ ;
  - split  $\tilde{D}$  into  $\tilde{D}_I$  and  $\tilde{D}_{II}$ ;
  - form  $\mathcal{D}^k$  from  $\mathcal{D}^{k-1}$  by removing  $\tilde{D}$  and adding  $\tilde{D}_I$  and  $\tilde{D}_{II}$ ;
  - form  $\mathcal{C}^k$  from  $\mathcal{D}^k$ ;
  - compute  $L_{\mathcal{D}^k}$  and  $U_{\mathcal{C}^k}$ ;
- return  $\mathcal{D}^{exit} := \mathcal{D}^k$ ,  $\mathcal{C}^{exit} := \mathcal{C}^k$ , and corresponding lower and upper bounds, Lyapunov functions, multipliers, and parameters.

Trivial but useful upper bounds for the value of  $\beta$  such that the conditions in Proposition 2.1 and  $\mathcal{P}_\beta \subseteq \Omega_V$  hold (and consequently on  $\beta_\Delta$ ) can be obtained by monitoring the value of  $p$  along divergent (not converging to the origin) trajectories of (7). These upper bounds are valid regardless of the form of the Lyapunov function as long as it satisfies conditions in Proposition 2.1. Tighter upper bounds for  $\beta_\Delta^*(\mathcal{V}_{poly}, \mathcal{S})$  can be obtained by relaxing (some of) the constraints in (14) and determining values of  $\beta$  for which such relaxations cannot be feasible. Such a relaxation can be obtained by modifying simulation-based convex relaxations for local stability analysis from [16]. Upper bounds obtained

this way are valid regardless of the degrees of the multipliers used in (14).

With the convention that  $\beta_D = 0$  if the problem in (13) is infeasible for some  $D \subseteq \Delta$ , the algorithm first tries to find a partition of  $\Delta$  such that for each cell there exists a local stability certificate. Then, it increases the computed value of  $L_D$  using finer partitions  $\mathcal{D}$ .

#### IV. POLYNOMIAL PARAMETRIC UNCERTAINTY

We now extend the development in section II to systems with polynomial parametric uncertainty. Specifically, we consider the system

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^m \delta_i f_i(x(t)) + g(\delta) f_{m+1}(x(t)), \quad (15)$$

where  $f_0, f_1, \dots, f_m$  are as before,  $g \in \mathbb{R}[\delta]$  is a scalar valued polynomial function, and  $f_{m+1} : \mathcal{R}^n \rightarrow \mathcal{R}^n$  is a vector valued polynomial function satisfying  $f_{m+1}(0) = 0$ . System in (15) has only one polynomial uncertainty. We later comment on the case with multiple polynomial uncertainties. Our approach is based on replacing  $g(\delta)$  by an artificial parameter  $\phi$ . Then, the dynamics in (15) can be written as

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^m \delta_i f_i(x(t)) + \phi f_{m+1}(x(t)), \quad (16)$$

which is affine in the parameters  $(\delta, \phi)$ . The graph of  $g$  is an  $m$ -dimensional manifold in  $\mathcal{R}^{m+1}$ . Next step is to cover this the graph of  $g$  by the polytope  $\Gamma \subset \mathcal{R}^{m+1}$ . At this point, results from section II are applicable to the system in (16) since the dependence on the parameters  $(\delta, \phi)$  is affine.

A polytope covering the graph of  $g$  can be obtained by bounding  $g$  by an affine function  $a_1^T \delta + b_1$  from below and by another affine function  $a_2^T \delta + b_2$  from above over the set  $\Delta$ . Here,  $a_1, a_2 \in \mathcal{R}^m$  and  $b_1, b_2 \in \mathcal{R}$ , and the volume of the polytope depends on the choices for the parameters  $a_1, a_2, b_1$ , and  $b_2$ . Then, the polytope with smallest volume among such covering polytopes can be computed through the optimization

$$\begin{aligned} & \min_{a_1, a_2, b_1, b_2} \text{Volume}(\Gamma(a_1, a_2, b_1, b_2)) \text{ such that} \\ & \Gamma(a_1, a_2, b_1, b_2) := \left\{ (\zeta, \psi) \in \mathcal{R}^{m+1} : \begin{array}{l} \delta \in \Delta, \\ a_1^T \zeta + b_1 \leq \psi \\ a_2^T \zeta + b_2 \geq \psi \end{array} \right\}, \\ & g(\delta) - (a_1^T \delta + b_1) \geq 0, \quad \forall \delta \in \Delta, \\ & g(\delta) - (a_2^T \delta + b_2) \leq 0, \quad \forall \delta \in \Delta. \end{aligned} \quad (17)$$

The function

$$\begin{aligned} & \text{Volume}(\Gamma(a_1, a_2, b_1, b_2)) \\ & := \int_{\Delta} (a_2 - a_1)^T \zeta + (b_2 - b_1) d\zeta_1 \dots d\zeta_m \\ & = (a_2 - a_1)^T \int_{\Delta} d\zeta \zeta + (b_2 - b_1) \int_{\Delta} d\zeta \end{aligned}$$

is linear in its arguments  $a_1, a_2, b_1$ , and  $b_2$ . Using the generalized S-procedure,<sup>2</sup> an upper bound for the optimal value in problem (17) can be computed by a linear SOS optimization problem. To this end, let affine functions  $h_i$ ,  $i = 1, \dots, N$ , provide an inequality description for  $\Delta$ , i.e.,

$$\Delta = \{\zeta \in \mathcal{R}^m : h_i(\zeta) \geq 0, i = 1, \dots, N\}.$$

*Proposition 4.1:* The optimal value of the optimization problem

$$\begin{aligned} & \min_{a_1, a_2, b_1, b_2, \sigma_{ui}, \sigma_{li}} \text{Volume}(\Gamma(a_1, a_2, b_1, b_2)) \text{ such that} \\ & g(\delta) - (a_1^T \delta + b_1) - \sum_{i=1}^N \sigma_{ui}(\delta) h_i(\delta) \in \Sigma[\delta], \\ & -g(\delta) + (a_2^T \delta + b_2) - \sum_{i=1}^N \sigma_{li}(\delta) h_i(\delta) \in \Sigma[\delta], \\ & \sigma_{ui} \in \mathcal{S}_{ui}, \sigma_{li} \in \mathcal{S}_{li}, i = 1, \dots, N, \end{aligned} \quad (18)$$

is an upper bound for the optimal value in problem (17). Here  $\mathcal{S}$ 's are finite dimensional subspaces of  $\Sigma[\delta]$ .

*Remarks 4.1:*

1) When there are  $m_{pu} \geq 1$  polynomial uncertainties  $g_1, \dots, g_{m_{pu}}$  in the description (15),  $m+1$  dimensional polytopes  $\Gamma_1, \dots, \Gamma_{m_{pu}}$  covering the graph of  $g_1, \dots, g_{m_{pu}}$ , respectively, can be determined using the procedure proposed in this section repeatedly. Then, a polytope covering the graph of  $(g_1, \dots, g_{m_{pu}})$  is the intersection  $\tilde{\Gamma}_1 \cap \dots \cap \tilde{\Gamma}_{m_{pu}}$ , where, for  $i = 1, \dots, m_{pu}$ ,

$$\tilde{\Gamma}_i := \{(\zeta, \psi) \in \mathcal{R}^{m+m_{pu}} : (\zeta, \psi_i) \in \Gamma_i\}.$$

2) In order to extend the applicability of the method explained in section II to systems with polynomial uncertainties, we proposed a procedure for covering the graph of a polynomial parametric uncertainty by a convex polytope. It may be possible to develop more efficient and/or less conservative graph covering strategies for a given polynomial uncertainty description.

3) Although we focused on polynomial uncertainties in this section, the method proposed in section II can be used for systems for which  $g$  in (15) is not polynomial as long as polytopic cover for its graph is provided.

#### V. COMPUTATIONALLY EFFICIENT IMPLEMENTATION

The number of constraints in (13) (consequently the number of decision variables since each new constraint includes new variables) increases exponentially with  $m$  (and  $m_{pu}$ ). This difficulty can be alleviated by computing suboptimal solutions for (13) in two steps [15]:

- compute a Lyapunov function for a particular system from the family of systems in (7) solving (13) (in this case, (8) is composed of one constraint);
- use this  $V$  as a local stability certificate for the entire family of systems and compute largest value of  $\gamma$  such that  $\Omega_{V, \gamma}$  (with  $\gamma \leq 1$ ) in the ROA for each system in (7).

<sup>2</sup>Higher order relaxations for semialgebraic set containment based on Positivstellensatz [11] can be used for less conservative results at the expense of increased computational cost.

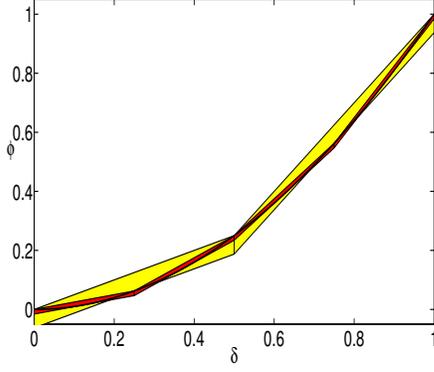


Fig. 1. Polytopic cover for the graph of  $\phi = \delta^2$  with 2 cells (red) and 4 cells (yellow). Thick black curve is the graph of  $\phi = \delta^2$ .

Once a Lyapunov function candidate is found in the first step, the second step involves solving smaller decoupled linear SDPs. We will illustrate this procedure in section VI.

As the state dimension increases, it may be challenging to solve the problem in (13) for even a singleton  $D$ . In this case, a recently developed methodology, merging formal proof techniques and information from simulations for computing invariant subsets of the ROA for systems with no uncertainty [16], can be used in the first step of the two-step procedure explained in the previous paragraph. Moreover, it is possible to generalize this method to generate Lyapunov function candidates for proving the local stability of the uncertain system using simulation for systems corresponding to finite sample of  $\Delta$ , such as  $\mathcal{E}_\Delta$ .

## VI. EXAMPLES

In all examples in this section,  $l_1(x) = l_2(x) = 10^{-6}x^T x$  and  $p(x) = x^T x$ .

### A. An example from the literature

Consider the system [6] governed by  $\dot{x}_1 = -x_1 + (-6x_2 + x_2^2 + x_1^3)\delta + (4x_2 - x_2^2)\delta^2$  and  $\dot{x}_2 = 3x_1 - 2x_2 + (-10x_1 + 6x_2 + x_1x_2)\delta + (12x_1 - 4x_2)\delta^2$ , where the uncertain scalar parameter  $\delta$  takes values in  $[0, 1]$ . Following the procedure from section IV, we replaced  $\delta^2$  by an artificial parameter  $\phi$ . Fig. 1 shows the set  $\{(\zeta, \psi) \in \mathcal{R}^2 : \zeta \in [0, 1], \psi = \zeta^2\}$  (black curve) and polytopic covers with 2-cell (yellow region in the small plot) and 4-cell (yellow region in the main plot) partitions (obtained during the application of the branch-and-bound refinement) of  $[0, 1]$ . With the initial partition  $\{[0, 1]\}$ , we applied the refinement for  $\partial(V) = 2$  and  $\partial(V) = 4$  using the two-step procedure. Upper and lower bounds for  $\beta_{[0,1]}^*$  are shown in Fig. 2 (top for  $\partial(V) = 2$  and bottom for  $\partial(V) = 4$ ). Upper bounds are obtained either monitoring the value of  $p$  along divergent trajectories or the value of  $\beta$  for which certain convex relaxations for ROA analysis proposed in the context of simulation-aided stability analysis in [16].

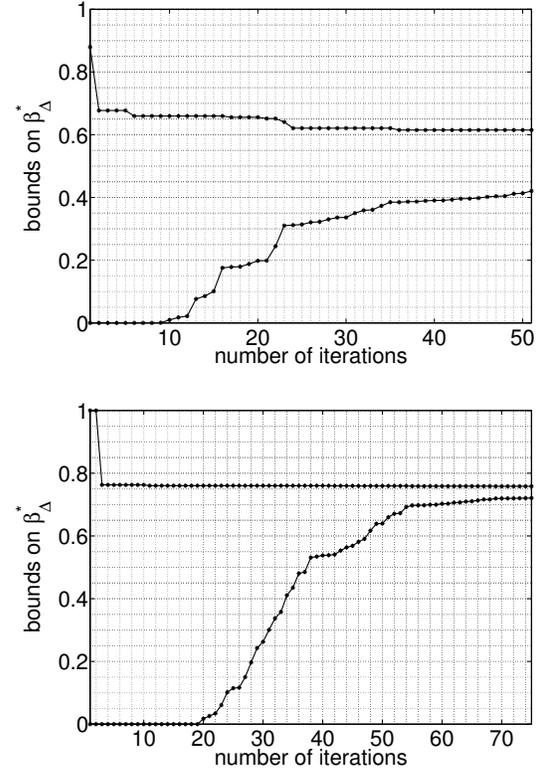


Fig. 2. Upper and lower bounds for  $\beta_{[0,1]}^*$  (top for  $\partial(V) = 2$  and bottom for  $\partial(V) = 4$ ).

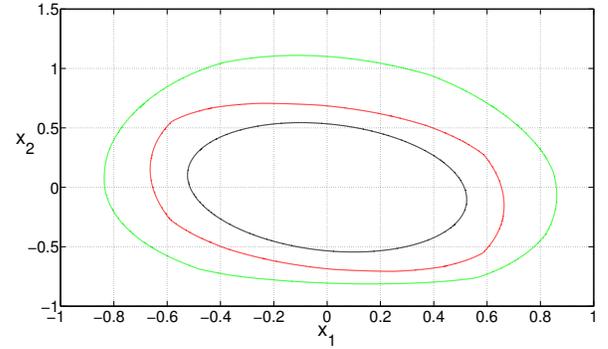


Fig. 3. Estimates of the robust ROA: from [6] (black), using the branch-and-bound based method for  $\partial(V) = 2$  (red) and  $\partial(V) = 4$  (green).

### B. Controlled short period aircraft dynamics

Consider the plant dynamics  $\dot{x}_1 = -3x_1 - 1.35x_2 - 0.56x_3 + (1 + \delta_1)(0.08x_1x_2 + 0.44x_2^2 + 0.01x_2x_3 + 0.22x_3^2) + (1.35 - 0.04x_2)u$ ,  $\dot{x}_2 = 0.91x_1 - 0.64x_2 - 0.02x_3 + (1 + \delta_2)(-0.05x_2^2 + 0.11x_2x_3 - 0.05x_3^2) + 0.4u$ ,  $\dot{x}_3 = x_1$ , and  $y = [x_1 \ x_3]^T$ , where  $x_1$ ,  $x_2$  and  $x_3$  are the pitch rate, the angle of attack, and the pitch angle, respectively. The input  $u$  is the elevator deflection and determined by the dynamics  $\dot{z}_1 = -0.60z_1 + 0.09z_2 - 0.06y_1 - 0.02y_2$ ,  $\dot{z}_2 = -0.75y_1 - 0.28y_2$ , and  $u = z_1 + 2.2z_2$ , where  $z$  is the controller state. Here,  $\delta_1$  and  $\delta_2$  are two uncertain

TABLE I

LOWER BOUNDS FOR  $\beta_D^*$ , WHERE IS  $D$  IS THE CORRESPONDING SUBSET OF  $[-0.1, 0.1] \times [-0.1, 0.1]$ .

$\delta_1 \backslash \delta_2$	$[-0.1, \frac{-0.1}{3}]$	$[\frac{-0.1}{3}, \frac{0.1}{3}]$	$[\frac{0.1}{3}, 0.1]$
$[-0.1, \frac{-0.1}{3}]$	12.7 9.1	12.5 8.5	11.6 7.9
$[\frac{-0.1}{3}, \frac{0.1}{3}]$	12.2 8.7	12.0 8.1	11.2 7.6
$[\frac{0.1}{3}, 0.1]$	11.1 8.4	11.1 7.8	11.1 7.3

TABLE II

LOWER BOUNDS FOR  $\beta_D^*$ , WHERE IS  $D$  IS THE CORRESPONDING SUBSET OF  $[-0.1, 0.1] \times [-0.1, 0.1]$ .

$\delta_1 \backslash \delta_2$	$[-0.1, \frac{-0.1}{3}]$	$[\frac{-0.1}{3}, \frac{0.1}{3}]$	$[\frac{0.1}{3}, 0.1]$
$[-0.1, \frac{-0.1}{3}]$	8.8 5.2	9.2 5.0	9.7 4.7
$[\frac{-0.1}{3}, \frac{0.1}{3}]$	9.0 4.9	8.4 4.7	8.3 4.5
$[\frac{0.1}{3}, 0.1]$	8.3 4.7	8.4 4.4	8.6 4.2

parameters introducing 10% uncertainty for the entries of the plant dynamics that are nonlinear in  $x$ , i.e.,  $\delta_1 \in [-0.1, 0.1]$  and  $\delta_2 \in [-0.1, 0.1]$ . Here, the entries of the above dynamics are shown up to three significant digits. The exact vector field used for this example is available at [1]. We partitioned  $\Delta = [-0.1, 0.1] \times [-0.1, 0.1]$  into 9 equal cubic sub-regions and solved the problem in (13) using computationally more efficient two-step implementation from section V. In the first step, we generated the Lyapunov function candidates using the simulation based method from [16] and found a diverging trajectory with  $p(x) = 14.0$  during the course of simulation runs. Table I shows the lower bounds for  $\beta_D^*$ , where is  $D$  is the corresponding subset of  $[-0.1, 0.1] \times [-0.1, 0.1]$ . The top value in each cell of Table I if for  $\partial(V) = 4$  and bottom value is for  $\partial(V) = 2$ .

### C. Controlled short period aircraft dynamics with time delay

In this example, we introduced a time delay of 0.53 units of time (which is 1/4 of the time delay margin of the dynamics linearized around the origin) between the controller and plant whose dynamics are given in section VI-B. We modeled the dynamics of the time delay by a first order Pade approximation and applied the analysis explained in section VI-B. Table II shows the lower bounds for  $\beta_{\Delta}^*$ , with  $\Delta = [-0.1, 0.1] \times [-0.1, 0.1]$ . An upper bound for  $\beta_{\{(0.1, 0.1)\}}$  is provided by a diverging trajectory with  $p(x) = 11.2$ .

## VII. CONCLUSIONS

We proposed a method to compute invariant subsets of the region-of-attraction for the asymptotically stable equilibrium points of polynomial dynamical systems with bounded parametric uncertainty. Parameter-independent Lyapunov functions were used to characterize invariant subsets of the robust region-of-attraction. A branch-and-bound type refinement

procedure was implemented to reduce the conservatism. We demonstrated the method on a two-state example from the literature and five-state controlled short period aircraft dynamics with and without time delay in the input to the plant.

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