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Forced Vibration Analysis of Flexible Euler-Bernoulli Beams with Geometrical Discontinuities

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Abstract— **This paper presents a novel framework for forced motion analysis of Euler-Bernoulli beam with multiple jumped discontinuities in the cross section. In this regard, the entire length of beam is partitioned into uniform segments between any two successive discontinuity points. Beam characteristics matrix can be derived based on the boundary conditions and the continuity conditions applied at the partitioned points. This matrix is particularly used to find beam natural frequencies and mode shapes. The governing ODE of motion and its state-space representation are then derived for the beam under a distributed dynamic loading condition. To clarify the implementation of the proposed method, a beam with two stepped discontinuities in the cross section is studied, and numerical simulations are provided to demonstrate the mode shapes and frequency response of beam for different stepped values. Results indicate that the added mass and stiffness significantly affects the mode shapes and natural frequencies.**

I. INTRODUCTION

Dynamic analysis of beam-like structures is significantly important in modeling real cases such as aircraft wings, spacecraft antennas, helicopter blades, robot arms and many other applications. In this respect, numerous studies can be found in the literature on the transverse vibration of uniform beams under different types of boundary conditions. However, in many real applications, the investigation of non-uniform cross-section beams may provide a realistic distribution of mass and stiffness desired for accurate structural analysis. Particularly, for structures with abrupt changes in cross section, the added mass, stiffness and geometrical discontinuities affect the modal behavior of structure which cannot be neglected. Some examples could include analysis of machining processes [1], design of road and railway bridges [2], and MEMS characterization [3]. In the last decades, many studies have been done on the transverse vibration of continuous Euler-Bernoulli beams [4, 5]. However, methods applied for continuous beams cannot be directly used for beams with sudden changes in the cross section. Partitioning method [1], finite difference approach [6], shear deformation theory [7], and transfer matrices approach [8] have been used to study free vibration of such structures. While bibliography on the free vibration of beams with one step change in the cross section is extensive [9-12], few studies have been focused on the analysis of beams with multiple jumps [13, 14]. Nevertheless, there is a lack of a straightforward framework for the forced vibration analysis of Euler-Bernoulli beam with any arbitrary number of step jumps in cross section, under general distributed dynamic loading. Added to this, graphical comparisons of beam mode shapes with and without geometrical discontinuities has not been well presented in the literature.

The present work is aimed at the forced vibration formulation and analysis of Euler-Bernoulli beam with an arbitrary number of step changes in cross section and properties. To obtain beam mode shapes, the entire length of beam is partitioned into the beam segments between any two successive discontinuity points. The characteristics matrix is then derived using applied boundary and continuity conditions. The natural frequencies of the beam and the parameters of the modes shapes can be obtained by imposing the non-trivial solution condition on the derived characteristics equation. Finally, using the assumed mode method, the governing ordinary differential equation (ODE) of beam and its state-space representation are derived under distributed vertical loading condition.

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II. EULER BERNOULLI BEAM WITH MULTIPLE STEPPED DISCONTINUITIES

Consider an initially straight non-uniform Euler-Bernoulli (EB) beam of length L, with variable cross section $A = A(x)$, variable stiffness $E = E(x)$, and variable moment of inertia $I = I(x)$. Let $x \in [0,\mathrm{L}]$ and $t \in [0,\infty)$ be the spatial and time variables, respectively. The governing equation for transverse vibration of beam with variable mass per unit length $m(x)$ and damping coefficient of $c(x)$ subjected to a vertical time varying distributed load $P(x,t)$ is a fourth order PDE expressed as:

$$\frac{\partial^2}{\partial x^2}(E(x)I(x)\frac{\partial^2 w}{\partial x^2}) + c(x)\frac{\partial w}{\partial t} + m(x)\frac{\partial^2 w}{\partial t^2} = P(x,t) \quad (1)$$

with $w(x,t)$ being the transversal displacement function. In order to obtain natural frequencies and natural modes of system, the eigenvalue problem associated with the transversal vibration of beam is obtained by applying free and un-damped conditions to Eq. (1) as follows:

$$\frac{\partial^2}{\partial x^2}(E(x)I(x)\frac{\partial^2 w}{\partial x^2}) = m(x)\frac{\partial^2 w}{\partial t^2} \quad (2)$$

Let's assume that the solution of Eq. (2) is separable in time and space domains,

$$w(x,t) = \Phi(x)Q(t) \quad (3)$$

where $\Phi(x)$ denotes the spatial mode shape function and $Q(t)$ represents the generalized time-dependent coordinate. Beam mode shapes are obtained by solving the spatial part of equation of motion written as:

$$\frac{d^2}{dx^2}(E(x)I(x)\frac{d^2\Phi(x)}{dx^2}) = \omega^2 m(x)\Phi(x) \quad (4)$$

where ω is a constant parameter considered as the natural frequency. For a beam with parametric discontinuities (e.g., jump in the moment of inertia or mass distribution) Eq. (4) cannot be solved using conventional approaches. An alternative method is to partition the beam into uniform segments between any two successive stepped points and apply the continuity conditions at these points. The next section discusses this technique in detail and proposes a framework for dynamic analysis of beams with jumped configuration.

A. Natural Modes Analysis of Stepped EB Beam

Figure 1 illustrates a straight axis EB beam with arbitrary boundary conditions and N jumped discontinuities in its spatial span. The beam considered in this study has a uniform cross section at each segment. Hence, Eq. (4) can be divided into n uniform equations expressed as:

$$(EI)_n \frac{d^4\phi_n(x)}{dx^4} = \omega^2 m_n \phi_n(x) \,, \quad l_{n-1} < x < l_n; \\ n = 1,2,3,...,N; \; l_0 = 0 \quad (5)$$

where $\phi_n(x)$, $(EI)_n$, and m_n are mode shape function, flexural stiffness, and mass per unit length of beam at the n^{th} segment. Let,

$$\beta_n^4 = \omega^2 \frac{m_n}{(EI)_n} \quad (6)$$

Eq. (5) can be rewritten in a more recognizable form

$$\frac{d^4\phi_n(x)}{dx^4} - \beta_n^4\phi_n(x) = 0 \quad (7)$$

with the following general solution

$$\phi_n(x) = A_n \sin(\beta_n x) + B_n \cos(\beta_n x) + \\ C_n \sinh(\beta_n x) + D_n \cosh(\beta_n x) \quad (8)$$

where A_n, B_n, C_n, D_n are the constants of integration determined by suitable boundary and continuity conditions. It is remarked that any classical boundary conditions can be applied to the beam; however, without the loss of generality, clamped-free conditions are chosen here for the boundaries. Applying the clamped condition at $x = 0$ requires:

$$\phi_1(0) = \frac{d\phi_1(0)}{dx} = 0 \quad (9)$$

Also, the continuity conditions for displacement, slope, bending moment, and shear force of beam at points of discontinuity are given by:

$$\phi_n(l_n) = \phi_{n+1}(l_n) \quad (10)$$

$$\frac{d\phi_n(l_n)}{dx} = \frac{d\phi_{n+1}(l_n)}{dx} \quad (11)$$

$$(EI)_n \frac{d^2\phi_n(l_n)}{dx^2} = (EI)_{n+1}\frac{d^2\phi_{n+1}(l_n)}{dx^2} \quad (12)$$

$$(EI)_n \frac{d^3\phi_n(l_n)}{dx^3} = (EI)_{n+1}\frac{d^3\phi_{n+1}(l_n)}{dx^3} \quad (13)$$

And, the free boundary condition at $x = L$ requires:

$$\frac{d^2\phi_N(l_N)}{dx^2} = \frac{d^3\phi_N(l_N)}{dx^3} = 0 \quad (14)$$

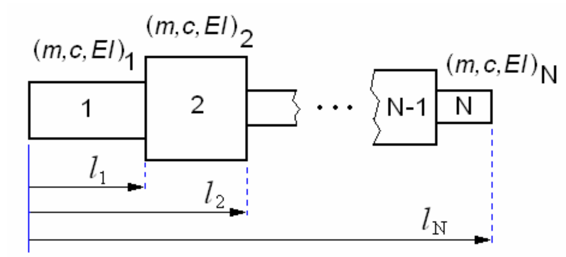

Figure 1. EB beam configuration with N jumped discontinuities.

The characteristics matrix of the system can be formed by applying Eqs. (9-14) to Eq. (8) at each point of discontinuity as well as at the boundaries. It is remarked that β_n's are functions of beam natural frequency with an explicit expression given in Eq. (6). Since the natural frequency is independent of segments indices and is considered for the entire length of beam, β_n's of different segments can be interrelated in terms of a single parameter β using Eq. (6):

$$\beta_n = \beta\alpha_n \quad (15)$$

where

$$\alpha_n = \left(\frac{m_n (EI)_1}{m_1 (EI)_n}\right)^{1/4} \quad (16)$$

Note that $\alpha_1 = 1$ and thus, $\beta = \beta_1$.

Therefore, the characteristics matrix becomes only function of a single parameter β. The characteristics equation is then given by:

$$\mathbf{J}_{4N\times 4N}\mathbf{P}_{4N\times 1} = 0 \quad (17)$$

where $\mathbf{J} = \mathbf{J}(\beta)$ is the characteristics matrix and **P** is the vector of mode shape coefficients:

$$\mathrm{P} = [A_1\ B_1\ C_1\ D_1\ A_2\ B_2\ C_2\ D_2\ \cdots\ A_N\ B_N\ C_N\ D_N]^{\mathrm{T}}_{1\times 4\mathrm{N}} \tag{18}$$

Matrix **J** is constructed based on three sets of equations. The first two rows and last two rows present the boundary conditions at $x = 0$ and $x = L$, respectively, and the middle part of matrix demonstrates the continuity conditions in the singularity points. Let's divide matrix **J** in three parts:

$$\mathbf{J} = \begin{bmatrix} \mathrm{J}_1 \\ \hline \mathrm{J}_2 \\ \hline \mathrm{J}_3 \end{bmatrix}_{4\mathrm{N}\times 4\mathrm{N}} \tag{19}$$

where $[\mathrm{J}_1]_{2\times 4N}$ represents the boundary conditions at the clamped end at $x = 0$ given by Eq. (9),

$$\mathrm{J}_2 = \begin{bmatrix} \overset{1}{[}\ \ J_2^{(1)}\ \ \overset{8}{]}_{4\times 8} & & & \\ & \overset{5}{[}\ \ J_2^{(2)}\ \ \overset{12}{]}_{4\times 8} & & 0 \\ & & \ddots & \\ 0 & & & \overset{4(N-2)}{[}\ \ J_2^{(N-1)}\ \ \overset{4N}{]}_{4\times 8} \end{bmatrix}_{4(N-1)\times 4N} \tag{20}$$

includes the continuity conditions given by Eqs. (10-13) at N-1 points of discontinuity with $J_2^{(n)}$ indicating the continuity conditions applied at the n[th] cross section, and $[\mathrm{J}_3]_{2\times 4N}$ represents the free boundary condition at $x = L$ given by Eq. (14).

In order to obtain a non-trivial solution for Eq. (17) and find the natural frequencies and mode shapes the determinant of matrix **J** must be set to zero

$$\det[\mathbf{J}(\beta)] = 0 \tag{21}$$

Since this matrix is a function of only parameter $\beta \in (0,\infty)$, its determinant can be numerically evaluated for its zero values. The values of β which satisfy Eq. (21) lead to the calculation of natural frequencies using a modified version of Eq. (6):

$$\omega_r^2 = \left(\beta^{(r)}\right)^4 \frac{(EI)_1}{m_1} = \left(\beta_n^{(r)}\right)^4 \frac{(EI)_n}{m_n} \tag{22}$$

where $\beta^{(r)}$'s are the solutions to Eq. (21) and ω_i s are the corresponding natural frequencies. Since the determinant of the matrix **J** has been set to zero for the selected values of β, Eq. (17) becomes underdetermined. However, with the integration of normalization condition, the obtained set of equations becomes solvable for the parameters vector **P**. Hence, the following normalization condition is applied to obtain coefficients of beam mode shapes.

$$\int_{l_0}^{l_N} m(x)\left(\phi^{(r)}(x)\right)^2 dx = 1 \tag{23}$$

where $\phi^{(r)}(x)$ is the r^{th} mode shape of beam.

The obtained mode shapes and natural frequencies are used to derive the ODE of motion for a beam under distributed dynamic excitation as will be discussed next.

B. Forced Motion Analysis of Stepped EB Beam

Using expansion theorem for the beam vibration analysis, the expression for the transversal displacement becomes:

$$w(x,t) = \sum_{r=1}^{\infty} \phi^{(r)}(x) q^{(r)}(t) \tag{24}$$

where $q^{(r)}(t)$'s are the generalized time-dependent coordinates. Now, consider beam is under a time-varying distributed vertical load $P(x,t)$ which can be expressed as:

$$P(x,t) = \sum_{r=1}^{\infty} \phi^{(r)}(x) p^{(r)}(t) \tag{25}$$

Substituting Eqs. (24) and (25) into PDE of motion Eq. (1) yields:

$$\begin{aligned} &\sum_{r=1}^{\infty}\Big\{\frac{d^2}{dx^2}\left[E(x)I(x)\frac{d^2\phi^{(r)}(x)}{dx^2}\right]q^{(r)}(t) + c(x)\phi^{(r)}(x)\dot{q}^{(r)}(t) \\ &\quad + m(x)\phi^{(r)}(x)\ddot{q}^{(r)}(t) - \phi^{(r)}(x)p^{(r)}(t)\Big\} = 0 \end{aligned} \tag{26}$$

To safely take the term $E(x)I(x)$ out of the bracket for the beam with multiple discontinuities, Eq. (26) is multiplied by an arbitrary mode shape $\phi^{(s)}(x)$ and is integrated over x:

$$\begin{aligned} &\sum_{r=1}^{\infty}\Big\{\int_{l_0}^{l_N}\Big(\frac{d^2}{dx^2}\left[E(x)I(x)\frac{d^2\phi^{(r)}(x)}{dx^2}\right]\phi^{(s)}(x)q^{(r)}(t) + \\ &c(x)\phi^{(r)}(x)\phi^{(s)}(x)\dot{q}^{(r)}(t) + m(x)\phi^{(r)}(x)\phi^{(s)}(x)\ddot{q}^{(r)}(t) \\ &-\phi^{(r)}(x)\phi^{(s)}(x)p^{(r)}(t)\Big)dx\Big\} = 0 \end{aligned} \tag{27}$$

Recall Eq. (5) which can be modified to

$$(EI)_n \frac{d^4\phi_n^{(r)}(x)}{dx^4} = \omega_r^2 m_n \phi_n^{(r)}(x) \tag{28}$$

Using Eq. (28) and dividing the spatial integral into N uniform segments, one can write:

$$\begin{aligned} &\int_{l_0}^{l_N}\Big(\frac{d^2}{dx^2}\left[E(x)I(x)\frac{d^2\phi^{(r)}(x)}{dx^2}\right]\phi^{(s)}(x)q^{(r)}(t) = \\ &\sum_{n=1}^{N}\Big[\int_{l_{n-1}}^{l_n}(EI)_n\frac{d^4\phi_n^{(r)}(x)}{dx^4}\phi_n^{(s)}(x)q^{(r)}(t)dx\Big] = \\ &\sum_{n=1}^{N}\Big[\int_{l_{n-1}}^{l_n}\omega_r^2 m_n\phi_n^{(r)}(x)\phi_n^{(s)}(x)q^{(r)}(t)dx\Big] = \\ &\omega_r^2 q^{(r)}(t)\int_{l_0}^{l_N} m(x)\phi^{(r)}(x)\phi^{(s)}(x)dx \end{aligned} \tag{29}$$

Applying beam orthogonally conditions given by:

$$\int_{l_0}^{l_N} m(x)\phi^{(r)}(x)\phi^{(s)}(x)dx = \int_{l_0}^{l_N} E(x)I(x)\phi^{(r)}(x)\phi^{(s)}(x)dx = 0, \quad r \neq s \tag{30}$$

and using Eqs. (23) and (29), Eq. (27) can be recast as follows:

$$\ddot{q}^{(r)}(t) + \sum_{r=1}^{\infty}\left\{\dot{q}^{(r)}(t)\int_{l_0}^{l_N} c(x)\phi^{(r)}(x)\phi^{(s)}(x)dx\right\} + \omega_r^2 q^{(r)}(t) = \sum_{r=1}^{\infty}\left\{p^{(r)}(t)\int_{l_0}^{l_N}\phi^{(r)}(x)\phi^{(s)}(x)dx\right\} \tag{31}$$

which can be simplified to

$$\ddot{q}^{(r)}(t) + \sum_{r=1}^{\infty}\left\{c_{rs}\dot{q}^{(r)}(t)\right\} + \omega_r^2 q^{(r)}(t) = \sum_{r=1}^{\infty}\left\{b_{rs}p^{(r)}(t)\right\} \tag{32}$$

with

$$c_{rs} = \int_{l_0}^{l_N} c(x)\phi^{(r)}(x)\phi^{(s)}(x)dx, \quad b_{rs} = \int_{l_0}^{l_N}\phi^{(r)}(x)\phi^{(s)}(x)dx \tag{33}$$

The truncated k-mode description of the beam Eq. (32) can now be presented in the following matrix form:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}\mathbf{u} \tag{34}$$

where

$$\begin{aligned} &\mathbf{M} = I_{k\times k}, \ \mathbf{C} = [c_{rs}]_{k\times k}, \ \mathbf{K} = [\omega_r^2\delta_{rs}]_{k\times k} \\ &\mathbf{F} = [b_{rs}]_{k\times k}, \ \mathbf{q} = [q^{(1)}(t), q^{(2)}(t), \ldots, q^{(k)}(t)]^T_{k\times 1} \\ &\mathbf{u} = [p^{(1)}(t), p^{(2)}(t), \ldots, p^{(k)}(t)]^T_{k\times k} \end{aligned} \tag{35}$$

Multiplying Eq. (25) by $\phi^{(s)}(x)$ and integrating over x yields:

$$\int_{l_0}^{l_N} m(x)P(x,t)\phi^{(s)}(x)dx = \sum_{r=1}^{\infty}\left\{p^{(r)}(t)\int_{l_0}^{l_N} m(x)\phi^{(r)}(x)\phi^{(s)}(x)dx\right\} = \sum_{r=1}^{\infty}\left\{p^{(r)}(t)\delta_{rs}\right\} \tag{36}$$

which leads to derivation of input vector $\mathbf{u}$ for a given load distribution as follows:

$$p^{(r)}(t) = \int_{l_0}^{l_N} m(x)P(x,t)\phi^{(r)}(x)dx \tag{37}$$

The state-space representation of Eq. (34) is given by:

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{Bu} \tag{38}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \quad \mathbf{B} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{F} \end{Bmatrix}, \quad \mathbf{X} = \begin{Bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Bmatrix} \tag{39}$$

The implementation of the proposed framework will be studied in the next section for a particular case of interest, where the EB beam has two stepped points in cross section and is subjected to a distributed dynamic excitation.

III. CASE STUDY: EB BEAM WITH TWO JUMPED DISCONTINUITY IN CROSS SECTION

A case of study is considered in thin section to demonstrate the implementation of the proposed method for forced vibration analysis of an EB beam with jump discontinuities. Figure 2 depicts a cantilever beam with clamped-free boundary conditions and two jump discontinuities in the cross-section subjected to a vertical load uniformly applied to the middle section. The objective is to derive and depict the mode shapes and frequency response of the beam for a finite number of modes. To observe the effects of the jump on the beam's mode shapes and system's frequency response, several thickness values are considered for the middle cross section as listed in Tables 1. It is assumed that cross-sections 1 and 3 have the same dimensions and properties, and only the thickness of the beam jumps in cross-section 2.

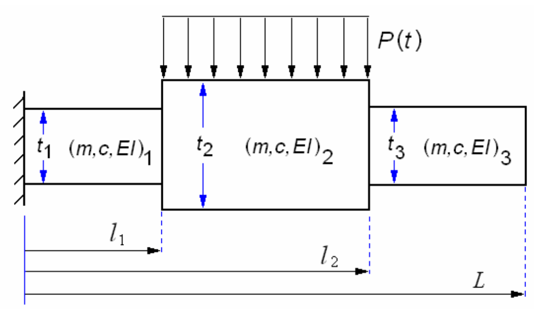

Figure 2. EB beam with two stepped discontinuities in cross section under distributed dynamic load.

Following the steps presented in the previous section, one can obtain the state-space representation of the system. However, the standard form of state-space Single-Input/Single-Output (SISO) representation of the system can be written as:

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{AX} + \mathbf{Bu} \\ \mathbf{Y} &= \mathbf{CX} \end{aligned} \tag{40}$$

Table 1. Beam parameters for numerical simulation of different *thickness* values in the middle section.

Config.	l_1(m)	l_2(m)	L(m)	t_1(m)	t_2(m)	t_3(m)	ω_1(rad/sec)	ω_2(rad/sec)	ω_3(rad/sec)	ω_4(rad/sec)
T1	0.1	0.2	0.3	0.001	0.001	0.001	57.1	357.9	1002.1	1963.7
T2	0.1	0.2	0.3	0.001	0.0015	0.001	58.1	431.9	1059.2	2293.9
T3	0.1	0.2	0.3	0.001	0.002	0.001	56.4	459.5	1052.1	2642.6
T4	0.1	0.2	0.3	0.001	0.0025	0.001	54.4	463.8	1030.3	2906.1
T5	0.1	0.2	0.3	0.001	0.003	0.001	52.5	459.3	1005.5	3070.2

Beam's other parameters:
Density: ρ = 7800(kg/m^3), width: b = 0.01(m), damping coefficient: c = 0.001(N.sec/m), module of elasticity: E = 200(Gpa)

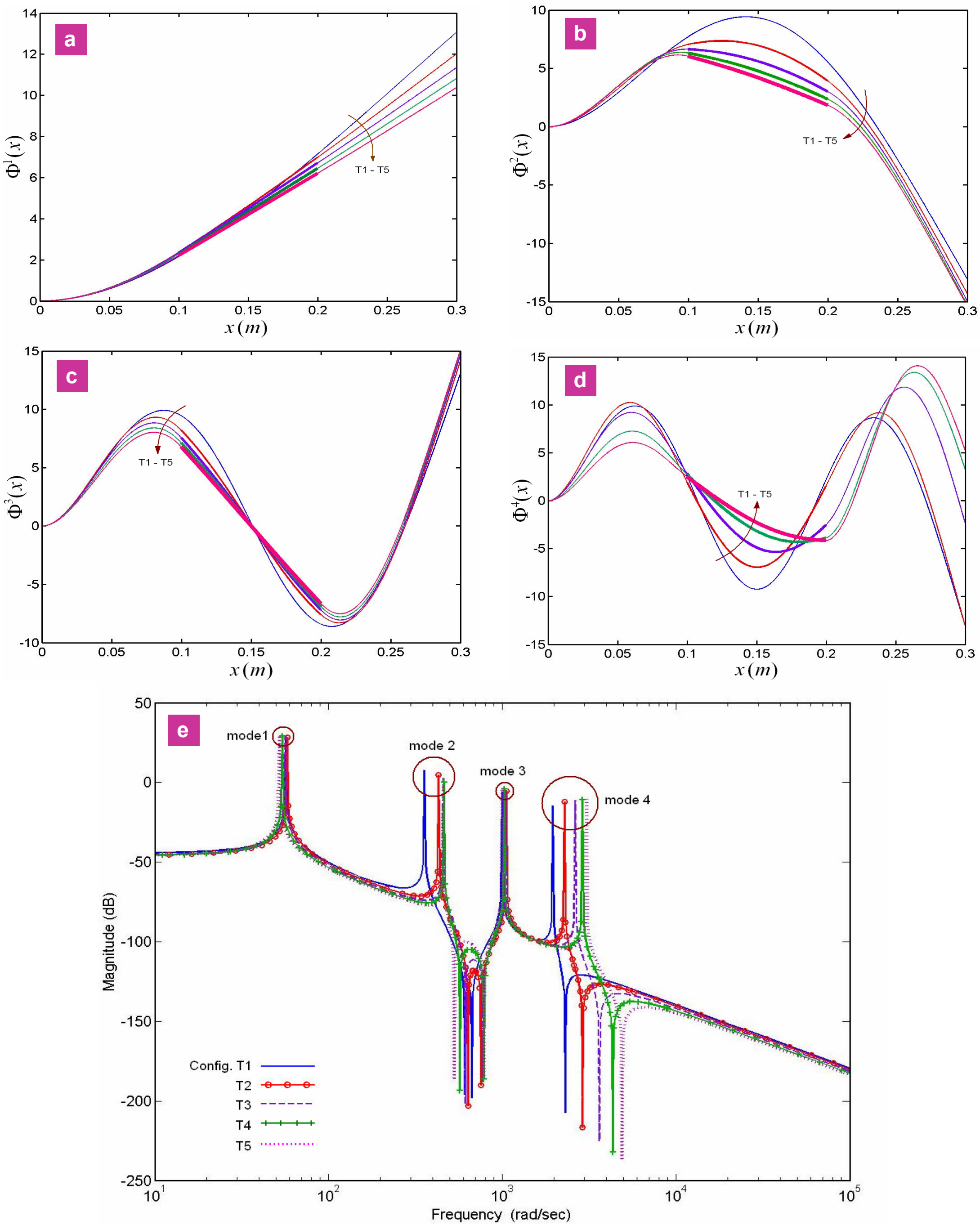

Figure 3. (a) First, (b) second, (c) third, and (d) fourth mode shapes of beams with five different middle section thicknesses, and (e) frequency response plot of beams' tip displacements.

where

$$\mathbf{C} = [\phi^{(1)}(L_0), \phi^{(2)}(L_0), ..., \phi^{(k)}(L_0), 0, ..., 0]_{1\times 2k} \qquad (41)$$

is the output matrix for measuring the displacement of point L_0. The frequency response of the system can now be plotted using beam's transfer function which can be obtained through the Laplace Transformation of its state-space model:

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B} \qquad (42)$$

Table 1 indicates the parameter values used for the simulations, where beam's thickness in middle cross-section varies. Beam's equation of motion has been truncated into four modes, and five different thickness values have been considered for the middle cross-section, one of which being a uniform beam without any jump in cross-section. Figure 3 depicts the mode shapes and frequency response of beam for different configurations. As seen from the figures, mode shapes of the beam significantly change as the thickness of the jump increases. Particularly, it is observable that such a change has more effect on mode shapes 2 and 4 compared to mode shapes 1 and 3. This reason perhaps is due to the fact that the location of the jump is in such a way that the resistance of the middle section against bending is more severe in modes 2 and 4. While in modes 1 and 3, the middle section is located on a fairly straight curvature. This affects not only beam's modes shapes but also its natural frequencies. The frequency response plot given in Figure 3 depicts that the first and third natural frequencies of beam for different jump configurations are localized, in contrast to the frequencies of second and fourth modes, where the frequency peaks are more scattered. The continuity of the mode shapes at jump points is an expected result of the analysis and can be clearly seen from the figures.

IV. Conclusion

This work presented a framework for derivation of mode shapes and state-space representation for EB beams with multiple jumped discontinuities in their cross section. To solve the PDE of motion, the beam was divided into uniform segments of constant parameters, and the continuity conditions were applied at the partitioned points. The characteristics matrix was then formulated using the beam boundary and continuity conditions. Natural frequencies of beam were obtained by setting the determinant of characteristics matrix to zero, and the beam mode shapes coefficients were obtained by integrating the beam characteristics equation and normalization condition. The governing ODE of motion and its state-space representation were then derived for the beam under a distributed dynamic loading condition. To clarify the proposed method, numerical simulations have been presented for a beam with two stepped discontinuities in the cross section. Results indicate that the effects of added mass and stiffness on the beam mode shapes and natural frequencies are significant. Hence, exact methods such as the proposed framework are required for practical implementation of such discontinuous structures.

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