2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Multivariable MRAC for Aircraft with Abrupt Damages

Yu Liu and *Gang Tao*

Abstract— **This paper addresses the compensation of aircraft damages using the direct adaptive control approach. The dynamical modeling of aircraft with damages is first introduced, which demonstrates the complex dynamics when asymmetric mass distribution occurs as the result of damages. An approximate model is proposed under certain flight conditions, and its linearization is performed, which captures the key dynamic features of the aircraft under asymmetric damages. Multivariable model reference adaptive control (MRAC) scheme is demonstrated for control of aircraft in both healthy and post-damage situations, by adapting controller parameters autonomously after the damages occur, without the knowledge of the damage time instants, nor the damage structure and values. Relaxation of design conditions is illustrated by expanding the controller structure and re-designing the adaptive law, to further reduce the need of the post-damage system knowledge.**

Keywords: Multivariable MRAC, aircraft damage compensation, flight control.

I. Introduction

Aviation safety under damaged conditions has attracted increasing research attention recently. Such damages, which are usually uncertain, may cause unknown changes to the aircraft mass, aerodynamic features, and the position of the center of gravity. Under asymmetric damages, the assumption of mass symmetry about x-z plane in aircraft body frame in standard aircraft modeling is no longer valid. Therefore, new aircraft modeling and control techniques are needed. The modeling and control of aircraft with asymmetrical mass distribution is addressed in [1], in which the aircraft dynamics with partial losses of left wing, vertical, and horizontal stabilizers is investigated. A neural network based adaptive control algorithm is introduced for the control of aircraft in the presence of structure uncertainties of damages. In [2], more detailed motion equations are introduced for aircraft with asymmetric mass loss. Simulation results are presented for the comparison between the developed motion equations and standard equations. In [3], we introduce a nonlinear aircraft model with partial wing damage. The motion equations are derived by assuming that the major aircraft mass concentrates in the aircraft fuselage, and the center of gravity shift is small and negligible. The linearization of such an aircraft model is illustrated. In [4], the real time identification of a damaged aircraft model is studied. A two-step identification process is introduced, which consists of an aircraft state estimation phase and an aerodynamic model identification step. With such a two-step process, the nonlinear part of the model identification is isolated in the first phase, and the aerodynamic parameter identification procedure is simplified to a linear one. A hybrid adaptive control method is given in [5], which is applied to aircraft with damages. The control design is based on a neural network parameter estimation blended with a direct adaptive law. A stability and convergence analysis is presented for this adaptive control methodology.

The uncertain damages can cause unknown variations to system structure and parameters, which complicates the control problem. Adaptive control is a suitable solution to this problem, with the capacity of controller adaptation to handle such system uncertainties. In this paper, we will show how multivariable model reference adaptive control (MRAC) can be applied to the control of aircraft with damages. Such a control design is expected to control the aircraft under both nominal and damage conditions without any detections and control switches. We will first introduce an aircraft model under asymmetric damages derived in [2]. We will also show that under certain flight conditions the aircraft dynamic model can be simplified, and a linearized aircraft model will be derived based on the simplified nonlinear model, which captures the main characteristics of the aircraft dynamics under asymmetric damages. We will then present a multivariable MRAC scheme and specify the conditions needed for it to be applicable to systems whose parameters may jump in values at a finite number of time instants. Closed-loop stability and asymptotic output tracking are ensured under both nominal and damage conditions without any detections and control switches. The key to a desirable adaptive design is to use as less as possible knowledge about the unknown system to be controlled, so that the class of systems admissible by the adaptive controller is as large as possible, leading to the effective handling of aircraft system damage conditions as adverse as possible.

The paper is organized as follows. In Section II, we present both the nonlinear and linearized models of aircraft with damages. In Section III, we develop the multivariable MRAC scheme for control of aircraft with damages. The adaptive control design is based on the decomposition of the system high frequency gain matrix (to use less its knowledge) and ensures the closed-loop stability and asymptotic output tracking under damage conditions (shown using a discontinuous Lyapunov function). Expansion of the controller structure and re-design of the adaptive law are discussed in Section IV to relax certain design conditions, for handling wider classes of adverse damage conditions of an aircraft system. In Section V, we discuss some ongoing research tasks.

This research was supported in part by NASA under the grant NNX08AB99A and NSF under the grant ECS0601475.

Yu Liu is with the Department of Electrical and Computer Engineering, University of Virginia, VA 22904 yuliu@virginia.edu

Gang Tao is with Faculty of Electrical and Computer Engineering, University of Virginia, VA 22904 gt9s@virginia.edu

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

II. Modeling of Aircraft with Damages

For control of damaged aircraft, the modeling of aircraft under damages is of particular importance. In this section, we will introduce a nonlinear model of aircraft with two wing-mounted engines with asymmetric damages. Its linearization will be performed under certain flight conditions.

A. Nonlinear Aircraft Model with Damages

A nonlinear aircraft model with asymmetric damages is developed in [2], by analyzing the three-dimensional kinetics of a rigid body. In this subsection, we present an introduction to this nonlinear model.

Force equations. Force equations can be achieved by applying Newton's second law to the shifted center of gravity of the aircraft after damages. They can be written as

$$\begin{aligned}&\dot{u} + qw - rv - (q^2 + r^2)\Delta x + (pq - \dot{r})\Delta y \\ &\quad +(pr + \dot{q})\Delta z = X/m - g\sin\theta + (T_L + T_R)/m, \quad (1)\\ &\dot{v} + ur - pw + (pq + \dot{r})\Delta x - (p^2 + r^2)\Delta y \\ &\quad +(qr - \dot{p})\Delta z = Y/m + g\cos\theta\sin\phi, \quad (2)\\ &\dot{w} + pv - uq + (pr - \dot{q})\Delta x + (\dot{p} + qr)\Delta y \\ &\quad -(p^2 + q^2)\Delta z = Z/m + g\cos\theta\cos\phi, \quad (3)\end{aligned}$$

where m is the mass of the aircraft, X, Y and Z are body-axis aerodynamic forces, θ and ϕ are Euler pitch and roll angle, u, v and w are the body-axis velocity components of the origin of the body-axis frame, p, q and r are the body-axis components of the angular velocity, $[\Delta x, \Delta y, \Delta z]^T$ are the coordinates of the shifted center of gravity in the body frame. The engine thrusts on two sides are denoted as T_L and T_R, and they are assumed to be parallel to the body x-axis. The mass m may have uncertain changes due to the damages, and X, Y and Z may also have unknown additional changes.

Moment equations. To derive the moment equations for the aircraft, we can consider the aircraft body as a collection of mass particles that are rigidly connected.

In standard aircraft modeling, the mass of the aircraft is assumed to be symmetric about x-z plane, so that the cross-products of inertia I_{xy} and I_{yz} are zero [6]. However, with asymmetric mass distribution due to damages, I_{xy} and I_{yz} become nonzero. By analyzing the rotational motions of the mass particles, one can obtain the following moment equations for asymmetric aircraft:

$$\begin{aligned}&I_x\dot{p} + I_{xy}\dot{q} + I_{xz}\dot{r} - I_{xy}pr - (I_z - I_y)qr + I_{yz}(q^2 - r^2) \\ &\quad + I_{xz}pq + m\Delta y(\dot{w} - qu + pv) + m\Delta z(-\dot{v} - ru + pw) \\ &\quad = L + mg\cos\theta\cos\phi\Delta y - mg\cos\theta\sin\phi\Delta z, \quad (4)\\ &I_{xy}\dot{p} + I_y\dot{q} + I_{yz}\dot{r} + (I_x - I_z)pr + I_{xy}qr + I_{xz}(r^2 - p^2) \\ &\quad - I_{yz}pq + m\Delta x(qu - \dot{w} - pv) + m\Delta z(\dot{u} - rv + qw) \\ &\quad = M - mg\sin\theta\Delta z - mg\cos\theta\cos\phi\Delta x \\ &\quad -\Delta z(T_L + T_R), \quad (5)\\ &I_{xz}\dot{p} + I_{yz}\dot{q} + I_z\dot{r} + (I_y - I_x)pq + I_{xy}(p^2 - q^2) - I_{xz}qr \\ &\quad + I_{yz}pr + m\Delta x(\dot{v} - pw + ru) + m\Delta y(rv - \dot{u} - qw) \\ &\quad = N + mg\cos\theta\sin\phi\Delta x + mg\sin\theta\Delta y + T_L(l + \Delta y) \\ &\quad -T_R(l - \Delta y), \quad (6)\end{aligned}$$

with I_i being the inertia moments and products in body axes. The force and moment equations (1)–(3) and (4)–(6) are much more complicated than their counterparts without damages. From control design point of view, it is desirable to develop simplified models under certain flight conditions, which still capture the main dynamic features of the aircraft in the presence of damages.

B. Model Simplification

In this study, we consider the following aircraft system and flight conditions. First, we assume that the center of gravity shift Δr is small, since major aircraft mass concentrates in the fuselage. A study in [1] shows that the magnitude of the shift along y-axis, Δy, is larger than Δx and Δz, and it is still less than 2.5% of the wing span when 50% of the left wing is lost. Second, we consider a transport aircraft flying within the neighborhood of a rectilinear flight, so that the angular velocities p, q, and r, as well as their rates, are in the neighborhood of zero.

To simplify the aircraft model under the above conditions, we will neglect the higher order terms involving the center of gravity shift, angular velocities, and angular accelerations. Such a principle leads to the following simplified force equations and moment equations.

Simplified force equations:

$$\begin{aligned}&\dot{u} + qw - rv = X/m - g\sin\theta + (T_L + T_R)/m \quad (7)\\ &\dot{v} + ur - pw = Y/m + g\cos\theta\sin\phi \quad (8)\\ &\dot{w} + pv - uq = Z/m + g\cos\theta\cos\phi. \quad (9)\end{aligned}$$

These force equations are similar to the force equations in the standard aircraft model. The two engine thrusts appear in the first equation because they are parallel to x-axis.

Simplified moment equations:

$$\begin{aligned}&I_x\dot{p} + I_{xy}\dot{q} + I_{xz}\dot{r} - I_{xy}pr - (I_z - I_y)qr + I_{yz}(q^2 - r^2) + I_{xz}pq \\ &\quad = L + mg\cos\theta\cos\phi\Delta y - mg\cos\theta\sin\phi\Delta z, \quad (10)\\ &I_{xy}\dot{p} + I_y\dot{q} + I_{yz}\dot{r} + (I_x - I_z)pr + I_{xz}(r^2 - p^2) + I_{xy}qr - I_{yz}pq \\ &\quad = M - mg\cos\theta\cos\phi\Delta x - mg\sin\theta\Delta z - \Delta z(T_L + T_R), \quad (11)\\ &I_{xz}\dot{p} + I_{yz}\dot{q} + I_z\dot{r} + (I_y - I_x)pq - I_{xz}qr + I_{xy}(p^2 - q^2) + I_{yz}pr \\ &\quad = N + mg\cos\theta\sin\phi\Delta x + mg\sin\theta\Delta y + T_L(l + \Delta y) \\ &\quad -T_R(l - \Delta y). \quad (12)\end{aligned}$$

The moment equations characterize the key effects of asymmetric aircraft mass distribution, i.e., the appearance of nonzero products of inertia I_{xy} and I_{yz}, and gravity and thrust moments due to the center of gravity shift.

C. Linearized Aircraft Model with Damages

In this subsection, we derive the linearized aircraft model from the above simplified nonlinear model. The state and control vectors for linearization are chosen as

$$x = [\,u \quad w \quad q \quad \theta \quad v \quad r \quad p \quad \phi \quad \psi\,]^T \quad (13)$$

$$U = [\,\delta_e \quad \delta_{t_l} \quad \delta_{t_r} \quad \delta_a \quad \delta_r\,]^T \quad (14)$$

where the notation "δ" has been dropped from δx and δU for simplicity of presentation. Thus u, v, and w represent the velocity perturbations along body axes, p, q and r are the angular velocity perturbations, θ, ϕ and ψ are the pitch, roll and yaw angle perturbations, and δ_e, δ_a, δ_r are the deflection perturbations of the elevator, aileron and rudder. δ_{t_l} and δ_{t_r} are the left and right throttle perturbations.

The equilibrium is chosen to be a rectilinear wing-level flight condition, which can include straight horizontal, ascending, or descending flight. Using small perturbation linearization [7], [10], we can obtain the linearized aircraft model with the following structure:

$$\dot{x} = \begin{bmatrix} A^{(1)}_{4\times 4} & A^{(2)}_{4\times 5} \\ A^{(3)}_{5\times 4} & A^{(4)}_{5\times 5} \end{bmatrix} x + \begin{bmatrix} B^{(1)}_{4\times 3} & B^{(2)}_{4\times 2} \\ B^{(3)}_{5\times 3} & B^{(4)}_{5\times 2} \end{bmatrix} U, \tag{15}$$

where

$$A^{(1)} = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial w} & \frac{\partial f_1}{\partial q} & -g\cos\theta_o \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial w} & \frac{\partial f_2}{\partial q} & -g\sin\theta_o \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial w} & \frac{\partial f_3}{\partial q} & \frac{\partial f_3}{\partial \theta} \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{16}$$

$$A^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial r} & \frac{\partial f_3}{\partial p} & \frac{\partial f_3}{\partial \phi} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{17}$$

$$A^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\partial f_6}{\partial u} & \frac{\partial f_6}{\partial w} & \frac{\partial f_6}{\partial q} & \frac{\partial f_6}{\partial \theta} \\ \frac{\partial f_7}{\partial u} & \frac{\partial f_7}{\partial w} & \frac{\partial f_7}{\partial q} & \frac{\partial f_7}{\partial \theta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{18}$$

$$A^{(4)} = \begin{bmatrix} \frac{\partial f_5}{\partial v} & \frac{\partial f_5}{\partial r} & \frac{\partial f_5}{\partial p} & g\cos\theta_o & 0 \\ \frac{\partial f_6}{\partial v} & \frac{\partial f_6}{\partial r} & \frac{\partial f_6}{\partial p} & \frac{\partial f_6}{\partial \phi} & 0 \\ \frac{\partial f_7}{\partial v} & \frac{\partial f_7}{\partial r} & \frac{\partial f_7}{\partial p} & \frac{\partial f_7}{\partial \phi} & 0 \\ 0 & \tan\theta_o & 1 & 0 & 0 \\ 0 & \frac{1}{\cos\theta_o} & 0 & 0 & 0 \end{bmatrix}, \tag{19}$$

$$B^{(1)} = \begin{bmatrix} \frac{\partial f_1}{\partial \delta_e} & \frac{\partial f_1}{\partial \delta_{tl}} & \frac{\partial f_1}{\partial \delta_{tr}} \\ \frac{\partial f_2}{\partial \delta_e} & 0 & 0 \\ \frac{\partial f_3}{\partial \delta_e} & \frac{\partial f_3}{\partial \delta_{tl}} & \frac{\partial f_3}{\partial \delta_{tr}} \\ 0 & 0 & 0 \end{bmatrix}, B^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\partial f_3}{\partial \delta_a} & \frac{\partial f_3}{\partial \delta_r} \\ 0 & 0 \end{bmatrix}, \tag{20}$$

$$B^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial f_6}{\partial \delta_e} & \frac{\partial f_6}{\partial \delta_{tl}} & \frac{\partial f_6}{\partial \delta_{tr}} \\ \frac{\partial f_7}{\partial \delta_e} & \frac{\partial f_7}{\partial \delta_{tl}} & \frac{\partial f_7}{\partial \delta_{tr}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B^{(4)} = \begin{bmatrix} \frac{\partial f_5}{\partial \delta_a} & \frac{\partial f_5}{\partial \delta_r} \\ \frac{\partial f_6}{\partial \delta_a} & \frac{\partial f_6}{\partial \delta_r} \\ \frac{\partial f_7}{\partial \delta_a} & \frac{\partial f_7}{\partial \delta_r} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{21}$$

For the interest of conciseness, the explicit expressions of the derivatives in A and B are not shown in the paper.

Comparing the developed aircraft model with the standard model in the literature [6], we can see that $A^{(1)}$, $A^{(4)}$, $B^{(1)}$, and $B^{(4)}$ have similar forms with those in standard models. However, all the derivatives in those matrices may be subject to unknown changes due to the changes of aerodynamic characteristics and aircraft mass caused by damages. Another feature of this model is that the matrices $A^{(2)}$, $A^{(3)}$, $B^{(2)}$, and $B^{(3)}$, which are zero in the nominal aircraft model, become nonzero under damages. The existence of the nonzero derivatives results from the asymmetric aircraft mass. Such nonzero derivatives may change from zero to nonzero when damages occur.

The existence of the coupling terms leads to an aircraft model with coupled longitudinal and lateral dynamics. With the changing and coupling derivatives, this model captures the essential flight dynamics under damages. It is critical to use such models for the design of adaptive control schemes for accommodating asymmetric aircraft damages.

III. Multivariable MRAC Design and Analysis

In this section, we present a multivariable model reference adaptive control design for aircraft with damages. Such a design is expected to handle the uncertainties of the aircraft system without and with damages, provided that certain design conditions are satisfied.

A. Problem Formulation

Consider a linear system of the form

$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) \tag{22}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $C \in R^{m\times n}$ are unknown parameter matrices, $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in R^m$ are the state, input and output vectors. To represent an aircraft model, we let A and B be expressed as

$$A = A_0 + \Delta A \tag{23}$$

$$B = B_0 + \Delta B \tag{24}$$

where A_0 and B_0 are the nominal parameter matrices for aircraft without damages, and ΔA and ΔB contain the unknown coupling terms and derivative changes caused by damages (they are zero when there are no damages).

The objective is to design a control vector signal $u(t)$ such that the plant output $y(t)$ tracks a given reference output

$$y_m(t) = W_m(s)[r](t) \in R^m \tag{25}$$

for a stable $m \times m$ transfer matrix $W_m(s)$ and a bounded reference signal $r(t) \in R^m$, despite the uncertain damages.

For control design, we make the following assumptions:

(A0): the parameter matrices A, B and C are piecewise constant, with a finite number of unknown and constant jumps (A_i, B_i, C_i), $i = 1, 2, \ldots, N$.

For each value (A_i, B_i, C_i) of (A, B, C), we define the transfer matrix $G_i(s) = C_i(sI - A_i)^{-1}B_i$ and assume:

(A1): All zeros of $G_i(s)$ are stable. (A2): An upper bound $\bar{\nu}$ on the observability index of $G_i(s)$ is known. (A3): $G_i(s)$ is strictly proper with full rank and has a known modified interactor matrix $\xi_m(s)$ such that $\lim_{s\to\infty} \xi_m(s)G_i(s) = K_{pi}$, the high frequency gain matrix of $G_i(s)$, is finite and nonsingular. (A4): $W_m(s) = \xi_m^{-1}(s)$. (A5): All leading principal minors of the matrix K_{pi} are nonzero and their signs are known and the same for each i.

These are basic assumptions for multivariable MRAC. For MRAC, the plants need to be minimum phase systems. The need of the uniform $\xi_m(s)$ of $G_i(s)$ and signs of the leading principal minors of K_{pi} is to show how a standard multivariable MRAC scheme can be used to handle the system piecewise-constant parameter variations which can occur in aircraft systems with damages. It should be noted that the system parameters are allowed to jump, and such jumps will not cause instability as shown in Section (III.C)

The cases with $\xi_m(s)$ unknown or $\xi_m(s)$ different for different $G_i(s)$, and with unknown or changed signs of leading principal minors of K_{pi} can also be handled, and will be addressed in Section IV.

B. Plant-Model Matching Controllers

For model reference adaptive control design, we need to define a nominal model reference controller which achieves the desired control objective when the system parameters A, B and C are known. Parameters of such a controller, which are unknown, are also utilized in deriving an error model needed for adaptation of an adaptive controller.

Since the system parameters (A, B, C) may take any of (A_i, B_i, C_i), $i = 1, 2, \ldots, N$, there is a set of such nominal controllers, and each of them has the structure

$$u^*(t) = \Theta_1^{*T}\omega_1(t) + \Theta_2^{*T}\omega_2(t) + \Theta_{20}^* y(t) + \Theta_3^*(t)r(t) \quad (26)$$

where $\omega_1(t) = F(s)[v](t)$, $\omega_2 = F(s)[y](t)$, $F(s) = \frac{A_F(s)}{\Lambda(s)}$, $A_F(s) = [I, sI, \ldots, s^{\bar{\nu}-2}I]^T$, $\Lambda(s)$ is a monic stable polynomial of degree $\bar{\nu} - 1$, with the upper bound $\bar{\nu}$ on the observability indices of $G_i(s)$. The nominal parameters $\Theta_1^* = [\Theta_{11}^*, \ldots, \Theta_{1\bar{\nu}-1}^*]^T$, $\Theta_2^* = [\Theta_{21}^*, \ldots, \Theta_{2\bar{\nu}-1}^*]^T$, Θ_{20}^*, Θ_3^*, $\Theta_{ij}^* \in R^{m\times m}$, $i = 1, 2$, $j = 1, \ldots, \bar{\nu} - 1$, are for plant-model matching, and are derived next.

We first introduce the following notation:

$$G_i(s) = C_i(sI - A_i)^{-1}B_i = Z_i(s)P_i^{-1}(s) \quad (27)$$

for some $m \times m$ right coprime polynomial matrices $Z_i(s)$ and $P_i(s)$ with $P_i(s)$ being column proper, $i = 1, 2, \ldots, N$.

With the specification of $\Lambda(s)$, $\xi_m(s)$, $P_i(s)$, $Z_i(s)$, there exist Θ_1^*, Θ_2^*, Θ_{20}^*, $\Theta_3^* = K_{pi}^{-1}$ such that

$$\begin{aligned} &\Theta_1^{*T} A_F(s)P_i(s) + (\Theta_2^{*T} A_F(s) + \Lambda(s)\Theta_{20}^*)Z_i(s) \\ = \;& \Lambda(s)(P_i(s) - \Theta_3^*\xi_m(s)Z_i(s)). \end{aligned} \quad (28)$$

Since $\Lambda(s)$ and $Z_i(s)$ are stable, we have the plant-model transfer matrix matching equation

$$I - \Theta_1^{*T}F(s) - \Theta_2^{*T}F(s)G_i(s) - \Theta_{20}^* G_i(s) = \Theta_3^* W_m^{-1}(s)G_i(s) \quad (29)$$

from which the plant-model matching parameters Θ_1^*, Θ_2^*, and Θ_{20}^* can be determined with $\Theta_3^* = K_{pi}^{-1}$. For each (A_i, B_i, C_i), $i = 1, 2, \ldots, N$, we can determine a set of constant parameters Θ_j^*, $j = 1, 2, 20, 3$. So for all the possible values of (A, B, C), plant-model matching parameters are piecewise constant, and the plant-model matching equation is also a piecewise equation.

C. Adaptive Control Scheme

To design the adaptive control scheme, a high frequency gain decomposition will be first introduced. As shown in [8] and [9], such a decomposition based design relies less on the *a priori* knowledge of the high frequency gain matrix. Then we develop a model reference adaptive control design, and establish the desired stability and tracking properties.

LDS decomposition of K_p [8], [9]. Let Δ_i, $i = 1, 2, \ldots, m$, denote the leading principal minors of the high frequency gain matrix $K_p \in R^{m\times m}$ and assume that $\Delta_i \neq 0$, $i = 1, 2, \ldots, m$. The gain matrix K_p then has a nonunique decomposition

$$K_p = L_s D_s S, \quad (30)$$

where $S \in R^{m\times m}$ is a symmetric and positive definite matrix, L_s is an $m \times m$ unit lower triangular matrix, and

$$\begin{aligned} D_s &= \mathrm{diag}\{s_1^*, s_2^*, \ldots, s_M^*\} \\ &= \mathrm{diag}\left\{\mathrm{sign}[\Delta_1]\gamma_1, \mathrm{sign}\left[\frac{\Delta_2}{\Delta_1}\right]\gamma_2, \ldots, \mathrm{sign}\left[\frac{\Delta_m}{\Delta_{m-1}}\right]\gamma_m\right\} \end{aligned} \quad (31)$$

such that $\gamma_i > 0$, $i = 1, \ldots, m$, may be arbitrary. All K_{pi} can have the same D_s based to Assumption (A5).

Adaptive controller. When plant parameters are uncertain, the controller parameters Θ_1^*, Θ_2^*, Θ_{20}^*, Θ_3^* are also unknown. As the adaptive version of (26), we use the controller

$$u(t) = \Theta_1^T(t)\omega_1(t) + \Theta_2^T(t)\omega_2(t) + \Theta_{20}(t)y(t) + \Theta_3(t)r(t) \quad (32)$$

where $\Theta_1(t)$, $\Theta_2(t)$, $\Theta_{20}(t)$, and $\Theta_3(t)$ are estimates of Θ_1^*, Θ_2^*, Θ_{20}^*, and Θ_3^*, and will be adaptively updated.

Error dynamics. From the plant-model transfer matrix matching equation (29), for any $u(t)$, we have

$$u(t) - \Theta_1^{*T}\omega_1(t) - \Theta_2^{*T}\omega_2(t) - \Theta_{20}^* y(t) = \Theta_3^* W_m^{-1}(s)[y](t), \quad (33)$$

from which, together with the reference model (25) and Assumption (A4), we obtain

$$\begin{aligned} &K_p\left(u(t) - \Theta_1^{*T}\omega_1(t) - \Theta_2^{*T}\omega_2(t) - \Theta_{20}^* y(t) - \Theta_3^* r(t)\right) \\ &= \xi_m(s)[y - y_m](t). \end{aligned} \quad (34)$$

With the LDS decomposition in (30), we express (34) as

$$\begin{aligned} &D_s S\left(u(t) - \Theta_1^{*T}\omega_1(t) - \Theta_2^{*T}\omega_2(t) - \Theta_{20}^* y(t) - \Theta_3^* r(t)\right) \\ &= L_s^{-1}\xi_m(s)[y - y_m](t). \end{aligned} \quad (35)$$

From (35) and the adaptive controller (32), we can have

$$\xi_m(s)[y - y_m](t) + \Theta_0^*\xi_m(s)[y - y_m](t) = D_s S\tilde{\Theta}^T(t)\omega(t), \quad (36)$$

where D_s is in (31), $S = S^T > 0$ in (30), $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ with $\Theta(t)$ being the estimate of $\Theta^* = [\Theta_1^{*T}, \Theta_2^{*T}, \Theta_{20}^*, \Theta_3^*]^T$, $\omega(t) = [\omega_1^T(t), \omega_2^T(t), y^T(t), r^T(t)]^T$, and, with L_s in (30), $\Theta_0^* = L_s^{-1} - I$ has a special form:

$$\Theta_0^* = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \theta_{21}^* & 0 & 0 & \cdots & 0 \\ \theta_{31}^* & \theta_{32}^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{m-1\,1}^* & \cdots & \theta_{m-1\,m-2}^* & 0 & 0 \\ \theta_{m\,1}^* & \cdots & \theta_{m\,m-2}^* & \theta_{m\,m-1}^* & 0 \end{bmatrix} \quad (37)$$

Given this structure of Θ_0^*, we define the parameter vectors

$$\begin{aligned} \theta_2^* &= \theta_{21}^* \in R, \\ \theta_3^* &= [\theta_{31}^*, \theta_{32}^*]^T \in R^2, \\ &\vdots \\ \theta_m^* &= [\theta_{m\,1}^*, \ldots, \theta_{m\,m-1}^*]^T \in R^{m-1} \end{aligned} \quad (38)$$

and let their estimates be $\theta_i(t)$, $i = 2, 3, \ldots, m$. Choose $f(s)$ as a stable and monic polynomial whose degree is equal to the maximum degree of $\xi_m(s)$, introduce the filter $h(s) = \frac{1}{f(s)}$, define the filtered tracking error

$$\bar{e}(t) = \xi_m(s)h(s)[e](t) = [\bar{e}_1(t), \ldots, \bar{e}_m(t)]^T \quad (39)$$

with $e(t) = y(t) - y_m(t)$, and denote

$$\eta_i(t) = [\bar{e}_1(t), \ldots, \bar{e}_{i-1}(t)]^T \in R^{i-1},\ i = 2, \ldots, m. \quad (40)$$

Operating both sides of (36) by $h(s)I_m$ leads to

$$\begin{aligned} &\bar{e}(t) + [0, \theta_2^{*T}\eta_2(t), \theta_3^{*T}\eta_3(t), \ldots, \theta_m^{*T}\eta_m(t)]^T \\ &= D_s S h(s)[\tilde{\Theta}^T\omega](t). \end{aligned} \quad (41)$$

Based on this equation, we define the estimation error

$$\begin{aligned} \epsilon(t) = &\bar{e}(t) + [0, \theta_2^T(t)\eta_2(t), \theta_3^T(t)\eta_3(t), \ldots, \theta_m^T(t)\eta_m(t)]^T \\ &+\Psi(t)\xi(t), \end{aligned} \quad (42)$$

where $\Psi(t)$ is the estimate of $\Psi^* = D_s S$, and

$$\xi(t) = \Theta^T(t)\zeta(t) - h(s)[\Theta^T\omega](t), \quad (43)$$

$$\zeta(t) = h(s)[\omega](t). \quad (44)$$

It then follows from (41)– (44) that

$$\begin{aligned} \epsilon(t) \ = \ &[0, \tilde{\theta}_2^T(t)\eta_2(t), \tilde{\theta}_3^T(t)\eta_3(t), \ldots, \tilde{\theta}_m^T(t)\eta_m(t)]^T \\ &+ D_s S\tilde{\Theta}(t)^T\zeta(t) + \tilde{\Psi}(t)\xi(t), \end{aligned} \quad (45)$$

where $\tilde{\theta}_i(t) = \theta_i(t) - \theta_i^*$, and $\tilde{\Psi}(t) = \Psi(t) - \Psi^*$.

Adaptive laws. We choose the adaptive laws

$$\dot{\theta}_i(t) \ = \ -\frac{\Gamma_{\theta i}\epsilon_i(t)\eta_i(t)}{m^2(t)},\ i = 2, 3, \ldots, m, \quad (46)$$

$$\dot{\Theta}^T(t) \ = \ -\frac{D_s\epsilon(t)\zeta^T(t)}{m^2(t)}, \quad (47)$$

$$\dot{\Psi}(t) \ = \ -\frac{\Gamma\epsilon(t)\xi^T(t)}{m^2(t)}, \quad (48)$$

where $\epsilon(t) = [\epsilon_1(t), \epsilon_2(t), \ldots, \epsilon_m(t)]^T$,

$$m^2(t) = 1 + \zeta^T(t)\zeta(t) + \xi^T(t)\xi(t) + \sum_{i=2}^{m}\eta_i^T(t)\eta_i(t), \quad (49)$$

and $\Gamma_{\theta i} = \Gamma_{\theta i}^T > 0$, $i = 2, 3, \ldots, m$, $\Gamma = \Gamma^T > 0$.

Stability analysis. To demonstrate the stability of the closed-loop system, we choose a piece-wise continuous Lyapunov function. Based on Assumption (A0), there are $N-1$ finite jumps due to the damages, and totally N choices of (A_i, B_i, C_i). Assuming that the asymmetric damage occurs at time instant t_j, $j = 1, 2, \ldots, N-1$, we choose the following Lyapunov-like function

$$V = \sum_{i=2}^{m}\tilde{\theta}_i^m\Gamma_{\theta i}^{-1}\tilde{\theta}_i + \mathrm{tr}[\tilde{\Psi}^T\Gamma^{-1}\tilde{\Psi}] + \mathrm{tr}[\tilde{\Theta}S\tilde{\Theta}^T] \quad (50)$$

for time intervals (t_{j-1}, t_j), $j = 1, \ldots, N$, with $t_0 = 0$ and $t_N = \infty$. Due to the changes of system parameters after the damages (which are finite), and the finite jumps of nominal parameters, there would be a finite jump of V for each jump of system parameters (A_i, B_i, C_i), i.e.,

$$V(t_j^+) - V(t_j^-) < \infty, j = 1, 2, \ldots, N-1. \quad (51)$$

From the adaptive laws (46)– (48), we obtain the time-derivative of V in each (t_{j-1}, t_j) as

$$\begin{aligned} \dot{V} = &-\frac{2}{m^2(t)}(\sum_{i=2}^{m}\tilde{\theta}^T(t)\epsilon_i(t)\eta_i(t) + \mathrm{tr}[\tilde{\Psi}^T\epsilon(t)\xi^T(t)] \\ &+\mathrm{tr}[\tilde{\Theta}(t)SD_s\epsilon(t)\zeta^T(t)]) = -\frac{2\epsilon^T(t)\epsilon(t)}{m^2(t)} \leq 0. \end{aligned} \quad (52)$$

Recall that V is not continuous at instant t_j, $j = 1, 2, \ldots, N-1$ and has only finite jumps at those instants. For each time interval (t_{j-1}, t_j), $j = 1, \ldots, N$, we have $\dot{V} \leq 0$, which implies that V is bounded. With fact that V only has finite jumps at finite time instants, we can conclude that V is bounded for $[0, \infty)$. So we can conclude that $\theta_i(t) \in L^\infty$, $\Theta(t) \in L^\infty$, and $\Psi(t) \in L^\infty$.

Integrating both sides of (52) for the time interval (t_{j-1}, t_j), we have

$$\int_{t_{j-1}}^{t_j}\frac{2\epsilon^T(\tau)\epsilon(\tau)}{m^2(\tau)}d\tau = V(t_{j-1}^+) - V(t_j^-). \quad (53)$$

For N intervals: $[0, t_1)$, (t_1, t_2), ..., (t_{N-1}, ∞), the above equation holds. Summing both side of (53) for $j = 1, \ldots, N$, we have

$$\begin{aligned} &\int_0^\infty\frac{2\epsilon^T(\tau)\epsilon(\tau)}{m^2(\tau)}d\tau \\ &= V(0) - V(t_1^-) + V(t_1^+) - V(t_2^-) + V(t_2^+) \\ &\quad - \cdots - V(t_j^-) + V(t_j^+) - V(t_{j+1}^-) + V(t_{j+1}^+) \\ &\quad - \cdots - V(t_{N-1}^-) + V(t_{N-1}^+) - V(\infty) \\ &= V(0) + \sum_{j=1}^{N-1}[V(t_j^+) - V(t_j^-)] - V(\infty) \end{aligned} \quad (54)$$

From (51), the condition that $V(t_j^+) - V(t_j^-)$ is finite, and (54), we obtain

$$\int_0^\infty\frac{2\epsilon^T(\tau)\epsilon(\tau)}{m^2(\tau)}d\tau \leq \infty, \quad (55)$$

from which we can obtain $\frac{\epsilon(t)}{m(t)} \in L^2 \cap L^\infty$. Similarly, we can obtain $\dot{\theta}_i(t) \in L^2 \cap L^\infty$, $i = 2, 3, \ldots, m$, $\dot{\Theta}(t) \in L^2 \cap L^\infty$, and $\dot{\Psi}(t) \in L^2 \cap L^\infty$. Based on these desired properties, we can establish the following results.

Theorem 1: The MRAC scheme consisting of (32), (46), (47) and (48), ensures closed-loop signal boundedness and asymptotic output tracking $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$, for the system (22) satisfying Assumptions (A0)–(A5).

The proof of this theorem can be obtained in a way similar to that in [8], based on the fact that there is a well-defined feedback structure for the closed-loop system which has a small loop gain, leading to closed-loop stability. Such a structure is developed from the feedback controller with bounded parameters and the plant with stable zeros. The smallness of its loop gain is ensured by the L^2 properties of the adaptive laws. The asymptotic tracking property follows from the complete parametrization of the error equation (42), the L^2 properties, and the signal boundedness of the closed-loop system. The piecewise continuous Lyapunov function technique provides a powerful tool to the stability analysis of MRAC for aircraft with multiple finite damages.

IV. **Relaxation of Design Conditions**

We now address the issue of relaxing or ensuring of the design conditions in Assumption (A2): the observability index upper bound $\bar{\nu}$ is known for $G_i(s)$; (A3): $\xi_m(s)$ is known and fixed for all $G_i(s)$; (A4): $W_m(s) = \xi_m^{-1}(s)$; and (A5): All leading principal minors of the matrix K_{pi} are nonzero and their signs are known and the same for each i.

Relaxation of Assumption (A2). For the set of parameter matrices (A_i, B_i, C_i), $i = 1, 2, \ldots, N$, we may obtain the upper bound on their observability indices for MRAC design. For aircraft control, an upper bound on observability index for both healthy and post-damage systems can be used for the adaptive controller structure.

Relaxation of Assumptions (A3) and (A4). In the proposed MRAC scheme, the interactor matrix $\xi_m(s)$, which characterizes the infinity zero structure of a multivariable system, is required to remain the same when the damages occur. If this condition is not satisfied, an expanded MRAC scheme can be used. In [11], a MRAC scheme is introduced with a new parametrization of the controller

$$u(t) = \Theta_1^T \omega_1(t) + \Theta_2^T \omega_2(t) + \Theta_{20} y(t) + \Theta_3 \omega_3(t) \quad (56)$$

where $\omega_1(t) = F(s)[v](t)$, $\omega_2 = F(s)[y](t)$, $\omega_3(t) = B_F(s)[y_m(t)]$, $F(s) = \frac{A_F(s)}{\Lambda(s)}$, $A_F(s) = [I, sI, \ldots, s^{\bar{\nu}-2}I]^T$, $B_F(s) = [I, sI, \ldots, s^{\bar{d}}I]^T$, with $\bar{d}$ being the upper bound on d (the maximum of the degrees of $\xi_{mi}(s)$ for each $G_i(s)$), Θ_1, Θ_2, Θ_{20}, and Θ_3 are controller parameters with proper dimensions, and $\Lambda(s)$ is a monic stable polynomial of degree $\bar{\nu} - 1$, with the upper bound $\bar{\nu}$ on the observability indices of $G_i(s)$. The signal $y_m(t)$ is generated from a new reference system which does not use the knowledge of $\xi_{mi}(s)$ for each $G_i(s)$ [11]. This approach does not require the exact information of $\xi_{mi}(s)$, and only the upper bound on the maximum degree of its elements is incorporated in the control design. The MRAC scheme ensures that all signals in the adaptive system are uniformly bounded and the tracking error goes to zero asymptotically. Thus, the design condition in Assumption (A3) can be relaxed.

Relaxation of Assumption (A5). As seen in (31) and (47), the knowledge of the signs of the leading principal minors of K_{pi} is used in the adaptive laws. To relax such knowledge, the adaptive laws can be modified by using the well-known Nussbaum gains in the place of those signs. A complete design of such a multivariable MRAC scheme is illustrated in [12].

V. **Conclusions**

In this paper we demonstrated how to design a multivariable model reference adaptive controller for aircraft with damages. We presented a systematic design procedure which consists of three technical parts: model specification of aircraft dynamics in the presence of damages, control law development for systems subject to parameter jumps caused by damages, and performance analysis for the closed-loop control system. An approximate aircraft model has been introduced as a simplification of the generic motion equations, which captures the main characteristics of the aircraft dynamics under asymmetric damages, resulted from the loss of mass symmetry. An effective analysis method for piecewise linear systems under MRAC, using a piecewise Lyapunov function, is used to show that desired stability and tracking properties are ensured, despite the jumping system parameter variations. Application of this result to aircraft control in the presence of damages is illustrated. Redesigns of control schemes for relaxation of some key design conditions have been proposed and discussed. This work shows that multivariable MRAC designs are potentially useful and effective for control of aircraft systems with damages and uncertainties. Our current research in this direction is focused on two aspects: a detailed simulation study of MRAC of aircraft systems with damages, and a thorough theoretical study of adaptive control redesign for handling larger classes of parameter and damage uncertainties.

References

[1] N. Nguyen, K. Krishnakumar, J. Kaneshige, and P. Nespeca, "Dynamics and adaptive control for stability recovery of damaged asymmetric aircraft," *Proc. of AIAA Guidance, Navigation, and Control Conference*, AIAA-2006-6049, 2006.

[2] B. J. Bacon and I. M. Gregory, "General equations of motion for a damaged asymmetric aircraft," *Proc. of AIAA Guidance, Navigation, and Control Conference*, AIAA-2007-6306, 2007.

[3] Y. Liu and G. Tao, "Model-based direct adaptive actuator failure compensation techniques with applications to aircraft flight control systems," *Proc. of AIAA Guidance, Navigation, and Control Conference*, AIAA-2006-6554, 2006.

[4] T. J. J. Lombaerts, Q. P. Chu, J. A. Mulder, and D. A. Joosten, "Real time damaged aircraft model identification for reconfiguring flight control," *Proc. of AIAA Atmospheric Flight Mechanics Conference,* AIAA-2007-6717, 2007.

[5] N. Nguyen, M. Bakhtiari-Nejad, and Y. Huang, "Hybrid adaptive flight control with bounded linear stability analysis," *Proc. of 2007 AIAA Guidance, Navigation and Control Conf.*, AIAA-2007-6422.

[6] A. E. Bryson, Jr., *Control of Spacecraft and Aircraft*, Princeton University Press, NJ, 1994.

[7] Y. Liu, X. Tang, G. Tao and S. M. Joshi, "Adaptive failure compensation for aircraft tracking control using engine differential based model," *Proc. American Control Conference*, pp. 5984–5989, 2006.

[8] G. Tao, *Adaptive Control Design and Analysis*, John Wiley & Sons, 2003.

[9] A. K. Imai, R. R. Costa, L. Hsu, G. Tao and P. V. Kokotović, "Multivariable MRAC using high frequency gain matrix factorization," *IEEE Trans. on Auto. Contr.*, vol. 49, no. 7, pp. 1152–1157, 2004.

[10] Y. Liu, X. Tang, G. Tao, and S. M. Joshi, "Adaptive compensation of aircraft actuation failures using an engine differential model," *to appear in IEEE Trans. on Control Systems Technology.*

[11] G. Tao, "Model reference adaptive control of multivariable plants with unknown interactor matrix," *Proc. of the 29th Conference on Decision and Control,* pp. 2730–2735, 1990.

[12] Y. Liu and G. Tao, "Multivariable MRAC using Nussbaum gains for aircraft with abrupt damages," *submitted to 47th IEEE Conference on Decision and Control*, 2008.