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# Belief Propagation in Feedback Systems: Connections to Bode and Observability

Sun Yu, Prashant G. Mehta, and Sean Meyn

*Abstract*— **This paper is concerned with uncertainty based characterizations of fundamental performance limitations (Bode formula) and input-output properties (observability) in feedback systems. A hidden Markov model (HMM) formulation is introduced, based on which a belief process (conditional pdf) is defined. The belief process expresses uncertainty in the state conditioned on history of noisy observations. Both the Bode formula and observability are related to the asymptotic dynamics of the belief process. These results are described for the case of Gaussian linear systems and for open-loop nonlinear systems with negative Lyapunov exponents.**

## I. Introduction

In recent years, several studies have considered information theoretic aspects of feedback control systems. Much of this work is spurred by advances in networks and the need for bridging communication with control; cf., [1]. Research has focussed on issues pertaining to control in the presence of communication constraints [2], stabilization in the presence of quantization [3], generalization of Bode like performance limitations to control over networks [4], [5], distributed control [6] *etc*. In many of these studies, information and entropy play a key role. As an example, one important result is the so-called data rate theorem which states that the rate of instability of an open-loop plant must be compensated by the information transmission rate over the communication channel in any stabilizing feedback [5].

The idea of an information theoretic framework for feedback systems has a rich history going back to Wiener [7] and Witsenhausen [8]. Although research on networks has provided a renewed impetus to study these questions [9], [10], there are many classical system-theoretic problems that can be viewed in such terms to both obtain new insights as well as new results. As an example, the data rate theorem is intimately related to the Bode formula. Using an information theoretic framework [11] presents Bode like results for disturbance rejection problems with linear plant and a general class of control.

This paper builds upon our earlier work [12] where we introduced a framework for characterizing fundamental limitations in control with the aid of belief propagation in an HMM setting. In this paper we utilize this framework to obtain a Bode like result for feedback systems with linear Gaussian dynamics. The result provides a conduit to examine input-output properties such as observability of states in terms of information or uncertainty. In classical settings, these properties are often understood in terms of energy, captured by a suitably defined $l^2$ norm. An initial condition is deemed relatively more (less) observable if it provides more (less) energy in the output, and there is a dual notion for controllability in terms of minimal energy control. We argue that uncertainty rather than energy is a more convenient concept to express input-output properties in nonlinear systems.

The Bode formula and observability are related concepts: The Bode formula provides intrinsic and control-independent uncertainty rates in terms of conditional entropy. The conditional entropy is the entropy of the belief process where the conditioning occurs with respect to the history of noisy observations. Thus, the belief process by itself should also provide an assessment of observability. The key point is that because Bode is control independent, one can hope to address observability of states in a control independent fashion. One immediate advantage is that we can then generalize these notions to open-loop unstable systems. Another advantage is one can obtain results for nonlinear systems.

The methodology of this paper is based on an ergodic theoretic viewpoint whereby a closed-loop dynamical system with an i.i.d disturbance is replaced by a certain Markov operator. The Markov operator is used to propagate probability densities on the joint space $(x, y)$, where $x$ is the state and $y$ is the output. With observations due to the output $y$ alone, this leads to an HMM. The conditional entropy of interest to the Bode problem is intimately related to the asymptotic dynamics of the so-called belief process of the HMM [13]. The belief process is constructed using a recursive Bayesian estimator and represents the conditional distribution in the state of the HMM given the history of observations. We sketch this construction together with the entropy estimate for a feedback system with linear Gaussian dynamics.

For observability, we again have the same setup and consider a belief process with noisy observations. For an anti-stable linear system (with all eigenvalues unstable), the observability is understood entirely in terms of asymptotic dynamics of the belief process. For an open-loop stable system, we characterize observability in terms of uncertainty of the (unknown) initial condition. For each of these, we write Lyapunov equations which are generalizations of the Lyapunov equations arising in classical settings. The uncertainty based considerations are subsequently extended to define and analyze observability in nonlinear settings. The

Sun Yu and Prashant G. Mehta are with the Coordinated Science Laboratory and the Dept of Mechanical Science & Engineering, University of Illinois at Urbana Champaign, 1206 W. Green Street, Urbana, IL 61801 mehtapg@uiuc.edu, yusun2@uiuc.edu

Sean Meyn is with the Coordinated Science Laboratory and the Dept. of Electrical and Computer Engineering, University of Illinois at Urbana Champaign,1406 W. Green St. Urbana, IL 61801 meyn@uiuc.edu

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results are presented for open-loop nonlinear systems with negative Lyapunov exponents. We recover the Lyapunov equation for observability when the model is specialized to be linear with Gaussian noise.

The remainder of the paper is organized as follows. Section II employs belief propagation to state and prove the Bode formula for the Linear Gaussian problem. This is also linked to observability of open-loop states with an arbitrary stabilizing feedback. Section III extends these considerations now to nonlinear systems. Examples and connections to the linear case are discussed. Finally, some conclusions are outlined in Sec. IV.

## II. Linear Gaussian Problem

### A. Problem Setup

In this section, we restrict our attention to the Linear Gaussian problem

$$\begin{aligned} x_{n+1} &= Ax_n + Bu_n, && (1)\\ y_n &= Cx_n + d_n, && (2)\\ u_n &= k(y_n), && (3) \end{aligned}$$

where $x_n \in X = \mathbb{R}^m$ is the state, $u_n \in Y = \mathbb{R}^p$ is the input, $y_n \in \mathbb{R}^q$ is the output, and $d_n \in \mathbb{R}^q$ is an i.i.d Gaussian disturbance. In the following, we discuss the Bode formula and observability for this system. Although observability results will be given for general MIMO settings, we will assume the feedback system to be SISO for the Bode formula.

### B. Bode Formula

The Bode integral formula states that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|S(e^{i\omega})|d\omega = \sum_k \log(|p_k|), \tag{4}$$

where $S(e^{i\omega})$ is the transfer function of the feedback loop from the disturbance $d_n$ to scalar output $y_n$, and $p_k$ are unstable poles ($|p_k| > 1$) of the open loop plant; cf. [14]. The input, output and disturbance here are assumed to be scalar signals. $S$ is referred to as the *sensitivity function* and for an open-loop plant $P$ and a stabilizing feedback control $C$, it is given by $S = \frac{1}{1+PC}$. Entropy of the signals in the feedback loop help provide another interpretation of the Bode integral formula [15]:

$$H_c(y) - H_c(d) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|S(e^{i\omega})|d\omega. \tag{5}$$

Here, $H_c(y)$ and $H_c(d)$ denote the conditional entropy (see [16]) of the random processes associated with the output $y$ and disturbance $d$ respectively. Combining Eq. (4) with Eq. (5), the open-loop unstable poles are seen to lead to a positive entropy rate.

In our earlier paper [12], we proposed a framework for obtaining $H_c(y)$ in terms of belief propagation. As in [12] we denote the joint space of state and output as $S \doteq X \times Y$, and the space of pdfs on $S$ as $\nabla S$. Belief process $\pi_n(x, y) \in \nabla_S$ is defined as the conditional pdf of joint process $s_n \in S$ conditioned on the history of observations $y_0^{n-1} = \{y_0, y_1, \ldots, y_{n-1}\}$. It represents the belief in $(x_n, y_n)$ given the history of past observations $y_0^{n-1}$. $f_n$ and $g_n$ denote the marginal of $\pi_n$ with respect to $X$ and $Y$ respectively. The belief propagation, $\pi_n \to \pi_{n+1}$, is carried out using a recursive Bayesian estimator; cf., [12].

Borrowing ideas from the HMM literature, the relative entropy $H_c(y)$ can be expressed in terms of asymptotic dynamics of the belief propagation. Let $\mu$ denote the invariant measure of the belief process then

$$H_c(y) = \int_{\nabla_s} h_y(\pi)d\mu(\pi) \tag{6}$$

where $h_y$ is the entropy function $-\int g(y)\ln g(y)dy$ where $g(y)$ is the marginal on the output space $Y$. This is referred to as the integral formulation of the entropy rate, originally due to D. Blackwell. Note that $\mu$ is an invariant measure *on* the space of pdfs $\nabla_S$; we refer the reader to [12] for details.

We assume the disturbance $\{d_n\} \sim N(0, r)$ to be i.i.d. and Gaussian and the pdf for the initial condition $f_0 = N(\hat{x}_0, P_0)$ is also Gaussian. This case is particularly interesting because belief propagation ($\pi_n \to \pi_{n+1}$) reduces to Kalman filtering equations. The simplification arises because $\pi_n$ and $f_n$ are all Gaussian in this case. Expressing $f_n = N(\hat{x}_n, P_n)$, the equations for belief propagation are:

$$\text{Conditioning:} \quad \left\{ \begin{array}{lcl} \hat{x}_n^+ &=& \hat{x}_n + K_n(y_n - C\hat{x}_n) \\ P_n^+ &=& (I - K_nC)P_n \end{array} \right. \tag{7}$$

$$\text{Dynamics:} \quad \left\{ \begin{array}{lcl} \hat{x}_{n+1} &=& A\hat{x}_n^+ + Bu_n \\ P_{n+1} &=& AP_n^+A^T \end{array} \right. \tag{8}$$

where $K_n = P_nC^T(CP_nC^T + rI)^{-1}$ is the Kalman gain. For a Gaussian random variable, entropy depends only upon the variance and one is thus interested in asymptotic values of $\{P_n\}$. If $(A, C)$ is observable, $P_n$ converges to a unique positive semi-definite solution of the Discrete Algebraic Riccati Equation (DARE):

$$P = A(P - PC^T(CPC^T + rI)^{-1}CP)A^T \tag{9}$$

In $\nabla_S$, the invariant measure is supported on a single point which renders the entropy calculations straightforward. We present the Bode formula in the following Theorem:

*Theorem 2.1:* Consider the closed-loop system (1)-(3) with linear dynamics. The disturbance $\{d_n\}$ is i.i.d with pdf $N(0, r)$ and initial condition $x_0$ is also uncertain with pdf $f_0 = N(\hat{x}_0, P_0)$. Then

$$H_c(y) = H(d) + \sum_k |p_k|, \tag{10}$$

where $|p_k| > 1$ are the unstable eigenvalues of $A$.

*Proof:* The entropy is obtained with respect to the asymptotic dynamics of (7)-(8), solution of DARE (9) in this case. The proof is carried out in three steps: (1) Consider a decomposition of $\mathbb{R}^m = \mathbb{R}^{m_s} \oplus \mathbb{R}^{m_u}$ into stable and unstable eigenspaces and write $A = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}$, $C = (C_s \;\; C_u)$. One can then show that $P = \begin{pmatrix} 0 & 0 \\ 0 & P_u \end{pmatrix}$ where $P_u \succ 0$. (2) With respect to unstable dynamics, the variance of $C_u x$ (output component due to the state) is given by the following simple formula

$$C_u P_u C_u^T = r(|A_u|^2 - 1). \tag{11}$$

(3) Finally, the entropy for the limiting Gaussian pdf is obtained as a simple calculation. We presents details of these steps next:

1) Suppose $\lambda$ is a simple eigenvalue of matrix $A$ with an eigenvector $v$ so $Av = \lambda v$. If $P \succeq 0$ is a semi positive-definite solution of the DARE (9) then

$$(1-|\lambda|^2)v^T P\bar{v} = -|\lambda|^2 v^T (PC_u^T(C_u PC_u^T+r)^{-1}C_u P)\bar{v} \tag{12}$$

Now if $|\lambda| < 1$ then this implies $v^T P\bar{v} \leq 0$. By positive semi-definiteness of $P$, we have $Pv = 0$. Thus the restriction of $P$ to stable eigenspace $\mathbb{R}^{m_s}$ is 0 and on account of symmetry,

$$P = \begin{pmatrix} O & O \\ O & P_u \end{pmatrix}, \tag{13}$$

where $P_u$ satisfies the DARE

$$P_u = A_u(P_u - P_u C_u^T(C_u P_u C_u^T + rI)^{-1}C_u P_u)A_u^T. \tag{14}$$

For repeated eigenvalues, a proof may be constructed in a standard manner by constructing an appropriate sequence.

2) The covariance matrix $P_u \succ 0$ and is a solution to the DARE in the unstable eigenspace:

$$P_u = A_u(P_u - P_u C_u^T(C_u P_u C_u^T + rI)^{-1}C_u P_u)A_u^T. \tag{15}$$

Using the Woodbury matrix identity, this leads to a Lyapunov equation

$$A_u^T P_u^{-1} A_u = P_u^{-1} + C_u^T r^{-1} C_u. \tag{16}$$

Taking determinant $|\cdot|$ on both sides and simplifying, one obtains

$$|A_u|^2 r^{m_u} = |rI + C_u^T C_u P_u|. \tag{17}$$

Now, $C_u^T C_u P_u$ is a rank 1 matrix so

$$|rI + C_u^T C_u P_u| = r^{m_u} + \text{trace}(C_u^T C_u P_u) r^{m_u - 1}. \tag{18}$$

Using (17),

$$\begin{aligned} (|A_u|^2 - 1)r^{m_u} &= \text{trace}(C_u^T C_u P_u) r^{m_u-1} \\ \therefore, \quad (|A_u|^2 - 1)r &= \text{trace}(C_u^T C_u P_u) = C_u P_u C_u^T. \end{aligned} \tag{19}$$

3) Finally, we compute the entropy for the limiting Gaussian pdf. From parts (1) and (2),

$$CPC^T = C_u P_u C_u^T = (|A_u|^2 - 1)r, \tag{20}$$

and the asymptotic covariance for the conditional pdf of the output $y_n$ is given by

$$\sigma_y^2 = CPC^T + r = r|A_u|^2. \tag{21}$$

As a result,

$$\begin{aligned} \lim_{n\to\infty} h_y(\pi_n) &= \frac{1}{2}\ln(2\pi e \sigma_y^2) \\ &= \ln(2\pi e r) + \ln|A_u| \\ &= H(d) + \ln|A_u| \end{aligned} \tag{22}$$

Using the integral formula (6)

$$\begin{aligned} H_c(y) &= \int_{\nabla_s} h_y(\pi) d\mu(\pi) \\ &= \int_{\nabla_s} (H(d) + \ln|A_u|) d\mu(\pi) \\ &= H(d) + \ln|A_u| \end{aligned} \tag{23}$$

■

### C. Observability

While the Bode formula provides a controller independent result for intrinsic uncertainty of the output process $\{y_n\}$, here we are interested in characterizing observability of states in terms of uncertainty. As with the Bode formula, the uncertainty is expressed with the aid of belief process in the presence of an i.i.d Gaussian disturbance.

Consider the feedback system (1)-(3) with noisy observations $\{y_0^n\}$. The output is not necessarily scalar now and the disturbance $d_n \sim N(0, R)$. The two cases with anti-stable and stable $A$ matrices are treated separately.

*1) Anti-stable case:* For the anti-stable case, all eigenvalue of $A$ are assumed to be outside unit circle. From part (1) in the proof of Bode formula, noisy observations lead to a perfect asymptotic belief in stable states and certain asymptotic uncertainty for unstable states. The latter is expressed in terms of the limiting conditional pdf $f_n = p(x_n|y_0^n)$ as $n \to \infty$ and will be used here to quantify observability of unstable states. Noting that $f_n = N(\hat{x}_n, P_n)$, the Kalman filtering equations for the variance are:

$$\begin{aligned} \text{Conditioning}: \quad P_n^+ &= [(P_n)^{-1} + C^T R^{-1} C]^{-1} \\ \text{Dynamics}: \quad P_{n+1} &= A P_n^+ A^T. \end{aligned} \tag{24}$$

Let $P \doteq \lim_{n\to\infty} P_n$, then $P$ is the solution of DARE:

$$P = A(P - PC^T(CPC^T + R)^{-1}CP)A^T \tag{25}$$

and using Woodbury matrix identity, we have

$$A^T P^{-1} A - P^{-1} = C^T R^{-1} C \tag{26}$$

Denote $S \doteq P^{-1}$, we obtain the Lyapunov equation

$$A^T S A - S - C^T R^{-1} C = 0. \tag{27}$$

$S$ is the so-called information matrix.

*2) Stable case:* For the stable case, all eigenvalue of $A$ are assumed to be inside unit circle. Since one has perfect asymptotic belief for such states, the observability is characterized using uncertainty for the initial condition $x_0$. This is expressed in terms of the conditional pdf $p(x_0|y_0^n)$ as $n \to \infty$. The reason for doing so is that noisy observations serve to provide information regarding the state process $\{x_0^n\}$. Without input noise, the uncertainty of this process

| | Conditional pdf | Lyapunov Equation |
|---|---|---|
| Anti-stable ($A_u$) | $\lim_{n\to\infty} p(x_n\|y_0^n)$ | $A_u^T S A_u - S = C^T R^{-1} C$ |
| Stable ($A_s$) | $\lim_{n\to\infty} p(x_0\|y_0^n)$ | $A_s^T S A_s - S = -C^T R^{-1} C$ |

TABLE I

OBSERVABILITY FOR LINEAR GAUSSIAN PROBLEM

arises only due to uncertainty in the initial condition $\{x_0\}$. This can also be seen by noting that $A^{-1}$ is anti-stable for $A$ stable. So the appropriate uncertainty limit to consider is in backward time.

The evolution of belief process now reduces to a smoothing problem, where $\pi_0 = p(x_0|y_0^n)$ represents the conditional pdf where the conditioning is due to future noisy observations $\{y_0^n\}$. As with the Bode formula, it suffices to consider the evolution of covariance matrix $P_n$ (or its inverse $S_n = P_n^{-1}$) where one traverses the time-line from $n$ backwards to 0. This is the so-called information filtering procedure with equations for the information matrix $S_n$:

$$\begin{aligned} \text{Dynamics}: \quad S_n &= A^T S_{n+1}^+ A \\ \text{Conditioning}: \quad S_n^+ &= S_n + C^T R^{-1} C, \end{aligned} \tag{28}$$

where dynamics precede the conditioning step because of backward recursion ($n+1 \to n$). This simplifies to

$$S_n^+ = A^T S_{n+1}^+ A + C^T R^{-1} C \quad 0 \le n \le N \tag{29}$$

This backward recursion continues till we find $S_0^+$. We denote $S \doteq S_0^+$ as the horizon $N \to \infty$. It is the solution to the following Lyapunov equation:

$$A^T S A - S + C^T R^{-1} C = 0 \tag{30}$$

and $P \doteq S^{-1}$ is the asymptotic covariance matrix for initial state.

*3) Discussion:* Table I summarizes the observability results for the an anti-stable and a stable system. With each of these, a Lyapunov equation characterizes observability of states in terms of information matrix $S$ (this is the inverse of the covariance matrix). For an anti-stable system, this matrix gives information regarding the asymptotic state while for a stable system, the matrix gives information on the initial condition. Since $S$ is symmetric and positive definite, one can easily obtain an eigen-decomposition of the state space in terms of directions with increasing information content. We note that the Lyapunov equation of the stable case is also the same as the Lyapunov equation with energy based considerations (with $R = I$). However, in terms of uncertainty, these considerations also extend to feedback systems with arbitrary feedback (just like Bode) as well as unstable system where input (due to feedback, say) is also being used for well-posedness of the closed-loop.

## III. OBSERVABILITY IN NONLINEAR SYSTEMS

### A. Problem Setup

We consider the nonlinear system

$$\text{state}: \quad x_{n+1} = \alpha(x_n), \tag{31}$$

$$\text{output}: \quad y_n = c(x_n) + d_n, \tag{32}$$

where $n$ is discrete time step, $x_n \in X \subset \mathbb{R}^m$ is the state, $y_n \in Y \subset \mathbb{R}^q$ is the output, $d_n \in \mathbb{R}^q$ is i.i.d. disturbance assumed here to be Gaussian $N(0, R)$. We assume open-loop settings here and make one additional assumption regarding the dynamics. In particular we assume that dynamics $\alpha(\cdot)$ have negative Lyapunov exponents. We recall that the *leading* Lyapunov exponent for $\alpha(\cdot)$ is defined as:

$$\lim_{n\to\infty} \frac{1}{n} \ln \rho(\alpha^{n'}(x_0)) \doteq \lambda(x_0) \tag{33}$$

where $\alpha^{n'}(x_0) \doteq \frac{d\alpha^n}{dx_0}(x_0)$ and $\rho(\cdot)$ denotes its spectral radius. The $m$ Lyapunov exponents are negative if and only if the leading one is. For a scalar map $\alpha(\cdot)$, there is only one Lyapunov exponent and it is given by

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \ln |\alpha^{n'}(x_0)|, \tag{34}$$

Typically Lyapunov exponents are constant – for instance when defined with respect to an Ergodic partition. We do not make this assumption as observability will be characterized with respect to an initial condition $x_0$. In linear settings, Lyapunov exponents are constant and are given by $\ln(\cdot)$ of the eigenvalues of the $A$ matrix. The leading one equals $\ln(\rho(A))$.

### B. Observability

As with the stable linear case, we characterize observability in terms of conditional pdf $p(x_0|y_0^n)$. This captures the uncertainty in initial condition. The challenge is that as opposed to the linear Gaussian case, belief propagation is by no means straightforward for the nonlinear case. The main idea of this section is that while the belief propagation is difficult, one can still obtain bounds for the variance of the estimate (for $x_0$) by using the concept of Fisher information; cf., [16].

We describe the method for an initial condition $x_0$ that is a priori unknown. Using (31)-(32), we have

$$y_m = c(\alpha^m(x_0)) + d_m. \tag{35}$$

The key to note is that $y_m$ are independent Gaussian random variables with mean $c(\alpha^m(x_0))$ and variance $R$. The Fisher information is the amount of information that an observed random variable ($y_m$) carries about the unknown ($x_0$). It is defined as variance of the score function:

$$V(y_m) = \frac{\partial}{\partial x_0} \left( \ln g(y_m; x_0) \right). \tag{36}$$

where $g$ is the Gaussian distribution with mean $c(\alpha^m(x_0))$ and variance $R$. An easy calculation then shows that the score function

$$V(y_m) = \left(\alpha^{m'}(x_0)\right)^T C_m^T R^{-1} \left(y_m - c(\alpha^m(x_0))\right), \tag{37}$$

where $C_m \doteq c'(\alpha^m(x_0))$ and the Fisher information

$$\begin{aligned} J^{(m)}(x_0) &= E[V(y_m)V^T(y_m)] \\ &= \left(\alpha^{m'}(x_0)\right)^T C_m^T R^{-1} C_m \left(\alpha^{m'}(x_0)\right) \end{aligned} \tag{38}$$

where $E[\cdot]$ denotes the expectation operator, and the superscript $(m)$ is used to draw attention to the fact that this is the information regarding $x_0$ contained in the $m^{\text{th}}$ observation. With negative Lyapunov exponents $|\alpha^{m'}(x_0)| \to 0$ as $m \to \infty$ and the observation contains less and less information regarding the initial condition $x_0$.

Since $\{y_0, y_1, \ldots, y_n\}$ are all independent, their joint pdf $g(y_0^n; x_0) = \Pi_{m=0}^n g(y_m; x_0)$ and one obtains the identity

$$J_n(x_0) = \sum_{m=0}^{n} J^{(m)}(x_0), \tag{39}$$

where $J_n$ is the Fisher information for the sequence $\{y_0^n\}$. Using (38), it is given by

$$J_n(x_0) = \sum_{m=0}^{n} \left(\alpha^{m'}(x_0)\right)^T C_m^T R^{-1} C_m \left(\alpha^{m'}(x_0)\right) \tag{40}$$

These considerations are useful because the Cramer-Rao inequality then allows one to write a lower bound on the variance of *any* unbiased estimator of the unknown $x_0$:

$$\text{Var}(x_0) \geq J_n^{-1}(x_0) \tag{41}$$

This is called the Cramer-Rao Lower Bound (CRLB). As $n \to \infty$, the series (40) converges for the case of negative Lyapunov exponents (for some sufficiently large $M$, there exists a non-negative constant $d < 1$ such that $\|\alpha^{m'}(x_0)\| < d^m$ for all $m > M$). Denoting $J(x_0) \doteq \lim_{n\to\infty} J_n(x_0)$, $\text{Var}(x_0) \geq J^{-1}(x_0)$.

*Example 3.1:* Consider first a Linear Gaussian setup where maps $\alpha$ and $c$ are linear:

$$\text{state}: \quad x_{n+1} = Ax_n, \tag{42}$$
$$\text{output}: \quad y_n = Cx_n + d_n, \tag{43}$$

and $\rho(A) < 1$. For an initial condition $x_0$,

$$y_m = CA^m x_0 + d_m. \tag{44}$$

The score function and the Fisher information (for $y_m$) are given by:

$$V(y_m) = (CA^m)^T R^{-1}(y_m - CA^m x_0), \tag{45}$$
$$J^{(m)}(x_0) = (CA^m)^T R^{-1}(CA^m). \tag{46}$$

The Fisher information for $y_0^n$ is

$$J_n(x_0) = \sum_{i=0}^{n} E_{x_0}[V(y_i)V^T(y_i)] = \sum_{i=0}^{n} (CA^i)^T R^{-1}(CA^i) \tag{47}$$

We note that the series converges because $\rho(A) < 1$. Denoting $J(x_0) \doteq \lim_{n\to\infty} J_n(x_0)$, it is easy to see that $J(x_0)$ is a solution to the following Lyapunov equation

$$J = A^T J A + C^T R^{-1} C, \tag{48}$$

i.e., the Fisher information matrix $J(x_0) = S = P^{-1}$, where $P$ is the covariance matrix found in the preceding section (for the stable case). The CRLB inequality is thus an equality in this case.

*Example 3.2:* Let $\alpha$ and $c$ be scalar nonlinear maps. We denote the variance of scalar Gaussian disturbance by $r$. In this case, the negative Lyapunov exponent property means:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \ln|\alpha^{n'}(x_0)| < 0. \tag{49}$$

We assume this. For an initial condition $x_0$,

$$y_m = c(\alpha^m(x_0)) + d_m. \tag{50}$$

The score function and the Fisher information are given by:

$$V(y_m) = \frac{1}{r}(y_m - c(\alpha^m(x_0))) \cdot c'(\alpha^m(x_0))\alpha^{m'}(x_0)$$
$$J^{(m)}(x_0) = \frac{|c'(\alpha^m(x_0))|^2}{r} |\alpha^{m'}(x_0)|^2. \tag{51}$$

The Fisher information for $y_0^n$ is thus

$$\begin{aligned} J_n(x_0) &= \sum_{m=0}^{n} E[V^2(y_m)] \\ &= \frac{1}{r}\sum_{m=0}^{n} |c'(\alpha^m(x_0))|^2 \cdot |\alpha^{m'}(x_0)|^2 \end{aligned} \tag{52}$$

Let $C \doteq \sup_{x\in X} |c'(x)|$ then

$$J_n(x_0) \leq \frac{C}{r}\sum_{m=0}^{n} |\alpha^{m'}(x_0)|^2 \tag{53}$$

and the series converges because the Lyapunov exponent is assumed negative (for some sufficiently large $M$, there exists a non-negative constant $d < 1$ such that $|\alpha^{m'}(x_0)| < d^m$ for all $m > M$). Denote $J(x_0) \doteq \lim_{n\to\infty} J_n(x_0)$ then CRLB implies that the variance of estimate $x_0$ is $\frac{1}{J(x_0)}$ or greater.

### C. Numerics

*Example 3.3:* Consider a nonlinear system

$$\begin{aligned} x_{n+1} &= 0.9x_n + \epsilon z_n^3, \\ z_{n+1} &= 0.9z_n, \end{aligned} \tag{54}$$

with both Lyapunov exponents equal to $\ln(0.9)$. The origin $(0, 0)$ is a globally asymptotically stable fixed-point. The output equation

$$y_n = x_n + d_n, \tag{55}$$

where $d_n \sim N(0, 1)$. With $\epsilon = 0$, the system is linear and the state $z$ is not observable. As $\epsilon$ increases, one gets more and more information about the state $z$ due to dynamics. However, as opposed to a linear system, the information now is also a function of the initial condition. One would expect the initial conditions with smaller $|z_0|$ to provide very little information regarding the state $z$ while the initial conditions with larger $|z_0|$ to provide more so.

The information matrix is obtained by using the Lyapunov equation (48) for $\epsilon = 0$, and the series formula (40) for the nonlinear case ($\epsilon = 0.1$). With $\epsilon = 0$, $J$ is a constant matrix independent of initial conditions. The smallest eigenvalue is 0 with an eigenvector along the $z$-direction. The conclusion is that the one has large (in this case infinite) uncertainty along the $z$ direction. This is consistent with the fact that $z$ is not observable for $\epsilon = 0$.

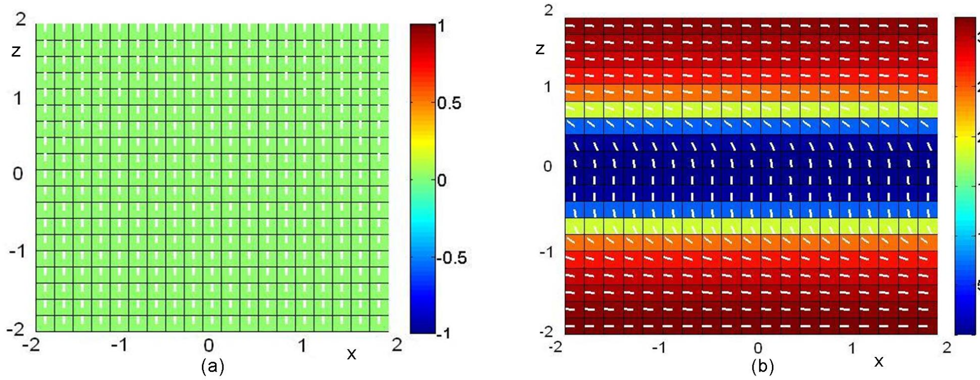


Fig. 1. Summary of observability computations for (a) linear ($\epsilon = 0$) and (b) nonlinear ($\epsilon = 0.1$) cases. The background is the smallest eigenvalue of $J(x_0, z_0)$ and the lines depict the eigen-direction (corresponds to smallest information or largest uncertainty directions).

Figure 1 summarizes the results of observability computations carried out for a grid of initial conditions in the phase space $X \doteq [-2, 2] \times [-2, 2]$. With $\epsilon = 0.1$, the information matrix $J(x_0, z_0)$ is now a function of the initial condition $(x_0, z_0)$. The system is now observable for all initial conditions in the sense that that $J(x_0, z_0)$ is non-singular. Fig. 1(b) depicts the smallest eigenvalue of $J(x_0, z_0)$ together with the eigen-direction (depicted as a short line). Consistent with our intuition, the smallest eigenvalue is relatively small for initial conditions with $z_0$ close to 0. It becomes larger as $|z_0|$ increases. This corresponds to the fact that initial conditions with larger value of $|z_0|$ are more observable than the ones with smaller value. The eigen-direction corresponding to large uncertainty is more interesting. It is close to the linear case for smaller $|z_0|$ but is along the $x$-direction for a sufficiently large value of $|z_0|$. This shows that the nonlinearity causes $z$ to become (relatively) more observable for sufficiently large values of $|z_0|$.

An interesting observation from the study is that the information matirx $J$ is independent of $x_0$ and depends only upon $z_0$. This is because $x$ enters the state and output equations in a linear fashion. So the information matrix (which is a function of only linearizations) is independent of $x_0$. As a result, $J$ is only a function of $z_0$ and the analysis above is independent of $x_0$.

## IV. Conclusion

In this paper we obtained uncertainty based characterizations of fundamental performance limitations (Bode formula) and observability. The uncertainty flow is studied using belief propagation in HMM settings. For the linear Gaussian case, the Bode formula was shown to be a simple consequence of Kalman filtering equations and naturally led to observability characterizations for both unstable and stable states. Furthermore, these uncertainty based characterizations of observability are independent of control much as the Bode formula is control independent. In open-loop stable settings, this lead to the well-known Lyapunov equation for observability. Uncertainty based characterizations also suggested a novel approach for observability of nonlinear systems in terms of Fisher information. We gave some generalizations for nonlinear systems with negative Lyapunov exponents.

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