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Feedforward Input Design for Minimum-Time/Energy, Output Transitions for Dual-Stage Systems

Dhanakorn Iamratanakul and Santosh Devasia

Abstract— **This article addresses the optimal (minimum-time/energy) trajectory design for changing the output from one value to another. The main contribution of this article is to establish existence of a solution to the problem when pre- and post-actuation are allowed. The method is illustrated with an experimental dual-stage actuator system.**

I. Introduction

This article solves the minimum-time/energy problem [1]-[3] for output transitions, i.e., for changing the output from one value $\underline{y}$ to another $\overline{y}$. The novelty of the proposed approach is that inputs are not applied just during the output-transition time interval $[0, t_f]$; rather, inputs are also applied before the beginning of (time $t < 0$) and after the end of (time $t > t_f$) the output-transition time interval — these inputs are called pre- and post-actuation. Recent works [4], [5] have shown that the addition of pre- and post-actuation inputs can improve the output-transition performance; however, these approaches only minimize the input energy and not the time needed for output transitions. This motivates the study of the minimum-time/energy, output-transition problem in the presence of pre and post-actuation. In addition to the ability for trade offs between time minimization and energy minimization, the minimum-time/energy approach avoids the computational difficulties associated with solving a purely minimum-time problem as well as the resulting, undesirable, bang-bang-type control inputs [1].

The major contribution of this work is to establish the existence of a solution to the optimal (minimal time/energy) output transition (OOT) problem in the presence of pre- and post-actuation — in contrast, the existence of a solution without pre- and post-actuation was established in [1]. Additionally, OOT solutions (with and without pre- and post-actuation) are comparatively evaluated using results from an experimental, dual-stage positioning system.

The use of pre- and post-actuation effectively increases the time available to apply inputs to the system with the OOT approach. In contrast, since the system is at rest (equilibrium) at the beginning when $t = 0$ and at the end $t = t_f$ of the output transition, pre- and post-actuation inputs are not used in the standard state-transition (SST) approach (e.g., [1],[6]-[8]) which transitions from from an initial state to a desired final state. Note that the output-transition time (t_f) is not increased with the OOT approach because the pre- and post-actuation inputs are constrained to maintain the output at the desired value outside the time interval $(0, t_f)$. Therefore, the availability of additional time to apply inputs (during pre- and post-actuation) tends to lower the amount of input needed. The resulting performance improvement is substantial for dual-stage (dual input single output) positioning systems because the actuator-redundancy provides flexibility to maintain the output at the desired value even as inputs are applied during the pre- and post-actuation [5]. Similarly, for the same amount of input, the use of pre- and post-actuation can lead to faster (smaller t_f) output transitions. Such tradeoff, between the amount of input and the transition time, is enabled with the minimum-time/energy approach to optimal output transitions (OOT) that is studied in this article.

A problem with early approaches for optimizing output transition (rather than state transition) is that they required the user to pre-specify the output trajectory [9], [10]. While the use of a pre-specified set of trajectories simplifies the output trajectory design, the challenge is that the pre-specification of the *best* output trajectory (e.g., the one that optimizes the output-tracking problem) is not intuitive. This requirement to pre-specify the output trajectory is avoided in more recent works [4], [11] that directly solve the minimum-energy output-transition problem and find the best output trajectory as part of the optimization procedure. These results, which minimize the input energy, are extended in the current article to solve the minimum-time/energy, output-transition problem — in the presence of pre and post-actuation.

II. OOT Problem Formulation

A. System Model

Consider a dual-stage positioning system for a flexible structure represented by the following transfer function with two inputs $\{u_1, u_2\}$ and a single-output y,

$$\begin{aligned} y(s) &= G_1(s)u_1(s) + G_2(s)u_2(s) \\ &= \frac{1}{D(s)}\left[N_1(s)u_1(s) + N_2(s)u_2(s)\right] \end{aligned} \quad (1)$$

where the vibrational resonances are defined by the n^{th} order polynomial $D(s)$, which is the least common multiple of the denominators of the strictly proper transfer functions $G_1(s)$ and $G_2(s)$.

Assumption 1: The denominator polynomial $D(s)$ has at least one root at the origin, i.e., the system (Eq. 1) includes rigid-body dynamics with multiple equilibrium points. ■

Assumption 2: The polynomials $N_1(s)$ and $N_2(s)$ do not share common roots, i.e., no common zero between the transfer functions $G_1(s)$ and $G_2(s)$. ■

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D. Iamratanakul is with Western Digital Inc., USA
S. Devasia is with the Mechanical Engineering Department, U. of Washington, Seattle, WA 98195, USA; Email: devasia@u.washington.edu

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Assumption 3: Without loss of generality, it is assumed that the relative degree [12] r of the transfer function $G_1(s)$ is not larger than the relative degree of transfer function $G_2(s)$, where $r < n$ since $G_1(s)$ and $G_2(s)$ are strictly proper. ■

Let the state-space representation of the system (Eq. 1) in the *minimal* form (controllable and observable) be

$$\begin{cases} \dot{x}(t) &= Ax(t) + B_1u_1(t) + B_2u_2(t) \\ y(t) &= Cx(t) \end{cases}, \tag{2}$$

where the system state is $x \in \Re^n$ and n is the order of the denominator polynomial $D(s)$. The relative degree assumption (Assumption 3) implies that (from [12]) $CA^{r-1}B_1 \neq 0$ and $CA^kB_1 = CA^kB_2 = 0$ for all $k = 0, 1, \ldots, r-2$.

B. *The Output-transition Problem*

We investigate the problem of changing the output from an initial value $\underline{y}$ to a final value $\overline{y}$ within a finite time-interval $[0, t_f]$, called the output-transition time-interval. It is noted that the output should be maintained constant (at the desired value) outside the output-transition time-interval. Formally, the output-transition problem [4], [11] is defined as follows.

*Definition 1 (**The output-transition problem**):* Given a pair of initial and final output values

$$\underline{y} = C\underline{x}, \qquad \overline{y} = C\overline{x} \tag{3}$$

corresponding to initial ($\underline{x}$) and final ($\overline{x}$) equilibrium configurations

$$A\underline{x} = 0 \quad \text{and} \quad A\overline{x} = 0 \tag{4}$$

of the system (Eq. 2), find bounded input-state trajectories $\{u_1(\cdot), u_2(\cdot), x_{ref}(\cdot)\}$ that satisfy the system equations (Eq. 2) and the following two conditions.

1. The output-transition condition: The output is transferred from an initial value $\underline{y}$ to a final value $\overline{y}$ within the output-transition time-interval $[0, t_f]$, and is maintained constant (at the desired value) before and after the output-transition, i.e.,

$$\begin{cases} y_{ref}(t) &= Cx_{ref}(t) &= \underline{y}, & \forall\ t \leq 0 \\ y_{ref}(t) &= Cx_{ref}(t) &= \overline{y}, & \forall\ t \geq t_f. \end{cases} \tag{5}$$

2. The delimiting-state condition: The state approaches the equilibrium state (rigid-body configuration) as time goes to (plus or minus) infinity, i.e.,

$$\begin{cases} x_{ref}(t) \to \underline{x} & \text{as } t \to -\infty \\ x_{ref}(t) \to \overline{x} & \text{as } t \to \infty. \end{cases} \quad ■ \tag{6}$$

C. *The Time/Energy Cost Function*

In this article, we find the *optimal* solution that minimizes the time/energy for dual-stage systems. This optimal output-transition (OOT) problem is stated below.

Definition 2: **The OOT Problem** is to find bounded input-state trajectories $\{u_1(\cdot), u_2(\cdot), x_{ref}(\cdot)\}$ and a transition time (t_f) that solve the output-transition problem (see Definition 1), and minimizes the following time/energy cost functional,

$$J\{t_f, u(\cdot)\} = \gamma t_f + \int_{-\infty}^{\infty} L(t)dt = \gamma t_f + J_E\{t_f, u(\cdot)\} \tag{7}$$

where $L(t) = \{u_1(t)^2 + \rho u_2(t)^2\}$, the positive constant ρ represents the relative weight between the two inputs, and the positive constant γ represents the relative weight between the output-transition time t_f and the input energy J_E. ■

D. *Pre-actuation, Transition, and Post-Actuation Costs*

The contribution of the input-energy J_E (in Eq. 7) can be partitioned into the pre-actuation cost ($t < 0$), the transition cost ($0 \leq t \leq t_f$), and the post-actuation cost ($t > t_f$), as

$$\begin{aligned} J_E &= \int_{-\infty}^{0} L(t)dt + \int_{0}^{t_f} L(t)dt + \int_{t_f}^{\infty} L(t)dt \\ &= J_{pre} + J_{tran} + J_{post}. \end{aligned} \tag{8}$$

E. *Solution Approach and Challenge*

The optimal output-transition (OOT) problem (Definition 2) is solved in two steps. In the first step, we consider the output-transition problem with a fixed transition time t_f and find and the input $u_{t_f}(\cdot)$ that minimizes the quadratic input-energy term J_E, (in Eq. 7), i.e.,

$$J_{E,opt}(t_f) := J_{E,opt}\{t_f, u_{E,opt}(\cdot)\} = \min_{u(\cdot)} [J_E\{t_f, u(\cdot)\}].$$

In the second step, the OOT problem is solved when the transition time t_f is free as follows

$$\begin{aligned} J^* = J\{t_f^*, u^*(\cdot)\} &:= \min_{t_f, u(\cdot)} J\{t_f, u(\cdot)\} \\ &= \min_{t_f} \left\{ \hat{J}(t_f) = \gamma t_f + J_{E,opt}(t_f) \right\}. \end{aligned} \tag{9}$$

The challenge is to show the existence of a minimum for this optimization process over all transition times t_f.

III. Fixed Transition Time t_f Case

A. *System Equations in Output Tracking Form*

The system can be re-written in the output-tracking form or normal form, see [12]. In particular, there exists

I) a state transformation Φ, defined by

$$\begin{aligned} x(t) &= \Phi \left[\xi(t)^T\ \eta(t)^T\right]^T \\ &:= [\Phi_\xi \mid \Phi_\eta] \left[\xi(t)^T \mid \eta(t)^T\right]^T \end{aligned} \tag{10}$$

where the component $\xi(t) := \left[y(t), \dot{y}(t), \ldots, y^{(r-1)}(t)\right]^T$ represents the output and its time-derivatives up to order $r-1$ (where r denotes the relative degree of the system), and the $n-r$ dimensional component $\eta(t)$ represents the internal state (see [12]); and

II) an input law that yields the exact-output tracking with the following general form,

$$u_{1,inv}(t) := C_\eta \eta(t) + D_\eta u_2(t) + D_{\mathbb{Y}} \mathbb{Y}_d(t) \tag{11}$$

where $\mathbb{Y}_d(t) := \left[\xi_d(t)^T,\ y_d^{(r)}(t)\right]^T$ and the subscript d denotes desired (or known) values,

III) such that the original system (Eq. 2) can be transformed into the following *output-tracking form*

$$\begin{aligned} \dot{\xi}(t) &= \dot{\xi}_d(t) \\ \dot{\eta}(t) &= A_\eta \eta(t) + B_\eta u_2(t) + B_{\mathbb{Y}} \mathbb{Y}_d(t). \\ A_\eta &= T_\eta \left(A - \frac{B_1 CA^r}{CA^{r-1}B_1}\right) \Phi_\eta \\ B_\eta &= T_\eta \left(-\frac{B_1 CA^{r-1}B_2}{CA^{r-1}B_1} + B_2\right) \\ B_{\mathbb{Y}} &= T_\eta \left[\left(A - \frac{B_1 CA^r}{CA^{r-1}B_1}\right) \Phi_\xi \mid \frac{1}{CA^{r-1}B_1}\right] \\ T_{\xi\eta} &:= \left[T_\xi^T \mid T_\eta^T\right]^T = [\Phi_\xi \mid \Phi_\eta]^{-1}. \end{aligned} \tag{12}$$

B. The Internal Dynamics

During pre-actuation ($t < 0$) and post-actuation ($t > t_f$), part of the system state is completely specified in terms of the constant desired output, i.e.,

$$\begin{aligned}\xi(t) &= \underline{\xi} := \left[\underline{y}, \dot{y}(t) = 0, \ldots, y^{(r-1)}(t) = 0\right]^T \quad \forall t \leq 0 \\ \xi(t) &= \overline{\xi} := \left[\overline{y}, \dot{y}(t) = 0, \ldots, y^{(r-1)}(t) = 0\right]^T \quad \forall t \geq t_f.\end{aligned} \tag{13}$$

Therefore, the only flexibility in the system state is in the internal dynamics $\eta(t)$ in Eq. (12), which is both controllable and observable as shown below.

Lemma 1: (A_η, B_η) is a controllable pair and (A_η, C_η) is an observable pair.

Proof: The system transfer function (Eq. 1) can be rewritten to obtain the inverse input $u_1 = u_{1,inv}$ needed to track a given desired output, $y = y_d$ as

$$u_{1,inv}(s) = -\frac{N_2(s)}{N_1(s)} u_2(s) + \frac{N_{\mathbb{Y}}(s)}{N_1(s)} \mathbb{Y}_d(s). \tag{14}$$

From Assumption 3, $\frac{N_{\mathbb{Y}}(s)}{N_1(s)}$ and $\frac{N_2(s)}{N_1(s)}$ are proper transfer functions. The lemma follows since the state space representation (in Eq. 12) of $\frac{N_2(s)}{N_1(s)}$ is minimal because $N_1(s)$ and $N_2(s)$ do not share common roots (Assumption 2). ∎

C. Rewrite Internal Dynamics

The initial ($\underline{\eta}$) and final ($\overline{\eta}$) equilibrium configurations of the internal dynamics which correspond to equilibrium states $\underline{x}$ and $\overline{x}$ are given by (from Eq. 10)

$$\underline{x} = \Phi\left[\underline{\xi}^T \ \underline{\eta}^T\right]^T; \qquad \overline{x} = \Phi\left[\overline{\xi}^T \ \overline{\eta}^T\right]^T \tag{15}$$

and satisfy the equilibrium condition (from Eqs. 4 and 12)

$$0 = A_\eta \underline{\eta} + B_{\mathbb{Y}} \underline{\mathbb{Y}}_d; \qquad 0 = A_\eta \overline{\eta} + B_{\mathbb{Y}} \overline{\mathbb{Y}}_d \tag{16}$$

where $\underline{\mathbb{Y}}_d := \left[\underline{\xi}^T, \ 0\right]^T$ and $\overline{\mathbb{Y}}_d := \left[\overline{\xi}^T, \ 0\right]^T$. To simplify the derivation, we represent the system in a new coordinate where the equilibrium is shifted to the origin. For example, during the post-actuation, the new state $\hat{x}(t) := x(t) - \overline{x}$. in transformed coordinates (ξ, η) are

$$\begin{bmatrix} \hat{\xi}(t) \\ \hat{\eta}(t) \end{bmatrix} = \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} - \begin{bmatrix} \overline{\xi} \\ \overline{\eta} \end{bmatrix}. \tag{17}$$

where the shifted component $\hat{\xi}$ is zero during the post-actuation (from Eq. 13). Then the internal dynamics (Eq. 12) during the post-actuation can be represented in the shifted coordinate as (using Eqs. 16 and 17)

$$\begin{aligned}\dot{\hat{\eta}}(t) &= A_\eta \hat{\eta}(t) + B_\eta u_2(t) + \left\{A_\eta \overline{\eta} + B_{\mathbb{Y}} \overline{\mathbb{Y}}_d\right\} \\ &= A_\eta \hat{\eta}(t) + B_\eta u_2(t) \quad \forall t \geq t_f\end{aligned} \tag{18}$$

and the output tracking input law (Eq. 11) becomes

$$\begin{aligned}u_{1,inv}(t) &= C_\eta \hat{\eta}(t) + D_\eta u_2(t) + \left[C_\eta \overline{\eta} + D_{\mathbb{Y}} \mathbb{Y}_d(t)\right] \\ &= C_\eta \hat{\eta}(t) + D_\eta u_2(t) \quad \forall t \geq t_f\end{aligned} \tag{19}$$

where the term in the brackets is zero since the equilibrium condition ($\hat{\eta} = 0$, $u_{1,inv} = 0$, $u_2 = 0$) satisfies the inverse law (Eq. 11). Moreover, during pre-actuation ($t \leq 0$) one obtains similar equations with $\hat{\eta}(t) = \eta(t) - \underline{\eta}(t)$, i.e.,

$$\left.\begin{aligned}\dot{\hat{\eta}}(t) &= A_\eta \hat{\eta}(t) + B_\eta u_2(t) \\ u_{1,inv}(t) &= C_\eta \hat{\eta}(t) + D_\eta u_2(t)\end{aligned}\right. \quad \forall t \leq 0. \tag{20}$$

D. Choice in the Boundary States:

The component ξ of the system state is completely specified, during pre-actuation ($t < 0$) and post-actuation ($t > t_f$) by Eq. (13); therefore, the only component that can be varied is the internal state η. Therefore, the boundary states $\{x_0 = x(0), x_{t_f} = x(t_f)\}$ must be chosen as

$$x_0 = \Phi\left[\underline{\xi}^T \mid \eta_0^T\right]^T \text{ and } x_{t_f} = \Phi\left[\overline{\xi}^T \mid \eta_{t_f}^T\right]^T \tag{21}$$

As a result, the available flexibility in the boundary states $\{x_0, x_{t_f}\}$ is in the choice of the boundary condition Ψ on the internal states

$$\Psi := \left[\eta_{t_f}^T \ \ \eta_0^T\right]^T. \tag{22}$$

E. Dependence on Boundary Condition Ψ

The choice of the boundary condition Ψ (Eq. 22) determines the cost for pre-actuation J_{pre}, transition J_{tran} and post-actuation J_{post} (in Eq. 8), as shown below. We begin with the input-energy cost during the post-actuation (J_{post} in Eq. 8), which can be rewritten by substituting for input $u_1 = u_{1,inv}$ (from Eq. 19) as

$$J_{post} := \int_{t_f}^{\infty} \left[\hat{\eta}^T Q \hat{\eta} + u_2^T N u_2 + 2 u_2^T S \hat{\eta}\right] dt. \tag{23}$$

where the matrices Q, S, and N are given by

$$\begin{aligned}Q &= C_\eta^T C_\eta; \quad S = D_\eta^T C_\eta = D_\eta C_\eta \\ N &= \rho + D_\eta^T D_\eta = \rho + D_\eta^2\end{aligned} \tag{24}$$

with $D_\eta^T = D_\eta$ since D_η (in Eq. 11) is a scalar. Thus, the optimization of the post-actuation cost (Eq. 23) is a standard linear-quadratic (LQ) optimal control problem subject to the internal dynamics (Eq. 18) and the choice of initial condition, i.e., the boundary state $\hat{\eta}(t_f)$ at the start of the post-actuation. The optimal post-actuation input can be obtained by using the positive definite solution W to an algebraic Riccati equation (ARE) under the following standard conditions, e.g., [13].

*Definition 3 (**LQ Solution Conditions**):*

1) The matrix $Q - S^T N^{-1} S$ is nonnegative definite,
2) the pair $(\tilde{A}_\eta, B_\eta)$ is controllable, where
$$\tilde{A}_\eta = A_\eta - B_\eta N^{-1} S \tag{25}$$
3) and the pair $(\tilde{A}_\eta, \tilde{C}_\eta)$ is observable, where $\tilde{C}_\eta$ is any matrix that satisfies $\tilde{C}_\eta \tilde{C}_\eta^T = Q - S^T N^{-1} S$. ∎

Lemma 2: The above LQ-solution conditions (in Definition 3) are satisfied under Assumptions 1-3.

Proof: The first condition follows from Eq. (24) as

$$Q - S^T N^{-1} S = C_\eta^T \frac{\rho}{\rho + D_\eta^2} C_\eta \geq 0 \tag{26}$$

since $\rho > 0$ (see Definition 2). The controllability of the pair $(\tilde{A}_\eta, B_\eta)$ is a consequence of the controllability of the pair (A_η, B_η), shown in Lemma 1, because controllability is preserved under the state feedback (Eq. 25), e.g., see [13], Lemma in Section 3.4, page 48. In general, state feedback does not preserve observability, however, the observability of $(\tilde{A}_\eta, \tilde{C}_\eta)$ follows from the observability of (A_η, C_η) shown in Lemma 1, because $\tilde{A}_\eta^T$ and $\tilde{C}_\eta^T$ can be written in terms of A_η^T and C_η^T,from Eqs. (24-26), as

$$\begin{aligned}\tilde{C}_\eta^T &= C_\eta^T \sqrt{\frac{\rho}{\rho + D_\eta^2}} \\ \tilde{A}_\eta^T &= A_\eta^T - C_\eta^T D_\eta (N^{-1})^T B_\eta^T\end{aligned} \tag{27}$$

Then the observability of $(\tilde{A}_\eta, \tilde{C}_\eta)$ follows from the observability of (A_η, C_η) — in Lemma 1. ∎

Lemma 3: **Optimization of Post-actuation Input** For any given transition time $t_f > 0$, the post-actuation input that satisfies the output-tracking conditions (Eqs. 5 and 6 in Definition 1) is uniquely specified in terms of the internal state η_{t_f} at the end of the output transition at time t_f as

$$u_{i,post}(t) \quad = \quad -K_{i,post}\left[e^{A_{cl,post}(t-t_f)}\right]\left\{\eta_{t_f}-\overline{\eta}\right\} \quad (28)$$

where $i = 1, 2$ and $\overline{\eta}$ is the internal state corresponding to the final equilibrium state $\overline{x}$, i.e., $[\overline{\xi}^T, \overline{\eta}^T]^T = \Phi^{-1}\overline{x}$. The gains $K_{1,post}$ and $K_{1,post}$ are defined as

$$\begin{aligned} K_{1,post} &:= \frac{CA^r\Phi_\eta}{CA^{r-1}B_1} - \frac{CA^{r-1}B_2}{CA^{r-1}B_1}K_{2,post} \\ K_{2,post} &:= N^{-1}\left(B_\eta{}^T W_{post} + S\right), \end{aligned}$$

W_{post} is the positive-definite solution to

$$\begin{aligned} &A_\eta^T W_{post} + W_{post}A_\eta + Q - \\ &\quad \left\{B_\eta^T W_{post} + S\right\}^T N^{-1}\left\{B_\eta^T W_{post} + S\right\} = 0, \end{aligned}$$

and the matrix $A_{cl,post}$ represents the post-actuation internal dynamics (with the post-actuation input) and is equal to

$$A_{cl,post} = A_\eta - B_\eta N^{-1}B_\eta^T W_{post} - B_\eta N^{-1}S.$$

Furthermore, the cost associated with the post-actuation input is quadratic in the boundary condition Ψ (in Eq. 22)

$$J_{post} \quad = \quad \left\{\eta_{t_f}-\overline{\eta}\right\}^T W_{post}\left\{\eta_{t_f}-\overline{\eta}\right\}. \quad (29)$$

Proof: Lemma 2 implies W_{post} is positive definite; the rest follows from standard optimal control theory [13]. ■

Lemma 4: **Optimization of Pre-actuation Input** Given the conditions in Lemma 3, the pre-actuation input is uniquely specified (in terms of η_0) as

$$u_{i,pre}(t) \quad = \quad -K_{i,pre}\left[e^{A_{cl,pre}(-t)}\right]\left\{\eta_0-\underline{\eta}\right\} \quad (30)$$

where $i = 1, 2$ and $\underline{\eta}$ is the internal state corresponding to the initial equilibrium state $\underline{x}$, i.e., $[\underline{\xi}^T, \underline{\eta}^T]^T = \Phi^{-1}\underline{x}$,

$$\begin{aligned} K_{1,pre} &:= \frac{CA^r\Phi_\eta}{CA^{r-1}B_1} - \frac{CA^{r-1}B_2}{CA^{r-1}B_1}K_{2,pre} \\ K_{2,pre} &:= N^{-1}\left(-B_\eta{}^T W_{pre} + S\right), \end{aligned}$$

the matrix W_{pre} is the positive-definite solution to

$$\begin{aligned} &-A_\eta^T W_{pre} - W_{pre}A_\eta + Q - \\ &\quad \left\{-B_\eta^T W_{pre} + S\right\}^T N^{-1}\left\{-B_\eta^T W_{pre} + S\right\} = 0. \end{aligned} \quad (31)$$

$$A_{cl,pre} = -A_\eta - B_\eta N^{-1}B_\eta^T W_{pre} + B_\eta N^{-1}S.$$

Furthermore, the cost associated with the pre-actuation input is quadratic in the boundary condition Ψ (in Eq. 22)

$$J_{pre} \quad = \quad \left\{\eta_0-\underline{\eta}\right\}^T W_{pre}\left\{\eta_0-\underline{\eta}\right\}. \quad (32)$$

Proof: The proof is omitted for brevity because it is similar to the post-actuation case. Note that by reversing the time axis, the pre-actuation problem becomes similar to the post-actuation problem, i.e., to drive the internal dynamics from $\hat{\eta}(0)$ to the initial equilibrium state [5]. ■

Lemma 5: **Optimization of Input During Transition Time** $(0, t_f)$ For any given transition time $t_f > 0$, the minimum input-energy input depends on the choice of the boundary condition Ψ (in Eq. 22). In particular, let $\{x_0, x_{t_f}\}$ be a pair of states related to the boundary condition Ψ through Eq. (21) Then, the minimum input-energy input that transfers the system from the initial state x_0 to the final state x_{t_f} within a transition time $T_{tran} = t_f$ is given by

$$u_{i,tran}(t) \quad = \quad \rho^{(1-i)}B_i^T e^{A^T(t_f-t)}G^{-1}d_x \quad (33)$$

for $i = 1, 2$, where G is the *invertible* controllability grammian, defined by

$$G = \int_0^{t_f} e^{A(t_f-\tau)}\left[B_1 \;\; B_2\right]\begin{bmatrix} 1 & 0 \\ 0 & \rho \end{bmatrix}^{-1}\begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} e^{A^T(t_f-\tau)}d\tau, \quad (34)$$

and d_x denotes the transition-state difference, given by

$$\begin{aligned} d_x &:= x_f - e^{At_f}x_0 \quad (35) \\ &:= H_1 f_x + H_2\Psi \quad (36) \end{aligned}$$

where $H_1 := [\Phi_\xi \mid -\Gamma_\xi]$, $H_2 := [\Phi_\eta \mid -\Gamma_\eta]$,

$$[\Gamma_\xi \mid \Gamma_\eta] := e^{At_f}\left[\Phi_\xi \mid \Phi_\eta\right], \text{ and } f_x := \left[\overline{\xi}^T \mid \underline{\xi}^T\right]^T.$$

Furthermore, the cost during the output transition when using this optimal state-transition control input is also quadratic in the boundary condition Ψ (in Eq. 22)

$$\begin{aligned} J_{tran} &= d_x^T G^{-1}d_x \\ &= (H_1 f_x + H_2\Psi)^T G^{-1}(H_1 f_x + H_2\Psi). \end{aligned} \quad (37)$$

Proof: This follows from standard optimal control theory, e.g., see [13]. ■

F. Minimum-Energy Output Transition

The input-energy cost $J_E(t_f)$, can be minimized by optimally choosing the boundary condition Ψ, as shown below.

Theorem 1 (Minimum-energy Output Transition): Given a transition time $t_f > 0$, the input energy J_E is optimized by the following choice of boundary condition Ψ_E and the corresponding optimal boundary states $\{x_{E,0}, x_{E,t_f}\}$ as

$$\Psi_E := \begin{bmatrix} \eta_{E,t_f} \\ \eta_{E,0} \end{bmatrix} = \Lambda^{-1}b,$$
$$x_{E,0} = \Phi\left[\underline{\xi}^T \mid \eta_{E,0}{}^T\right]^T, \quad x_{E,t_f} = \Phi\left[\overline{\xi}^T \mid \eta_{E,t_f}{}^T\right]^T. \quad (38)$$

where the matrix Λ is defined by

$$\Lambda := \begin{bmatrix} W_{post} & 0 \\ 0 & W_{pre} \end{bmatrix} + H_2^T G^{-1}H_2. \quad (39)$$

and the minimal-energy input u_E is given by (for $i = 1, 2$)

$$u_{E,i} \;=\; \begin{cases} -K_{i,pre}\left[e^{A_{cl,pre}(-t)}\right]\left\{\eta_{E,0}-\underline{\eta}\right\} & \forall\, t<0 \\ -K_{i,post}\left[e^{A_{cl,post}(t-t_f)}\right]\left\{\eta_{E,t_f}-\overline{\eta}\right\} & \forall\, t>t_f \\ \rho^{(1-i)}B_i^T e^{A^T(t_f-t)}G^{-1}\left[x_{E,t_f}-e^{At_f}x_{E,0}\right] & \\ \qquad \forall\, 0\le t\le t_f, & \end{cases}$$

and input-energy cost is

$$J_{E,opt}(t_f) := \Psi_E{}^T\Lambda\Psi_E - 2\Psi_E{}^T b + c = -b^T\Lambda^{-1}b + c \quad (40)$$

$$\text{where } b := \begin{bmatrix} W_{post}\overline{\eta} \\ W_{pre}\underline{\eta} \end{bmatrix} - H_2^T G^{-1}H_1 f_x,$$

$$c := \overline{\eta}^T W_{post}\overline{\eta} + \underline{\eta}^T W_{pre}\underline{\eta} + f_x^T H_1^T G^{-1}H_1 f_x.$$

Proof: For a given boundary condition Ψ, the minimal input-energy cost J_E (in Eq. 8) can be found from J_{post} in Eq. (29), J_{pre} in Eq. (32), and J_{tran} in Eq. (37) as

$$J_E(t_f, \Psi) := \Psi^T\Lambda\Psi - 2\Psi^T b + c \quad (41)$$

which is quadratic in Ψ. Therefore, the optimal value of the boundary condition Ψ (Ψ_E in Eq. 38) follows since Λ (in Eq. 39) is positive definite because W_{post}, W_{pre} are positive definite from Lemmas 3 and 4, and $H_2^T G^{-1}H_2$ (in Eq. 39) is positive semi-definite by symmetry and invertibility of G. The rest follows by setting $\Psi = \Psi_E$ in Lemmas 3-5. ■

IV. OOT: Transition Time t_f Free Case

Lemma 6: The cost function $\hat{J}(t_f)$ in Eq. (9) is a continuous function of the transition time t_f for all $t_f > 0$.

Proof: This follows since $\hat{J}(t_f)$ is the composition of products and sums of functions that are continuous in t_f. ■

Lemma 7: If the transition is between two distinct outputs, i.e., $\underline{y} \neq \overline{y}$, then the time/energy cost $\hat{J}(t_f)$ in Eq. (9) tends to infinity as the transition time t_f decreases to zero ($t_f \to 0^+$), and as the transition time increases to infinity.

Proof: The time/energy cost $\hat{J}(t_f)$ tends to infinity as the transition time increases because the term γt_f (in Eq. 9) tends to infinity as $t_f \to \infty$. Next, as the transition time t_f approaches zero, we show that the input-energy term $J_{E,opt}(t_f)$ (in Eq. 9), and the time/energy cost $\hat{J}(t_f)$, tend to infinity. Towards this, the transition part of the the input-energy (J_{tran} in Eq. 37) is bounded from below as

$$\begin{aligned} J_{tran} &\geq \lambda_{min}[G(t_f)^{-1}]\; \|d_x(t_f)\|^2 \\ &= \{\lambda_{max}[G(t_f)]\}^{-1} \|d_x(t_f)\|^2 \end{aligned} \tag{42}$$

where $\|\cdot\|$ denotes the standard 2-norm, and $\lambda_{min}[\cdot], \lambda_{max}[\cdot]$ are the smallest and largest eigenvalues, respectively. Furthermore, an upper bound on maximum eigenvalue $\lambda_{max}[G(t_f)]$ of the controllability grammian $G(t_f)$ can be found as [14]

$$\begin{aligned} \lambda_{max}[G(t_f)] &\leq \int_0^{t_f} \|e^{A(t_f-\tau)} B R^{-1} B^T e^{A^T(t_f-\tau)}\| d\tau \\ &\leq \left\{ \max_{\upsilon \in [0,t_f]} \|e^{A\upsilon} B R^{-1} B^T e^{A^T \upsilon}\| \right\} \int_0^{t_f} d\tau \\ &= \kappa t_f \quad \text{where} \quad 0 < \kappa < \infty. \end{aligned} \tag{43}$$

Substituting $\lambda_{max}[G(t_f)]$ from Eq. (43) into Eq. (42) yields

$$J_{tran} \geq \frac{\|d_x(t_f)\|^2}{\kappa t_f}, \tag{44}$$

which implies that this cost tends to infinity as the transition time $t_f \to 0$ unless the state difference $d_x(t_f) \to 0$ as well. However, we show that if the state-difference $d_x(t_f)$ approaches zero then the pre-actuation cost J_{pre} will go to infinity. From Eqs. (10, 21, 35)

$$\begin{aligned} d_x(t_f) &= x_{t_f}(t_f) - e^{At_f} x_0(t_f) \\ &= \Phi \left\{ \begin{bmatrix} \overline{\xi} \\ \eta_{t_f}(t_f) \end{bmatrix} - \Phi^{-1} e^{At_f} \Phi \begin{bmatrix} \underline{\xi} \\ \eta_0(t_f) \end{bmatrix} \right\} \end{aligned} \tag{45}$$

where the transition matrix $\Phi^{-1} e^{At_f} \Phi$ is continuous in the transition time t_f and is partitioned (corresponding to the coordinates (ξ, η)) as

$$\Phi^{-1} e^{At_f} \Phi := \begin{bmatrix} \Theta_{\xi\xi}(t_f) & \Theta_{\xi\eta}(t_f) \\ \Theta_{\eta\xi}(t_f) & \Theta_{\eta\eta}(t_f) \end{bmatrix} \tag{46}$$

with $\lim_{t_f \to 0} \Phi^{-1} e^{At_f} \Phi = I$. Note that, since the transition matrix Φ is invertible, the state-difference $d_x(t_f)$ will only approach zero if the difference in the output-related term ξ (in Eq. 10) approaches zero, i.e. from Eqs. (45,46),

$$\lim_{t_f \to 0} \{\overline{\xi} - \Theta_{\xi\xi}(t_f)\underline{\xi} - \Theta_{\xi\eta}(t_f)\eta_0(t_f)\} = 0. \tag{47}$$

This limit (in Eq. 47) will only be true if $\|\eta_0(t_f)\|$ approaches infinity since

$$\lim_{t_f \to 0} \Theta_{\xi\xi}(t_f) = I, \quad \lim_{t_f \to 0} \Theta_{\xi\eta}(t_f) = 0, \tag{48}$$

and $\overline{\xi} \neq \underline{\xi}$ since $\overline{y} \neq \underline{y}$. However, if $\|\eta_0(t_f)\|$ approaches infinity then $\|\eta_0(t_f) - \underline{\eta}\|$ also approaches infinity where $\underline{\eta}$ is finite and corresponds to the initial equilibrium state $\underline{x}$. Therefore, the pre-actuation cost approaches infinity since (from Eq. 32)

$$J_{pre} = \{\eta_0 - \underline{\eta}\}^T W_{pre} \{\eta_0 - \underline{\eta}\} \geq \underline{\lambda}_{W_{pre}} \|\eta_0(t_f) - \underline{\eta}\|^2 ,$$

where $\underline{\lambda}_{W_{pre}}$ the smallest eigenvalue of W_{pre} is greater than zero because W_{pre} is positive definite (Lemma 4). Thus, either the pre-actuation cost J_{pre} or the transition cost J_{tran} will approach infinity as the transition time $t_f \to 0$. Therefore, the total time/energy cost $\hat{J}(t_f)$ (in Eq. 9) also approaches infinity as the transition time $t_f \to 0$. ■

Theorem 2: There exists a solution to the minimum-time/energy output-transition problem in Definition 2.

Proof: If the initial and final output are the same, i.e. $\underline{y} = \overline{y}$, then the optimal transition time is $t_f^* = 0$ with zero input. Consider the non-trivial case, i.e., $\underline{y} \neq \overline{y}$ and let the minimum transition cost be $\hat{J}(t_1) = K_1/2$ where K_1 is a positive scalar for some transition time $t_f = t_1$. Then, there exists a nonzero transition time $t_{f,\alpha}$ such that the time/energy cost $\hat{J}(t_f) \geq K_1$ whenever the transition time t_f is sufficiently small, i.e., less than $t_{f,\alpha}$ because $\hat{J}(t_f) \to \infty$ as $t_f \to 0+$ as shown in Lemma 7. Furthermore, from Eq. (9), the time/energy cost $\hat{J}(t_f) \geq K_1$ whenever the transition time t_f is sufficiently large, in particular when $t_f > t_{f,\beta}$ where $t_{f,\beta} = K_1/\gamma$. Therefore, the time/energy cost function $\hat{J}(t_f)$ is larger than K_1 outside the closed and bounded time interval $[t_{f,\alpha}, t_{f,\beta}]$, which is nonempty since the transition time of $t_f = t_1$ must be in this time interval as illustrated in Figure 1.

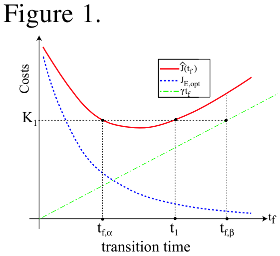

Fig. 1. Schematic sketch of transition-time cost γt_f, energy cost $J_{E,opt}$ and total time/energy cost $\hat{J}(t_f)$ as functions of transition-time t_f.

The existence of a minimum in this time interval $[t_{f,\alpha}, t_{f,\beta}]$ follows from the continuity of the cost function $\hat{J}(t_f)$ in the transition time t_f since a minima exists for any continuous function on a compact (closed and bounded) interval, e.g., see Theorem 3.17.21 in [15]. ■

V. Example: Dual-Stage Actuator Systems

We illustrate the advantages of using the proposed optimal output-transition approach on an experimental dual-stage system shown in Figure 2. The experimental system is similar to dual-stage actuators in emerging disk drive applications. The actuator in the first stage is a voice coil motor (VCM) and the second stage is a piezo-actuator (PZT). The tip-position of the flexible arm is the output, which is measured using an eddy-current inductive sensor. Additional details of the experimental system is provided in Ref. [16].

Results: To evaluate the effectiveness of the use of pre- and post-actuation in the proposed control scheme, we compare the results of minimum-time/energy output-transition solution (OOT) to the one obtained from standard minimum-time energy solution (SST) without pre- and post-actuation [1].

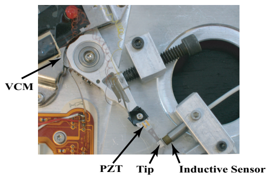

Fig. 2. The dual-stage actuator experimental system.

Cost comparison between OOT and SST: The optimal OOT cost, using pre- and post-actuation, is always less than or equal the SST cost because the inputs considered in the output-transition optimization (Definition 1) includes the optimal input from the SST approach with zero pre- and post-actuation. For example, the cost comparison for output-transition of 2.5 micrometers between using the OOT approach (which uses pre- and post-actuation) and the SST approach (which does not use pre- and post-actuation) is shown in Figure 3. It is noted that the cost component due to the input-energy ($J_{E,opt}$) increases as the output-transition time t_f decreases, as expected. Furthermore, for any given output-transition time t_f, the minimum time/energy cost (J_{OOT}) from the OOT approach (with pre- and post-actuation) is less than the total time/energy cost (J_{SST}) from the SST approach without pre- and post-actuation. We achieve a 19.7% reduction in the optimal time/energy cost by using the OOT approach ($J^*_{oot} = 1.85 \times 10^{-3}$) when compared to the SST approach ($J^*_{sst} = 2.30 \times 10^{-3}$). In addition, the optimal output-transition time is reduced by 23% (from 4.8 milliseconds by using SST approach to 3.7 milliseconds) compared to the result from standard optimal control technique, i.e., we can achieve faster output transition by using the pre- and post-actuation.

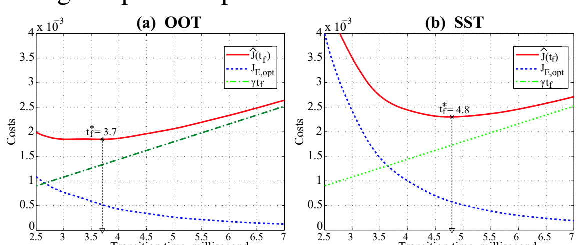

Fig. 3. Comparison of time/input-energy cost: OOT vs SST.

Experimental results: The experimental results of the minimum-time/input-energy output-transition solution is presented in Figure 4(a) to verify the output transitions are achieved with the computed inputs. The comparison of the VCM and PZT input trajectories between using the OOT and SST approaches are shown in Figures 4(c) and 4(d). Note that both OOT and SST approaches achieve the output transition in the same output-transition time; however, the OOT achieves this output transition with a lower cost. The weighting factors γ and ρ can be varied to find the fastest achievable output-transition without exceeding a given constraint on the VCM and PZT inputs. For example, if the VCM input is to be kept below $15mv$ and the PZT input below $10V$ then, the fastest seek time that can be achieved by using OOT approach is 3.7 millisecond, over different choices of the weighting factors γ and ρ[16].

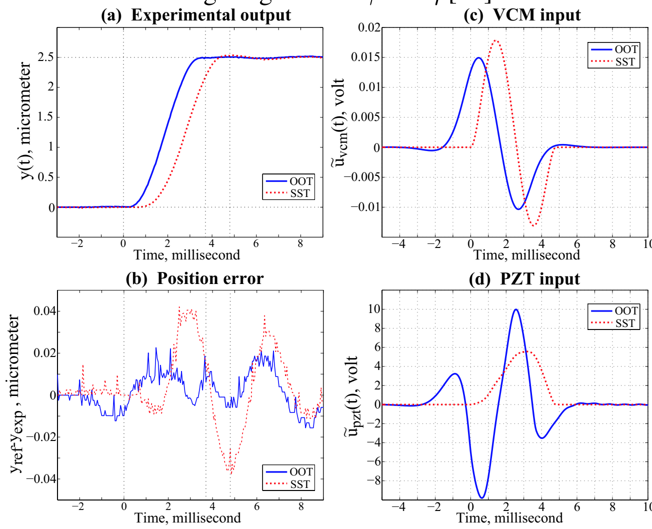

Fig. 4. Experimental results for 2.5-micrometer output-transition: (a) Output trajectory, (b) Position error with OOT approach, (c) VCM-input trajectory, and (d) PZT-input trajectory.

VI. Conclusion

The minimum-time/input-energy output-transition problem for dual-stage systems was posed and solved in this article. The approach was applied to a dual-stage actuator system, and experimental results were presented.

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