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On Stochastic LQR Design and Polynomial Chaos

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***Abstract*— In this paper we develop a novel theoretical framework for linear quadratic regulator design for linear systems with probabilistic uncertainty in the parameters. The framework is built on the generalized polynomial chaos theory, which can handle Gaussian, uniform, beta and gamma distributions. In this framework, the stochastic dynamics is transformed into deterministic dynamics in higher dimensional state space, and the controller is designed in the expanded state space. The proposed design framework results in a family of controllers, parameterized by the associated random variables. The theoretical results are applied to a controller design problem based on stochastic linear, longitudinal F16 model. The performance of the stochastic design shows excellent consistency with the results obtained from Monte-Carlo based designs, in a statistical sense.**

I. Introduction

Recently there has been a growing interest in combining robust control approaches with stochastic control methods to develop the so called field of *probabilistic robust control*. In traditional robust control, uncertainty is restricted to uniform distributions only and robustness is a binary notion. In probabilistic robust control, the risk adjusted robust performance of the closed-loop system is of interest, which is more practical and possibly less conservative. Research on feedback control of stochastic systems have been focussed on deterministic systems with stochastic forcing. The system dynamics is assumed to be perfectly known and is excited by a noise with certain statistical properties. For linear systems and Gaussian white noise excitation, methods such as $\mathcal{H}_2$ (LQG) optimal control algorithms are quite matured. Previous work on stochastic control for linear systems haven focussed on regulating the moment response of stochastic systems with either stochastic forcing or probabilistic uncertainty on system parameters. Problems with stochastic forcing have almost always assumed Gaussian distribution. Non Gaussian forcing is usually approximated using Gaussian closure. Skelton *et al.* [1] have addressed the problem of covariance control with Gaussian excitation. Barmish *et al.* [2], [3] and Stengel *et al.* [4] have addressed robustness with probabilistic uncertainty using sampling or Monte-Carlo based approaches. Recently Polyak *et al.* [5] and Fujisaki *et al.* [6] have addressed LQR design with probabilistic uncertainty for linear and linear parameter varying systems respectively.

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Till date, methods that address probabilistic uncertainty in system parameters, use sampling based approaches to solve the stochastic problem in a deterministic setting, which results in very large scale problems for sufficiently accurate representation of uncertainty. The complexity is reduced if design is restricted to uncertainty with uniform distribution, and worst-case design philosophy is adopted. The novelty of the framework presented in this paper is that non-sampling based methods are used to accurately capture the evolution of uncertainty in state trajectories due to uncertainty in system parameters, which can potentially lead to less conservative designs. The framework is built on the generalized polynomial chaos theory which translates stochastic dynamics into deterministic dynamics, in higher dimensional state space. This increase in dimensionality, however, is significantly lower than the sampling based methods for comparably accurate representation of uncertainty.

Polynomial chaos (PC) was first introduced by Wiener [7] where Hermite polynomials were used to model stochastic processes with Gaussian random variables. According to Cameron and Martin [8] such an expansion converges in the $\mathcal{L}_2$ sense for any arbitrary stochastic process with finite second moment. This applies to most physical systems. Xiu *et al.* [9] generalized the result of Cameron-Martin to various continuous and discrete distributions using orthogonal polynomials from the so called Askey-scheme [10] and demonstrate $\mathcal{L}_2$ convergence in the corresponding Hilbert functional space. This is popularly known as the generalized polynomial chaos (gPC) framework.

The gPC framework has been applied to applications including stochastic fluid dynamics [11], [12], [13], stochastic finite elements [14], and solid mechanics [15], [16]. However, application of gPC to control related problems has been surprisingly limited. The work of Hover *et al.* [17] addresses stability & control of a bilinear dynamical system, with probabilistic uncertainty on the system parameters. The controller design problem considered involved determining a family of proportional gains to minimize a finite time integral cost functional. In this research work we focus on optimal control in the $\mathcal{L}_2$ sense for linear systems with probabilistic uncertainty in system parameters. It is assumed that the probability density functions of these parameters are known and these parameters may enter the system dynamics in any manner. We generalize these results for minimum expectation and variance cost, and

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determine appropriate control laws that solves the corresponding stochastic linear optimal control problems.

II. Wiener-Askey Polynomial Chaos

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of the subsets of Ω, and P is the probability measure. Let $\Delta(\omega) = (\Delta_1(\omega), \cdots, \Delta_d(\omega)) : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}^d)$ be an $\mathbb{R}^d$-valued continuous random variable, where $d \in \mathbb{N}$, and $\mathcal{B}^d$ is the σ-algebra of Borel subsets of $\mathbb{R}^d$. A general second order process $X(\omega) \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$ can be expressed by polynomial chaos as

$$X(\omega) = \sum_{i=0}^{\infty} x_i \phi_i(\Delta(\omega)), \quad (1)$$

where ω is the random event and $\phi_i(\Delta(\omega))$ denotes the gPC basis of degree p in terms of the random variables $\Delta(\omega)$. The functions $\{\phi_i\}$ are a family of orthogonal basis in $\mathcal{L}_2(\Omega, \mathcal{F}, P)$ satisfying the relation

$$\mathbf{E}[\phi_i \phi_j] = \mathbf{E}[\phi_i^2]\delta_{ij}, \quad (2)$$

where δ_{ij} is the Kronecker delta and $\mathbf{E}[\cdot]$ denotes the expectation with respect to the probability measure $dP(\omega) = f(\Delta(\omega))d\omega$ and probability density function $f(\Delta(\omega))$. Henceforth, we will use Δ to represent $\Delta(\omega)$.

For random variables Δ with certain distributions, the family of orthogonal basis functions $\{\phi_i\}$ can be chosen in such a way that its weight functions has the same form as the probability density function $f(\Delta)$. These orthogonal polynomials are the members of the Askey-scheme of polynomials [10], which forms a complete basis in the Hilbert space determined by their corresponding support. Table I summarizes the correspondence between the choice of polynomials for a given distribution of Δ [9].

Random Variable Δ	$\phi_i(\Delta)$ of the Wiener-Askey Scheme
Gaussian	Hermite
Uniform	Legendre
Gamma	Laguerre
Beta	Jacobi

TABLE I

Correspondence between choice of polynomials and given distribution of $\Delta(\omega)$.

III. Stochastic Linear Dynamics and Polynomial Chaos

To frame the optimal control problem, consider stochastic linear systems of the form

$$\dot{x}(t,\Delta) = A(\Delta)x(t,\Delta) + B(\Delta)u(t), \quad (3)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$. The system has probabilistic uncertainty in the system parameters, characterized by $A(\Delta), B(\Delta)$, which are matrix functions of random variable $\Delta \equiv \Delta(\omega) \in \mathbb{R}^d$ with certain *stationary* distributions. Due to the stochastic nature of (A, B), the system trajectory will also be stochastic. We do not consider stochastic forcing in this paper, but this framework can be easily extended to include stochastic forcing and will be addressed in future publications.

Let us represent components of $x(t,\Delta), A(\Delta)$ and $B(\Delta)$ as,

$$x(t,\Delta) = [x_1(t,\Delta) \cdots x_n(t,\Delta)]^T, \quad (4)$$

$$A(\Delta) = \begin{bmatrix} A_{11}(\Delta) & \cdots & A_{1n}(\Delta) \\ \vdots & & \vdots \\ A_{n1}(\Delta) & \cdots & A_{nn}(\Delta) \end{bmatrix}, \quad (5)$$

$$B(\Delta) = \begin{bmatrix} B_{11}(\Delta) & \cdots & B_{1m}(\Delta) \\ \vdots & & \vdots \\ B_{n1}(\Delta) & \cdots & B_{nm}(\Delta) \end{bmatrix}. \quad (6)$$

By applying the Wiener-Askey gPC expansion to $x_i(t,\Delta), A_{ij}(\Delta)$ and $B_{ij}(\Delta)$, we get

$$x_i(t,\Delta) = \sum_{k=0}^{p} x_{i,k}(t)\phi_k(\Delta) = \mathbf{x}_i(t)^T \Phi(\Delta), \quad (7)$$

$$A_{ij}(\Delta) = \sum_{k=0}^{p} a_{ij,k}\phi_k(\Delta) = \mathbf{a}_{ij}^T \Phi(\Delta), \quad (8)$$

$$B_{ij}(\Delta) = \sum_{k=0}^{p} b_{ij,k}\phi_k(\Delta) = \mathbf{b}_{ij}^T \Phi(\Delta), \quad (9)$$

where $\mathbf{x}_i(t), \mathbf{a}_{ij}, \mathbf{b}_{ij}, \Phi(\Delta) \in \mathbb{R}^p$ are defined by

$$\begin{aligned} \mathbf{x}_i(t) &= [x_{i,0}(t) \cdots x_{i,p}(t)]^T, \\ \mathbf{a}_{ij} &= [a_{ij,0}(t) \cdots a_{ij,p}(t)]^T, \\ \mathbf{b}_{ij} &= [b_{ij,0}(t) \cdots b_{ij,p}(t)]^T, \\ \Phi(\Delta) &= [\phi_0(\Delta) \cdots \phi_p(\Delta)]^T. \end{aligned}$$

Furthermore, without loss of generality, define

$$u_i(t,\Delta) = \sum_{k=0}^{p} u_{i,k}(t)\phi_k(\Delta) = \mathbf{u}_i(t)^T \Phi(\Delta) \quad (10)$$

with

$$\mathbf{u}_i(t) = [u_{i,0}(t) \cdots u_{i,p}(t)]^T.$$

When $u_i(t,\Delta)$ is a feedback control, two possibilities exist. It could be stochastic, the design in that case generates a family of control trajectories, parameterized by Δ. If control is chosen to be deterministic, i.e. we are interested in a single control trajectory for all possible $\Delta \in \mathcal{D}_\Delta = \{\Delta(\omega) : \omega \in \Omega\}$, $u_i(t) = u_{i,0}(t)$ with all other coefficients as zero.

The number of terms $p+1$ is determined by the dimension d of Δ and the order r of the orthogonal polynomials $\{\phi_k\}$, satisfying $p + 1 = \frac{(d+r)!}{d!r!}$. The coefficients $a_{ij,k}$ and $b_{ij,k}$ are obtained via Gelarkin projection onto $\{\phi_k\}_{k=0}^{p}$ given by

$$a_{ij,k} = \frac{\langle A_{ij}(\Delta), \phi_k(\Delta)\rangle}{\langle \phi_k(\Delta)^2 \rangle}, \quad (11)$$

$$b_{ij,k} = \frac{\langle B_{ij}(\Delta), \phi_k(\Delta)\rangle}{\langle \phi_k(\Delta)^2 \rangle}. \quad (12)$$

The $n(p+1)$ time varying coefficients, $\{x_{i,k}(t)\}; i = 1,\cdots,n; k = 0,\cdots,p$, are obtained by substituting the approximated solution in the governing equation (eqn.(3)) and conducting Gelarkin projection onto $\{\phi_k\}_{k=0}^p$, to yield $n(p+1)$ *deterministic* linear differential equations, given by

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U}, \tag{13}$$

with $\mathbf{X} \in \mathbb{R}^{n(p+1)}$; $\mathbf{A} \in \mathbb{R}^{n(p+1)\times n(p+1)}$; $\mathbf{B} \in \mathbb{R}^{n(p+1)\times m}$ and

$$\mathbf{X} = [\mathbf{x}_1^T \ \mathbf{x}_2^T \ \cdots \ \mathbf{x}_n^T]^T, \tag{14}$$

$$\mathbf{U} = [\mathbf{u}_1^T \ \mathbf{u}_2^T \ \cdots \ \mathbf{u}_m^T]^T. \tag{15}$$

While it is possible to derive many forms for the $\mathbf{A}$ and $\mathbf{B}$ matrices, a convenient form can be obtained in the following manner. Define $\hat{e}_{ijk} = \frac{\langle \phi_i, \phi_j \phi_k \rangle}{\langle \phi_i^2 \rangle}$. The linear dynamics can then be expressed as

$$\begin{aligned} \dot{x}_{i,l} \quad = \quad & \sum_{j=1}^{n} \sum_{k=0}^{p} \sum_{q=0}^{p} a_{ij,k} x_{j,q} \hat{e}_{lkq} + \\ & \sum_{j=1}^{m} \sum_{k=0}^{p} \sum_{q=0}^{p} b_{ij,k} u_{j,q} \hat{e}_{lkq}. \end{aligned}$$

Define the matrix Φ_k as

$$\Phi_k = \begin{bmatrix} \hat{e}_{1k1} & \hat{e}_{1k2} & \cdots & \hat{e}_{1kp} \\ \hat{e}_{1k2} & \hat{e}_{2k2} & \cdots & \hat{e}_{2kp} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}_{1kp} & \hat{e}_{2kp} & \cdots & \hat{e}_{pkp} \end{bmatrix}. \tag{16}$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ can be written as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1n} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n1} & \mathbf{A}_{n2} & \cdots & \mathbf{A}_{nn} \end{bmatrix}, \tag{17}$$

$$\mathbf{A}_{ij} = \sum_{k=0}^{p} a_{ij,k} \Phi_k, \tag{18}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1m} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \cdots & \mathbf{B}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{n1} & \mathbf{B}_{n2} & \cdots & \mathbf{B}_{nm} \end{bmatrix}, \tag{19}$$

$$\mathbf{B}_{ij} = \sum_{k=0}^{p} b_{ij,k} \Phi_k. \tag{20}$$

More convenient expressions for $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} \quad = \quad \sum_{k=0}^{p} A_k \otimes \Phi_k, \tag{21}$$

$$\mathbf{B} \quad = \quad \sum_{k=0}^{p} B_k \otimes \Phi_k, \tag{22}$$

where $\otimes$ is the Kronecker product and A_k, B_k are the projections of $A(\Delta), B(\Delta)$ on $\{\phi_k(\Delta)\}_{k=0}^P$ respectively. Therefore, transformation of a stochastic linear system with $x \in \mathbb{R}^n, u \in \mathbb{R}^m$, with p^{th} order gPC expansion, results in a *deterministic* linear system with increased dimensionality equal to $n(p+1)$.

IV. Stochastic LQR Design

The objective of this work is to determine the conditions for optimality of systems with probabilistic uncertainty in their coefficients. This enables us to solve stochastic optimal control problems in a deterministic framework. In this section, we derive a standard cost function encountered in stochastic optimal control problems in terms of the polynomial chaos expansions. Here we consider minimum expectation cost function. We will examine the solution to this problem for both deterministic and stochastic feedback control laws, and discuss the benefits/limitations of each.

A. Minimum Expectation Control

Minimum expectation optimal trajectories are obtained by minimizing the following cost function, analogous to the Bolza form,

$$\min_u \mathbf{E}\left[\int_0^\infty (x^T Q x + u^T R u) dt\right], \tag{23}$$

where $x \equiv x(t) \in \mathbb{R}^n, u \equiv u(t) \in \mathbb{R}^m$, $Q = Q^T > 0, R = R^T > 0, S = S^T > 0$. For scalar x, the quantity $\mathbf{E}[x^2]$ in terms of its gPC expansions is given by

$$\mathbf{E}[x^2] = \sum_{i=0}^{p} \sum_{j=0}^{p} x_i x_j \int_{\mathcal{D}_\Delta} \phi_i \phi_j f d\Delta = \mathbf{x}^T W \mathbf{x}, \tag{24}$$

where $\mathcal{D}_\Delta$ is the domain of Δ , x_i are the gPC expansions of x, $f \equiv f(\Delta)$ is the probability distribution of Δ; $W \in \mathbb{R}^{(p+1)\times(p+1)} = \{w_{ij}\}$, with $w_{ij} = \int_{\mathcal{D}_\Delta} \phi_i \phi_j f d\Delta$; and $\mathbf{x} = (x_0 \ \cdots \ x_p)^T$. The expression $\mathbf{E}[x^2]$ can be generalized for $x \in \mathbb{R}^n$ where $\mathbf{E}[x^T x]$ is given by

$$\mathbf{E}[x^T x] = \mathbf{X}^T (I_n \otimes W) \mathbf{X}, \tag{25}$$

$I_n \in \mathbb{R}^{n\times n}$ is the identity matrix and $\otimes$ is the Kronecker product, and $\mathbf{X}$ is given by eqn.(14). The cost function in eqn.(23) can now be written in terms of the gPC expansions as

$$\min_u J = \min_u \int_0^\infty \left(\mathbf{X}^T Q_{\bar{x}} \mathbf{X} + \mathbf{E}\left[u^T R u\right]\right) dt, \tag{26}$$

where $Q_{\bar{x}} = Q \otimes W$. The expected value of $u^T R u$ will depend upon the control implementation discussed in section IV-B.

B. Feedback Solutions

In this section, we will discuss conditions for optimality for various feedback structures as they apply to a quadratic cost of the form developed in the previous section.

1) Augmented Deterministic State Feedback with Constant Gain: The first implementation we will discuss involves the assumption that the control is probabilistic and augmented state vector $\mathbf{X}$ is used for feedback. If we assume $u = \sum_{k=0}^{p} u_{i,k}(t) \phi_k(\Delta)$

$$\mathbf{E}\left[u^T R u\right] = \mathbf{U}^T R_{\bar{u}} \mathbf{U}, \tag{27}$$

where $R_{\bar{u}} = R \otimes W$.

Proposition 1: The cost function in eqn. (26) is minimized with a control of the form

$$\mathbf{U} = -R_{\bar{u}}^{-1}\mathbf{B}^T P\mathbf{X}, \quad (28)$$

where P is the solution to the Riccati equation

$$\mathbf{A}^T P + P\mathbf{A} - P\mathbf{B}R_{\bar{u}}^{-1}\mathbf{B}^T P + Q_{\bar{x}} = 0. \quad (29)$$

Proof: As per the usual solution to the LQR problem, we can write the cost function as $J = \mathbf{X}^T P\mathbf{X}$. From Euler-Lagrange (by substituting for the Lagrange multiplier), we obtain $0 = R_{\bar{u}}\mathbf{U} + \mathbf{B}^T P\mathbf{X}$, giving

$$\mathbf{U} = -R_{\bar{u}}^{-1}\mathbf{B}^T P\mathbf{X},$$

where $\mathbf{U}$ is defined by eqn.(15). Substituting this into the cost function and taking the derivative of both sides gives

$$\dot{P} + \mathbf{A}^T P - P\mathbf{B}R_{\bar{u}}^{-1}\mathbf{B}^T P + P\mathbf{A} - P\mathbf{B}R_{\bar{u}}^{-1}\mathbf{B}^T P = -Q_{\bar{x}} - P\mathbf{B}R_{\bar{u}}^{-1}\mathbf{B}^T P.$$

For the infinite horizon problem $\dot{P} = 0$, completing the proof. ■

Remark 1: The solution to this expression yields a constant gain matrix, but implementation requires knowledge of gPC expansions of the states. The control vector $\mathbf{U} = -R_{\bar{u}}^{-1}\mathbf{B}^T P\mathbf{X}$ defines $u(t,\Delta)$, a family of control laws, parameterized by Δ. Appropriate $u(t)$ can be determined based on the knowledge of Δ, as a result Δ must also be known during implementation.

2) Stochastic State Feedback with Constant Gain: In this formulation, the state trajectory $x(t,\Delta)$ is used to generate the control law that is not explicitly parameterized by Δ. This approach does not require estimation of the gPC expansions of the state and hence doesn't require the knowledge of Δ. We propose feedback of the form

$$u(t,\Delta) = Kx(t,\Delta). \quad (30)$$

Once again the control is stochastic, due to stochastic state trajectory, and enters the cost function as $\mathbf{E}\left[u^T R u\right]$. The control vector in gPC framework then becomes

$$\mathbf{U} = (K \otimes I_p)\,\mathbf{X}. \quad (31)$$

In this manner, we are selecting a feedback structure that results in a problem similar to the output feedback problem in traditional control, for the system in eqn.(13). The modified cost function becomes

$$J = \int_0^\infty \mathbf{X}^T \left(Q_{\bar{x}} + \left(K^T \otimes I_p\right) R_{\bar{u}} \left(K \otimes I_p\right)\right) \mathbf{X} dt. \quad (32)$$

Proposition 2: For a feedback law of the form in eqn.(30), the cost function in eqn.(32) is minimized for a matrix $K \in \mathbb{R}^{m\times n}$ solving

$$\mathbf{A}^T P + P\mathbf{A} + P\mathbf{B}(K\otimes I_p) + (K^T \otimes I_p)\mathbf{B}^T P + Q_{\bar{x}} + (K^T \otimes I_p)R_{\bar{u}}(K\otimes I_p) = 0, \quad (33)$$

subject to $P = P^T > 0$. Furthermore, a solution exists for some $Q_{\bar{x}}$ and $R_{\bar{u}}$ if the feasibility condition

$$\mathbf{A}^T P + P\mathbf{A} + (K^T \otimes I_p)\mathbf{B}^T P + P\mathbf{B}(K\otimes I_p) < 0, \quad (34)$$

is satisfied.

Proof: Let $J = \mathbf{X}^T P\mathbf{X}$. Taking the derivative of the cost function gives rise to the matrix equation

$$\dot{P} + P\mathbf{A} + P\mathbf{B}(K\otimes I_p) + \mathbf{A}^T P + (K^T \otimes I_p)\mathbf{B}^T P = -Q_{\bar{x}} - (K^T \otimes I_p)R_{\bar{u}}(K\otimes I_p).$$

For an infinite time interval, let $\dot{P} \to 0$, giving the first condition. Now, we must show the second part of the proposition. The feasibility condition implies that we can select some stabilizing gain, K and that we can select some $M = M^T > 0$, and find a $P = P^T > 0$ such that

$$\mathbf{A}^T P + P\mathbf{A} + (K^T \otimes I_p)\mathbf{B}^T P + P\mathbf{B}(K\otimes I_p) = -M.$$

Select $M = \hat{M} \otimes W$. Let $\hat{M} = Q + K^T R K$. Because K make the system Hurwitz, use of Lyapunov's theorem guarantees the existence of a P. This completes the proof. ■

Remark 2: The bilinear matrix inequality (BMI) in eqn.(34) does not have any analytical solution and must be solved numerically to obtain K and P. The BMI can be solved using solvers such as PENBMI [18].

Remark 3: Unlike, in the previous design, the variation in the state trajectories directly maps to a corresponding deterministic control and does not require explicit knowledge of Δ. This can lead to computational benefits during implementation. This feedback structure mimics the traditional robust control approach where a single controller guarantees robust performance for the entire range of parameter variation. The advantage here is that it admits any arbitrary distribution, where traditional robust control is limited to uniform distribution only.

3) Stochastic State Feedback with Stochastic Gain: This section deals with the optimality of a control law that involves feedback of the form $u = K(\Delta)x(t,\Delta)$. This feedback structure is also analogous to output feedback control, but with increased degree of freedom. Implementation of this control law requires knowledge of Δ. To determine the values $u_{i,j}$, we project the control onto the polynomial subspace

$$u_{i,l} = \frac{1}{\langle \phi_l^2 \rangle} \sum_{j=1}^{n}\sum_{k=0}^{p}\sum_{q=0}^{p} k_{ij,k} x_{j,q} \langle \phi_l, \phi_k \phi_q \rangle,$$

giving

$$\mathbf{U} = \left(\sum_{k=0}^{p} K_k \otimes \Phi_k\right)\mathbf{X} = \mathbf{K}\mathbf{X}. \quad (35)$$

When the control K is not a function of Δ, this corresponds to $K_{ij,k} = 0$ for $k \geq 1$. The matrix $\Phi_0 = I_p$, so the previous case is recovered. The cost function is written in terms of this feedback strategy as

$$J = \int_0^\infty \mathbf{X}^T \left(Q_{\bar{x}} + \mathbf{K}^T R_{\bar{u}} \mathbf{K}\right)\mathbf{X} dt. \quad (36)$$

Proposition 3: The feedback law in eqn.(35) optimally drives the system to the origin with respect to the cost function in eqn.(36) for $K(\Delta)$ solving

$$\mathbf{A}^T P + P\mathbf{A} + P\mathbf{B}\mathbf{K} + \mathbf{K}^T\mathbf{B}^T P + Q_{\bar{x}} + \mathbf{K}^T R_{\bar{u}}\mathbf{K} = 0, \quad (37)$$

subject to $P = P^T > 0$. Furthermore, a solution exists for some $Q_{\bar{x}}$ and $R_{\bar{u}}$ if the feasibility condition

$$\mathbf{A}^T P + P\mathbf{A} + \mathbf{K}^T \mathbf{B}^T P + P\mathbf{B}\mathbf{K} < 0, \tag{38}$$

is satisfied.

Proof: The proof is similar to the previous proposition and is therefore omitted. ■

Remark 4: This control strategy provides more flexibility for solving the necessary condition for optimality at the expense of more complexity in implementation, i.e. the necessity for knowledge of Δ.

4) Deterministic Control with Augmented State Feedback: In this feedback structure, the augmented gPC states of the stochastic system is used to derive a deterministic control. This corresponds to a control with $u_i(t,\Delta) = u_{i,0}$. As a result, the system $\mathbf{B}$ matrix becomes

$$\hat{\mathbf{B}} = \begin{bmatrix} b_{11,1} & b_{12,1} & \cdots & b_{1m,1} \\ b_{11,2} & b_{12,2} & \cdots & b_{1m,2} \\ \vdots & \vdots & & \vdots \\ b_{11,p} & b_{12,p} & \cdots & b_{1m,p} \\ b_{21,1} & b_{22,1} & \cdots & b_{2m,1} \\ \vdots & \vdots & & \vdots \\ b_{n1,p} & b_{n2,p} & \cdots & b_{nm,p} \end{bmatrix},$$

or $\hat{\mathbf{B}}$ can be written in the form of eqn.(19), where

$$\hat{\mathbf{B}}_{ij} = \sum_{k=0}^{p} b_{ij,k}\delta_{1k}.$$

where δ_{1k} is a vector of zeros with a 1 at the k^{th} position. Since u is a deterministic control,

$$\mathbf{E}[u^T R u] = u^T R u. \tag{39}$$

Unlike previous cases, the dimension of $\hat{\mathbf{B}}$ is $n(p+1) \times m$ instead of $n(p+1) \times m(p+1)$. The optimal control problem for this case involves selecting a control structure of the form

$$u = K\mathbf{X}, \tag{40}$$

where $K \in \mathbb{R}^{m \times n(p+1)}$.

Proposition 4: Assume the matrix pair $(\mathbf{A}, \hat{\mathbf{B}})$ is controllable. The control law in eqn.(40) with a gain given by

$$K = -R^{-1}\hat{\mathbf{B}}^T P, \tag{41}$$

where $P = P^T > 0$ is the solution of the algebraic Riccati equation

$$\mathbf{A}^T P + P\mathbf{A} - P\hat{\mathbf{B}}R^{-1}\hat{\mathbf{B}}^T P + Q_{\bar{x}} = 0, \tag{42}$$

and optimizes the performance index in eqn.(26) for a deterministic feedback law.

Proof: This is the solution to the standard LQR problem. ■

Remark 5: The solution to this control problem maps $\mathbf{X}$, the gPC expansions of the states, directly to deterministic control $u(t)$. Hence, knowledge of Δ is necessary to compute $\mathbf{X}$ during implementation.

V. Example: Stochastic Control of an F-16 Aircraft Model

As a simple example, consider the following model of an F-16 aircraft at high angle of attack

$$\dot{x} = Ax + Bu,$$

with states $x = [V \;\; \alpha \;\; q \;\; \theta \;\; T]^T$ where V is the velocity, α the angle of attack, q the pitch rate, θ its angle, and T is the thrust. The controls, $u = [\delta_{th} \;\; \delta_e]^T$, are the elevator deflection, δ_e, and the throttle δ_{th}. Note that these values are all perturbation from trim. The A and B matrices are given by

$$A = \begin{bmatrix} 0.1658 & -13.1013 & (-7.2748) & -32.1739 & 0.2780 \\ 0.0018 & -0.1301 & (0.9276) & 0 & -0.0012 \\ 0 & -0.6436 & -0.4763 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -0.0706 \\ 0 & -0.0004 \\ 0 & -0.0157 \\ 0 & 0 \\ 64.94 & 0 \end{bmatrix}$$

Similar to the analysis of Lu[19], the terms in parenthesis in the A matrix are assumed to be stochastic and are functions of a *single* random variable, Δ. The uncertainty in these terms is assumed to be distributed uniformly by $\pm 20\%$ about the nominal values -7.2748 and 0.9276 respectively. This uncertainty corresponds to the uncertainty in the damping term C_{xq}. The control design problem is to keep the aircraft at trim, given some perturbation in the initial condition, in the presence of parametric uncertainty. This is accomplished with an LQR design, using the control law in eqn.(28). This design generates a family of controllers, parameterized by Δ. We compare the performance of the stochastic LQR design with Monte-Carlo designs, where LQR designs were performed for a large sample of uniformly distributed Δ. The cost function for the Monte-Carlo designs is kept identical to that in the stochastic design, i.e. matrices Q and R were same for all the designs. Figure 1 shows the performance of the Monte-Carlo LQR designs, represented in gray, as well as the performance of the probabilistic design. The variance and mean of the state trajectories, computed from the gPC expansions, are shown as dashed and solid line respectively. We observe that the statistics obtained from stochastic LQR design is consistent with the Monte-Carlo simulations. The key is that the stochastic control design problem is solved *deterministically* and by a *single* design. The controller obtained is statistically similar to the family of LQR designs over the sample set of Δ. An implementation of such a control design in practice would require knowledge of the parameter, Δ. Additional research is required to determine methodologies for estimating Δ. The polynomial chaos based framework can also be used for statistical verification of the robustness of the controller with probabilistic uncertainty in system parameters.

VI. Limitations of Polynomial Chaos

The gPC framework is well suited for evaluating short term statistics of dynamical systems. However, their performance

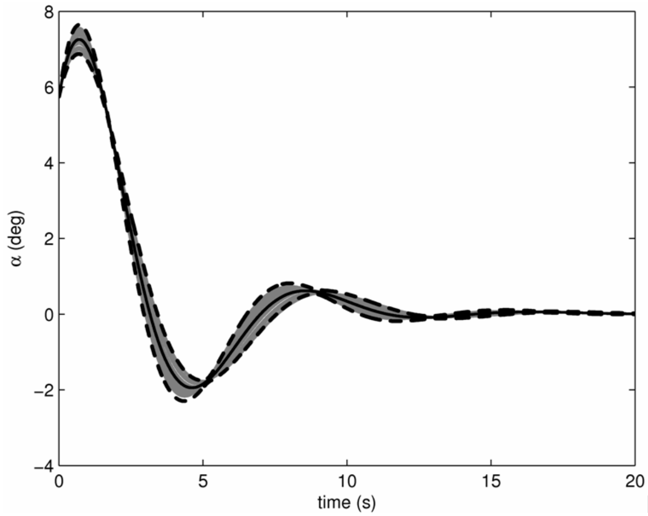

Fig. 1. Probabalistic response of α.

degrades upon long term integration. This degradation arises due to finite dimensional approximation of the probability space $(\Omega, \mathcal{F}, P)$. Several methods have been proposed to reduce this degradation, including adaptive [20] and multi-element approximation techniques [21]. We will incorporate them in our future work on stochastic LQR design.

VII. SUMMARY

In this paper we have presented a framework for designing linear quadratic regulators for systems with probabilistic parametric uncertainty. The framework is built on the generalized polynomial chaos theory and can handle Gaussian, uniform, beta and gamma probability distribution functions. The framework is different from other research performed in this area, in the characterization of parametric uncertainty and its propagation due to the dynamics of the system. From a single design, the proposed framework generates a family of controllers, parameterized by the associated random variables. The design process is validated on a simplified longitudinal linear F16 model, and the results were consistent with the Monte-Carlo based designs in the statistical sense.

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REFERENCES

[1] Robert E. Skelton, Tetsuya Iwasaki, and Karolos M. Grigoriadis. *A Unified Algebraic Approach to Linear Control Design*. CRC Pr I Llc, 1997.

[2] B. R. Barmish, C. M. Lagoa, and P. S. Shcherbakov. Probabilistic Enhancement of Robustness Margins Provided by Linear Matrix Inequalities. *Proceedings of the Allerton Conference on Communication, Control and Computing, Monticello*, 1996.

[3] B. R. Barmish. A Probabilistic Robustness Result for a Multilinearly Parameterized H_∞ Norm. *American Control Conference*, 5:3309–3310, June 2000.

[4] R. F. Stengel and L.E. Ryan. Stochastic Robustness of Linear Time-Invariant Control Systems. *IEEE Transactions on Automatic Control*, 36:82–87, Jan 1991.

[5] B. T. Polyak and R. Tempo. Probabilistic Robust Design with Linear Quadratic Regulators. *Proceedings of the 39th IEEE Conference on Decision and Control*, 2:1037–1042, Dec 2000.

[6] Yasumasa Fujisaki, Fabrizio Dabbene, and Roberto Tempo. Probabilistic Design of LPV Control Systems. *Automatica*, 39:1323–1337, Dec 2003.

[7] N. Wiener. The Homogeneous Chaos. *American Journal of Mathematics*, 60(4):897–936, 1938.

[8] R. H. Cameron and W. T. Martin. The Orthogonal Development of Non-Linear Functionals in Series of Fourier-Hermite Functionals. *The Annals of Mathematics*, 48(2):385–392, 1947.

[9] Dongbin Xiu and George Em Karniadakis. The Wiener–Askey Polynomial Chaos for Stochastic Differential Equations. *SIAM J. Sci. Comput.*, 24(2):619–644, 2002.

[10] R. Askey and J. Wilson. Some Basic Hypergeometric Polynomials that Generalize Jacobi Polynomials. *Memoirs Amer. Math. Soc.*, 319, 1985.

[11] Thomas Y. Hou, Wuan Luo, Boris Rozovskii, and Hao-Min Zhou. Wiener Chaos Expansions and Numerical Solutions of Randomly Forced Equations of Fluid Mechanics. *J. Comput. Phys.*, 216(2):687–706, 2006.

[12] Dongbin Xiu and George Em Karniadakis. Modeling Uncertainty in Flow Simulations via Generalized Polynomial Chaos. *J. Comput. Phys.*, 187(1):137–167, 2003.

[13] Xiaoliang Wan, Dongbin Xiu, and George Em Karniadakis. Stochastic solutions for the two-dimensional advection-diffusion equation. *SIAM J. Sci. Comput.*, 26(2):578–590, 2005.

[14] Roger G. Ghanem and Pol D. Spanos. *Stochastic Finite Elements: A Spectral Approach*. Springer-Verlag New York, Inc., New York, NY, USA, 1991.

[15] Roger Ghanem and John Red-Horse. Propagation of Probabilistic Uncertainty in Complex Physical Systems Using a Stochastic Finite Element Approach. *Phys. D*, 133(1-4):137–144, 1999.

[16] R.G. Ghanem. Ingredients for a General Purpose Stochastic Finite Elements Implementation. *Comput. Methods Appl. Mech. Eng.*, 168(1-4):19–34, 1999.

[17] Franz S. Hover and Michael S. Triantafyllou. Application of Polynomial Chaos in Stability and Control. *Automatica*, 42(5):789–795, 2006.

[18] Michal Kocvara and Michael Stingl. PENBMI. PENOPT GbR,Hauptstrasse 31A, 91338 Igensdorf OT Stckach, Germany.

[19] Bei Lu and Fen Wu. Probabilistic robust control design for an f-16 aircraft. In *Proceedings of the AIAA Guidance, Navigation, and Control Conference and Exhibit*, San Francisco, CA, August 2005. AIAA.

[20] R. Li and R. Ghanem. Adaptive Polynomial Chaos Expansions Applied to Statistics of Extremes in Nonlinear Random Vibration. *Probabilistic Engineering Mechanics*, 13:125–136, April 1998.

[21] Xiaoliang Wan and George Em Karniadakis. An Adaptive Multi-Element Generalized Polynomial Chaos Method for Stochastic Differential Equations. *J. Comput. Phys.*, 209(2):617–642, 2005.