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Integrated actuator placement and fault tolerant controller design for a class of distributed parameter systems

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Abstract— **In the current work we present an integrated actuator placement and fault tolerant controller design based on the concept of spatial H_2 norm and combined with a fault tolerant controller design concepts for DPS presented in earlier work by the authors. Specifically, a nonlinear optimization problem is formulated to identify and rank spatially independent actuator groups that may control a distributed process with minimal actuator power. The optimization problem is based on the concept of the spatial H_2 norm and guarantees a minimum amount of controllability for each group. The augmented fault monitoring scheme employs a time varying threshold in order to minimize detection time, and the fault accommodating scheme simply switches off the faulty actuator and activates a healthy group without reconfiguring the control signal, as the latter by design is the same for all possible actuator groups. The simplicity of the proposed methodology lies in the unique property of transport reaction processes that properly chosen actuators at physically different locations have an identical effect on the system long term behavior.**

I. Introduction

The issue of control and optimization of transport-reaction processes has received considerable attention in the last two decades [12], [5]. One research direction has focused on the design of nonlinear controllers and robust controllers that are specifically tailored to circumvent the computational requirements associated to the infinite dimensional nature of the mathematical description based on the property of parabolic PDEs that the eigenspectrum of the spatial operator can be partitioned into a finite size set of eigenvalues that are close to the imaginary axis and an infinite size set of eigenvalues that are far in the left half plane, implying that the dominant behavior of the system can be accurately captured by a finite number of eigenmodes [33]. Controllers are subsequently designed based on the reduced-order models [2], [3], [4], [11], [12], [17]. Another approach for controller design is based on the notions of passivity [38] and on the definitions of spatial $\mathcal{H}_2$ norms [21].

An important issue for the controller design methodology for distributed processes, is the actuator placement such that the system exhibits desired system theoretic properties such as enhanced controllability. A complexity in this endeavor lies in the spatial dependence of the concepts of controllability and observability which prompted a number of researchers to address the important topic of actuator and sensor placement, [1], [6]. The conventional approach to actuator placement is to select the locations based on open-loop considerations to ensure that the necessary controllability, reachability or power factor requirements are satisfied [36], [28], [31], [10]. For further work on different aspects of actuator and sensor placement based on controllability/observability, the reader is directed to the survey papers [35] and [23].

At the same time, actuator, sensor and component failures have too often plagued chemical processes, often leading to deteriorating product quality and potentially dangerous process operation, such as runaway conditions. Motivated by the importance of the aforementioned failures, the issue of fault tolerant and fault accommodating controller design has been an active research topic in the chemical engineering community for open-loop stable and open-loop unstable processes [7], [8], [25], [26], [37]. However, while there has been extensive research from the control community on fault detection and diagnosis of finite dimensional systems using model-based robust and adaptive control techniques, see for example the books [30], [32] and references therein, little work can be found for similar treatment of infinite dimensional systems.

To address the issue of fault tolerance within the prism of an integrated design of actuator selection, we embark on a completely new direction, whereby the actuator optimization metric is not defined in terms of enhanced controllability, improved performance or enhanced robustness with respect to disturbances or unmodeled dynamics, *but* with respect to fault tolerability. By taking advantage of the spatial variability that transport-reaction processes naturally enjoy, we endeavor on an entirely new concept of optimizing actuator locations, being physically apart, and yet requiring the same feedback gain! This artifact is only applicable to distributed parameter systems wherein two or more different locations within the spatial domain of definition can provide the same level of controllability and at the same time the same feedback gain can be applicable to all such locations. We capitalize on this property of spatially distributed parameter systems to find locations for different groups of actuators, each of which resulting in a similar controllability level, and when a given actuator group fails, then simply deactivating the faulty actuator group and activating another actuator group constitutes the fault accommodation policy. The control signal in this case is not changed, a situation that cannot be implemented in lumped parameter systems.

Therefore the major conceptual contribution of this work is to search for a set of "optimal" actuator locations that have the same level of controllability and design a single feedback controller for the archetypal actuator group. Continuing,

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consider the situation where the currently activated actuator group fails and built a on-line supervisor that monitors for possible faults by comparing the plant's performance with that of the nominal performance. The instance that a deviation is noticed, then a fault is declared, and hence fault detection. The fault accommodation in this special case does not require a control reconfiguration, *but simply* the deactivation of the current actuator group and the activation of another healthy actuator group. Such a simple fault accommodation significantly reduces the costs associated with process supervision with the obvious economic and performance savings.

The mathematical formulation of the class of PDEs under consideration is presented in Section II where some earlier concepts of spatial controllability are also presented. The proposed actuator placement scheme is presented in Section III which introduces the added feature of spillover consideration. The fault detection scheme is presented in Section IV where a time varying threshold in incorporated in the monitoring scheme in order to minimize the fault detection time. Utilizing the earlier results on actuator positioning, the proposed overly simplified fault accommodation scheme is also given in Section IV. Numerical results depicting the novel approach of the proposed fault tolerant scheme with simplified accommodation are given in Section V.

II. MATHEMATICAL FORMULATION

We consider the problem of computationally identifying the optimal locations of actuators for processes that can be mathematically described by parabolic partial differential equation (PDE) systems of the form:

$$\frac{\partial}{\partial t}x(t,\xi) = \mathcal{A}(\xi)x(t,\xi) + b(\xi;\xi_a)u(t), \tag{1}$$

where $x(\cdot,\xi) \in \mathbb{R}$ is the state, $t \in \mathbb{R}^+$ is the time, $\xi \in \Omega$ is the spatial coordinate and Ω is a bounded smooth domain in $\mathbb{R}^n$ ($n = 1, 2$ or 3), and $u(t) \in \mathbb{R}^K$ is the manipulated variable vector. $\mathcal{A}$ is a second order (strongly) elliptic operator [14] of the form:

$$\mathcal{A}\phi = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial}{\partial \xi_j}(\alpha_{jk}(\xi)\frac{\partial \phi}{\partial \xi_k}) + \sum_{j=1}^{n}\alpha_j(\xi)\frac{\partial \phi}{\partial \xi_j} + \alpha_0(\xi)\phi,$$

for $\xi \in \Omega$, and $b(\xi;\xi_a) = [b_1(\xi;\xi_{a,1}) \ \dots \ b_K(\xi;\xi_{a,K})]$, where $b_j(\xi;\xi_{a,j})$ denotes the spatial distribution of the j^{th} actuating device (e.g., boundary, distributed and/or pointwise) placed at location $\xi_{a,j}$, the j^{th} component of the vector $\xi_a = \{\xi_{a,1}, \xi_{a,2}, \dots, \xi_{a,K}\} \in \Omega_\alpha$; $\Omega_\alpha \subseteq \prod^K \Omega$ denotes the domain of permissible actuator locations. All three cases of boundary conditions for the above PDE system may be considered: mixed (Robin), Neumann or Dirichlet [15], where the boundary, denoted by $\partial\Omega$, can be decomposed to $\partial\Omega = \Gamma_a \cup \Gamma_b$ with Γ_a denoting the part of the boundary where the actuator(s) *may* be placed and Γ_b $(= \partial\Omega \setminus \Gamma_a)$ the remainder of the boundary where actuators are not desired or allowed to be placed. Defining an appropriate Hilbert space $\mathcal{H} = L_2(\Omega)$ with inner product

$$\langle \psi_1, \psi_2 \rangle = \int_\Omega \psi_1^*(\xi)\psi_2(\xi)\,d\xi, \tag{2}$$

and norm $\|\psi\|_L = \sqrt{\langle \psi_1, \psi_2 \rangle}$, the PDE system can be equivalently written in the following abstract form [14]:

$$\dot{x}(t) = \mathcal{A}x(t) + Bu(t), \tag{3}$$

where $x(t,\cdot) = x(t) \in \mathcal{H}$ is the state and B denotes the input operator either on the part of the boundary Γ_a where actuation is desired or permissible, or on the permissible part of the interior of the domain Ω_α.

The PDE of (1) can be solved independently for each mode by using the orthogonality properties of the eigenfunctions of the spatial operator $\mathcal{A}$ [13]. In this case

$$\int_\Omega \phi_j^*(\xi)\mathcal{A}\phi_i(\xi)\,d\xi = \lambda_i\delta_{ji}, \ \int_\Omega \phi_j^*(\xi)\phi_i(\xi)\,d\xi = \delta_{ji}, \tag{4}$$

where λ_i denotes the i^{th} eigenvalue, and δ_{ji} denotes the Kronecker delta. We assume that the eigenvalues are ordered such that $\lambda_{i+1} \le \lambda_i, \forall i = 1, \dots, \infty$. Using the computed eigenfunctions as a basis function set for $\mathcal{H}$, $x(t,\xi)$ can be equivalently expressed via the expansion $x(t,\xi) = \sum_{i=1}^{\infty}\phi_i(\xi)x_i(t)$. Employing Laplace transforms [16] with $\mathcal{L}[x_i(t)] = X_i(s)$, the system is represented in the s domain

$$X_i(s;\xi_a) = \frac{1}{(s-\lambda_i)}\int_\Omega \phi_i^*(\xi)b(\xi;\xi_a)\,d\xi\,U(s) \tag{5}$$

with $X(s,\xi;\xi_a) = \sum_{i=1}^{\infty}\phi_i(\xi)X_i(s;\xi_a)$. We define the K-dimensional row vector

$$B_i(\xi_a) \ \triangleq \ \int_\Omega \phi_i^*(\xi)b(\xi;\xi_a)\,d\xi$$

and the transfer function matrix of the i^{th} eigenmode as

$$G_i(s;\xi_a) \triangleq \frac{1}{(s-\lambda_i)}B_i(\xi_a). \tag{6}$$

The Laplace transform of the spatial distributed state can then be represented as the infinite sum

$$\begin{aligned} X(s,\xi;\xi_a) &= \textstyle\sum_{i=1}^{\infty}\phi_i(\xi)\,X_i(s;\xi_a) \\ &= \textstyle\sum_{i=1}^{\infty}\phi_i(\xi)\,G_i(s,\xi_a)\,U(s) = \sum_{i=1}^{\infty}\phi_i(\xi)\frac{1}{(s-\lambda_i)}B_i(\xi_a)\,U(s). \end{aligned} \tag{7}$$

The resulting K-input/distributed(infinite)-output transfer function is then given by

$$G(s,\xi;\xi_a) = \sum_{i=1}^{\infty}\phi_i(\xi)\frac{1}{(s-\lambda_i)}B_i(\xi_a) = \sum_{i=1}^{\infty}\phi_i(\xi)G_i(s;\xi_a). \tag{8}$$

A property of strongly elliptic operators is that their eigenspectrum can be decomposed into a finite number of eigenvalues that are close to the imaginary axis and an infinite one that lies far in the left half complex plane, which implies that the long term dynamics of the process can be accurately captured by only a finite number of eigenmodes, while the infinite complement eigenmodes relax to their steady-state values fast. This property will be employed in the next subsection to formulate a computationally tractable optimization problem.

III. ACTUATOR PLACEMENT

We now pose the computation of the optimal actuator locations problem as an optimization one, which is constrained in a set of admissible actuator locations and which ensures certain open-loop objectives are satisfied. We first introduce the mathematical background leading to the definition of the objective function and constraint functions.

A. Spatial norms and controllability measures

The spatial $\mathcal{H}_2$ norm [27] of the transfer function $G(s,\xi;\xi_a)$ in (8) is defined as

$$\|G\|^2_{\mathcal{H}_2} \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\Omega} tr\{G^*(j\omega,\xi;\xi_a)G(j\omega,\xi;\xi_a)\}\, d\xi\, d\omega.$$

Using the orthogonality property of the eigenfunctions, the above norm simplifies to

$$\begin{aligned}\|G\|^2_{\mathcal{H}_2} &= \textstyle\sum_{i=1}^{\infty} \|G_i(s,\xi_a)\|_2^2 \\ &= \textstyle\sum_{i=1}^{\infty}\int_{-\infty}^{\infty} tr\{G_i^*(j\omega;\xi_a)G_i(j\omega;\xi_a)\}\, d\xi.\end{aligned} \tag{9}$$

With the aid of this spatial $\mathcal{H}_2$ norm, a number of measures of actuator placement effectiveness can be proposed. We define

$$f_i(\xi_a) \triangleq \|G_i(s;\xi_a)\|_2 = \left\|\frac{1}{(s-\lambda_i)}B_i(\xi_a)\right\|_2. \tag{10}$$

Capitalizing on the previously mentioned property of strong elliptic operators that the higher modes become progressively more stable ($\lambda_{i+1} \leq \lambda_i$ and $\lambda_i \to \infty$ as $i \to \infty$), we only need consider a finite number of modes to compute an accurate approximation of the spatial $\mathcal{H}_2$ norm:

$$\begin{aligned}\|G(s,\xi;\xi_a)\|^2_{\mathcal{H}_2} &= \textstyle\sum_{i=1}^{\infty}\|G_i(s;\xi_a)\|_2^2 = \sum_{i=1}^{\infty} f_i^2(\xi_a) \\ &= \textstyle\sum_{i=1}^{N} f_i^2(\xi_a) + \sum_{i=N+1}^{\infty} f_i^2(\xi_a) \\ &= \mathbf{H}^2(\xi_a) + \mathbf{S}^2(\xi_a),\end{aligned} \tag{11}$$

where $\mathbf{H}^2(\xi_a) \triangleq \sum_{i=1}^{N}\|G_i(s;\xi_a)\|_2^2 = \sum_{i=1}^{N} f_i^2(\xi_a)$ denotes the truncation of the $\mathcal{H}_2$ spatial norm to the first N modes and $\mathbf{S}^2(\xi_a) \triangleq \sum_{i=N+1}^{\infty}\|G_i(s;\xi_a)\|_2^2 = \sum_{i=N+1}^{\infty} f_i^2(\xi_a)$ denotes the "spillover" of the actuator action to the higher modes. We further assume that this spillover effect on a finite number of higher modes (medium range modes), $i = N+1,\dots,M$ need only be considered, i.e. $\mathbf{S}^2(\xi_a) \simeq \sum_{i=N+1}^{M} f_i^2(\xi_a)$.

Note that $\mathbf{H}^2(\xi_a)$ *at location* ξ_a *is a measure of controller authority placed at location* ξ_a *over the entire spatial domain in an average sense (averaged over the first N modes).*

Following [20], we may now define the i^{th} *modal controllability* at actuator locations ξ_a as follows.

Definition 1: [i^{th} modal controllability] The i^{th} modal controllability at actuator locations $\xi_a = \{\xi_{a,1}, \xi_{a,2}, \dots, \xi_{a,K}\}$ is defined as

$$\mathcal{M}_i(\xi_a) = \frac{f_i(\xi_a)}{\max_{\zeta\in\Omega_\alpha} f_i(\zeta)} \times 100\%, \quad i = 1,2,\dots,N, \tag{12}$$

and describes the total controller authority of all actuators at locations ξ_a over the i^{th} mode.

If the modal controllability at specific locations ξ_a of a given mode is zero, it means that none of the controllers has any authority over that mode. This also coincides with the notion of approximate controllability for the class of PDEs with Riesz-spectral operators [14]. The requirement for that in this case is $\langle b, \phi_i\rangle \neq 0$, $\forall i$. When the i^{th} mode has a zero modal controllability at location ξ_a, it means that $f_i(\xi_a) \equiv 0$ and hence $B_i(\xi_a) = 0$, or that

$$\int_{\Omega} \phi_i(\xi)^* b(\xi;\xi_a)\, d\xi = 0.$$

Furthermore, we also consider the *cumulative spatial controllability* [27].

Even though the above definitions allow us to formulate the optimal actuator placement problem, in order to gain better control over the placement of each individual actuator, we also define the i^{th} *modal controllability* of the j^{th} actuator at location $\xi_{a,j}$.

Definition 2: [i^{th} modal controllability of the j^{th} actuator at location $\xi_{a,j}$] The i^{th} modal controllability of the j^{th} actuator at location $\xi_{a,j}$ is defined as

$$\begin{aligned}\mathcal{M}_{i,j}(\xi_a) &= \frac{f_{i,j}(\xi_a)}{\max_{\zeta\in\Omega_\alpha} f_{i,j}(\zeta)} \times 100\%, \quad i = 1,\dots,N, \\ f_{i,j}(\xi_a) &\triangleq \left\|\tfrac{1}{(s-\lambda_i)}\int_{\Omega}\phi_i^*(\xi) b_j(\xi;\xi_{a,j})\, d\xi\right\|_2.\end{aligned} \tag{13}$$

The i^{th} modal controllability of the j^{th} actuator at location $\xi_{a,j}$ describes the controller authority of the j^{th} actuator over the i^{th} mode. If the modal controllability at a specific actuator location of a given mode is zero, it means that the *specific controller has no authority over that mode.* Note that from the definition of $f_{i,j}(\xi_a)$ and $f_i(\xi_a)$ we have

$$f_i^2(\xi_a) = \sum_{j=1}^{K} f_{i,j}^2(\xi_a).$$

The component spatial controllability may be defined similarly as the j^{th} actuator authority over the entire spatial domain in an average sense.

Remark 3.1: An issue which often arises during the search of optimal actuator locations using modal methods is the weight in the objective function that should be assigned to each mode. In the current formulation the weight that is assigned to each mode is dependent upon the eigenvalue of the specific eigenmode. As a result, modes that are less stable have a greater contribution to $\mathbf{H}^2(\xi_a)$ and as a consequence the actuator placement search will gravitate towards locations that assign greater force on the specific modes.

B. Spillover effects

While the notions of spatial and modal controllability allow one to choose actuator locations that cater to specific (primarily dominant low) modes, care must be exercised in order to avoid choosing locations that might excite medium range modes, that lead to performance deterioration; medium range modes are modes that are at immediate proximity of the first N modes. The effect of the actuators on the higher modes is mathematically described by the term $\mathbf{S}(\xi_a)$ in (11), which, in general, cannot be computed since it is an infinite sum of modal controllabilities. Capitalizing again on the property of strongly elliptic spatial operators, the spatial controllability for these medium range modes can be similarly defined as:

Definition 3: [Cumulative spillover spatial controllability] The cumulative spillover effect of K actuators placed at locations ξ_a to higher modes is defined as

$$\mathcal{S}_M^N(\xi_a) \triangleq \frac{\mathbf{S}(\xi_a)}{\max_{\zeta\in\Omega_\alpha}\mathbf{S}(\zeta)} \approx \frac{\sqrt{\sum_{i=N+1}^{M} f_i^2(\xi_a)}}{\max_{\zeta\in\Omega_\alpha}\sqrt{\sum_{i=N+1}^{M} f_i^2(\zeta)}} \times 100\%. \tag{14}$$

The component spillover effect of the j^{th} actuator can be similarly defined.

C. Multiple location issues

A problem commonly encountered during the placement of multiple actuators is the issue of coupling effects between actuators. An excellent exposure of the optimization and computational issues associated with multiple actuators, including that of actuator clusterization, can be found in the book by Uciński [34]. While the actuator placement problem can be formulated as an optimization one for a single actuator using the definitions of the previous subsection, multiple actuators pose additional challenges. One such issue is placing two actuators *at* or *near* the same location. Additionally, one must avoid placing actuators at locations that provide certain *spatial symmetry* (i.e., they affect the system modes in a similar fashion) while placed at *different* locations. This issue was exposed and addressed in detail in [6]. For completeness we present the mathematical foundation; the interested reader may refer to [6] for further details.

To address this issue, with the aid of (13), we define the *spatial controllability matrix* $\mathcal{M} \in \mathbb{R}^{N\times K}$ as:

$$\mathcal{M}(\xi_a) = \begin{bmatrix} \mathcal{M}_{1,1}(\xi_a) & \mathcal{M}_{1,2}(\xi_a) & \dots & \mathcal{M}_{1,K}(\xi_a) \\ \mathcal{M}_{2,1}(\xi_a) & \mathcal{M}_{2,1}(\xi_a) & \dots & \mathcal{M}_{2,K}(\xi_a) \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{M}_{N,1}(\xi_a) & \mathcal{M}_{N,2}(\xi_a) & \dots & \mathcal{M}_{N,K}(\xi_a) \end{bmatrix}. \quad (15)$$

The proximity, in an L_2 sense, of the actuator locations can be identified through the singular values of $\mathcal{M}$, which are equal to the number of locations K (assuming that the number of modes that are of interest are higher than the number of available actuators $N \geq K$). Specifically, if the columns of $\mathcal{M}$ are linearly dependent then at least one singular value attains the value of zero. The proximity of the columns of $\mathcal{M}$ to linear dependence can be identified through the ratio of the largest singular value to the smallest one, which represents the condition number of $\mathcal{M}$, [19]. This implies that there is a direct relationship between the value of the condition number and the redundancy in the actuator network.

An in depth presentation of the physical interpretation of the specific measure as well as the Selection of bound c on the condition number that represents the desired minimum distance between actuators, $\delta\xi$, in *both a physical and an L_2 sense*, can be found in [6].

D. Optimization problem formulation

The first objective in the proposed integrated actuator placement/fault tolerant controller design methodology is to identify L actuator groups that comprise of K actuators, ranked in terms of their spatial controllability performance and spillover controllability detrimental effect. Specifically, each group should consist of K actuator locations, denoted in vector form by $\xi_a \in \Omega_\alpha$, that maximize the cumulative spatial controllability and minimize the cumulative spillover controllability of the system while at the same time maintain a reasonable level of controller authority over each mode. However, an issue with the use of the concept of spatial and spillover controllabilities is that in order to be computed, one should already know the location of maximum effect $\xi_{a,m}$ in the K-dimensional subspace Ω_α, the objective of the optimization problem. Furthermore, a large spillover spatial controllability value may not signify an undesirable location, as the "worst possible" location may still have an insignificant effect on the intermediate system modes. To circumvent these issues, in the optimization problem, the objective function is computed based directly on the spatial norm.

The constrained optimization problem for actuator group l is then formulated as:

$$\begin{aligned} &\xi_l^{opt} = \arg\max_{\xi_a\in\Omega_\alpha} \left\{\mathbf{H}^2(\xi_a) - \omega_s \mathbf{S}^2(\xi_a)\right\} \\ &\qquad\qquad \text{s.t.} \\ &cond(\mathcal{M}(\xi_a)) \leq c, \\ &\max_{j=1,\dots,K} \mathcal{M}_{i,j}(\xi_a) \geq \beta_i, \ \forall i = 1,2,\dots,N, \\ &|\xi_k^{opt} \otimes I_K - I_K \otimes \xi_a| \geq \delta\xi I_K \otimes I_K, \ k = 1,\dots,l-1. \end{aligned} \quad (16)$$

where with $\otimes$ we denote the Kronecker product, and with I_K we denote the unit matrix of dimensions $K\times 1$ (a matrix consisting of all 1s) and with $|\cdot|$ the element by element absolute value. The use of component modal controllabilities over total modal controllabilities in the optimization problem is preferred, to account for possible solutions where one actuator is redundant (i.e., with low modal controllability values for all the modes) in which case the optimization problem is infeasible. We address the issue of actuator spillover that inadvertently affects the closed-loop process response through the inclusion of term $\mathbf{S}^2(\xi_a)$. The relative importance of minimizing spillover effects compared to obtaining locations that maximize the spatial controllability can be further tuned through the selection of the weight ω_s. The last constraint in the optimization problem of (16) imposes a minimum physical distance of $\delta\xi$ between all actuators of *all* l actuator groups.

The formulated optimization problem of (16) can be solved using standard search algorithms such as Newton-based, interior-point or direct-search methods [9]. Due to the nonlinear nature of the objective function and the inequality constraints, global optimization methods are preferable, including αBB [18], particle swarm optimization [22] and simulated annealing. Another issue is the "rugged" topology of the objective function; algorithms that efficiently handle such objective functions have been developed, including funneling [24], derivative free and direct search [9] optimization algorithms. In the event the set Ω_a is disjoint, branch and bound algorithms [9] and genetic algorithms [29] can also be used to obtain the optimal actuator locations. In the current work a 2-level optimization scheme was employed, comprising of an inner sequential quadratic programming algorithm (guaranteeing local optimality), and a simulated annealing outer structure, to identify the optimal locations.

The proposed algorithm to identify L actuator groups consisting of K actuators each, and rank them according to the group spatial authority over the slow modes of the system of (3) can now be presented in pseudocode form:

1) Define number of actuator groups L, actuators per group K.

2) Define the dimensionality of the slow and intermediate subsystems, N, M, the minimum distance between the actuators in an L_2 sense, c, and physical sense, $\delta\xi$, level of model controllabilities β_i.
3) Set $l = 1$.
4) Solve the optimization problem of (16).
5) If $l < L$, then $l = l+1$, repeat step 3.

Remark 3.2: To prevent formulating an optimization problem that is infeasible due to constraints on the L_2 proximity of the actuators we may impose soft constraints on the minimum distance between the actuators, avoiding clustering effects, through the augmentation of the term $-\omega_c\, cond(\mathcal{M}(\xi_a))$ in the objective function, resulting in $\max_{\xi_a\in\Omega_\alpha} \{\mathbf{H}^2(\xi_a) - \omega_s \mathbf{S}^2(\xi_a) - \omega_c\, cond(\mathcal{M}(\xi_a))\}$. The relative importance of clustering to maximizing the spatial controllability is controlled through the appropriate choice of values for ω_c. Note that due to the wide range of values that the condition number may have and the fact that it is always larger than one, it is preferable to use the logarithm of the condition number. Alternatively, the inverse of the condition number may be used in a minor modification of the formulated problem.

Remark 3.3: Even though the problem formulation necessitates the computationally intensive step of calculating the singular values of $\mathcal{M}(\xi_a)$ at each iteration, for the usual size of the problem under consideration, the computational requirements of the used search and the numerical algebra algorithms are well within the capabilities of current processors.

IV. Robust Detection using time-varying thresholds on residual signals

The process under consideration is written as follows

$$\dot{x}_s(t) = A_s x_s(t) + \mathbf{B}_s \Delta_s(t) u(t) \tag{17}$$

where $\mathbf{B}_s = \left[\begin{array}{cc} \alpha(t)B_{s1} & (1-\alpha(t))B_{s2} \end{array}\right]$,

$$\Delta_s(t) = \left[\begin{array}{cc} \beta(t)\Delta_{s1} & (1-\beta(t))\Delta_{s2} \end{array}\right]^T,$$

where $\alpha(t) \in \{0,1\}$ is the effectiveness factor for the current actuator in use and $\beta(t)$ is the supervisor-chosen actuator activation function. The former depends on the health of the actuator group and the latter is chosen by the supervisor. In fact, the effectiveness factor $\alpha(t)$ coincides with the fault time profile; when the actuator fault is abrupt, then $\alpha(t) = 1 - H(t-T_a)$. The matrix B_{s1} corresponds to the projection onto the slow subsystem of the input operator associated with the first actuator group and similarly B_{s2} corresponds to the input operator associated with the second actuator group.

Under healthy operating conditions, the system is assumed to employ the first actuator group and hence $\alpha(t) = 1$. The supervisor assumes also that the system starts as healthy and hence sets $\beta(t) = 1$. It should be noted that the product $B_{s1}\Delta_1 = B_{s2}\Delta_2 = B_{s0}$ by design. Then the control signal $u(t)$ is chosen only once and is based on the matrix B_{s0}. If the first actuator group is used, then the slow subsystem becomes

$$\dot{x}_s(t) = A_s x_s(t) + B_{s1}\Delta_{s1} u(t) = A_s x_s(t) + B_{s0} u(t),$$

and when the second actuator group is used, the plant equation is given by

$$\dot{x}_s(t) = A_s x_s(t) + B_{s2}\Delta_{s2} u(t) = A_s x_s(t) + B_{s0} u(t).$$

The above demonstrates that regardless of the actuator group used, the closed loop system behaves the same.

The plant equation can be compactly written as

$$\begin{aligned} &\dot{x}_s(t) = A_s x_s(t) + \\ &(B_{s1}\beta(t)\Delta_{s1}\alpha(t) + B_{s2}((1-\beta(t))\Delta_{s2}(1-\alpha(t)))\, u(t), \end{aligned} \tag{18}$$

where $\alpha(t)$is either one or zero at any time. Since the actuator effectiveness factor $\alpha(t) \in \{0,1\}$, then at any time the term $\mathbf{B}_s\Delta_s(t)$ must be either $B_{s1}\Delta_{s1}$ or $B_{s2}\Delta_{s2}$. For simplicity, it is assumed that the healthy actuator group is the first one and hence $\mathbf{B}_s\Delta_s(t) = B_{s1}\Delta_{s1}$ prior to the fault occurrence T_a. Hence $\alpha(t) = 1$ for $t < T_a$. When the actuator fault occurs, the value of $\alpha(t)$ becomes $\alpha(t) = 0$ for $t \geq T_a$. The supervisor should have knowledge of this and reconfigure the controller by switching to the other actuator group. This can be done by setting $\beta(t)$ to zero. *However*, the supervisor will not immediately detect the fault in the system, and must find a way to declare the fault immediately after the fault occurrence T_a. While $\alpha(t) = 0$ for $t \geq T_a$, the supervisor will still assume nominal conditions and maintain $\beta(t) = 1$. After the onset of the fault and prior to a possible detection, the system will be given by

$$\dot{x}_s = A_s x_s + (B_{s1} 1 \Delta_{s1} 0 + B_{s2}(1-1)\Delta_2(1-0))\, u = A_s x_s$$

A. Fault Detection

In order to monitor the system and estimate the time at which this fault occurs, a fault detection observer is required. This takes the form

$$\dot{\hat{x}}_s = A_s \hat{x}_s(t) + B_{s1}\Delta_{s1} u(t) + L_s(x_s(t) - \hat{x}_s(t)). \tag{19}$$

Prior to the fault occurrence (i.e. for $t < T_a$), we have that the detection error $e(t) = x_s(t) - \hat{x}_s(t)$ is governed by

$$\dot{e}_s(t) = (A_s - L_s) e_s(t), \qquad e_s(0) = e_{s0}. \tag{20}$$

After the occurrence of the actuator fault, the plant will have no operational actuator while the monitoring scheme would (erroneously) assume that the first actuator group is functional. Thus, the detection error will be governed by

$$\dot{e}_s(t) = (A-L) e_s(t) + (0 - B_{s1}\Delta_{s1}) u(t), \quad t \geq T_a. \tag{21}$$

One would like to utilize the residual signal to declare a fault in the system, and make the reconfiguration by switching to the other actuator group. The residual signal is chosen as the norm of the detection error and thus

$$\begin{aligned} r(t) &= |e_s(t)| \\ &= |e^{(A_s-L_s)t} e(0) - \textstyle\int_0^t e^{(A_s-L_s)(t-\tau)} B_{s1}\Delta_{s1} u(\tau)\, d\tau| \\ &\leq |e^{(A_s-L_s)t} e(0)| + |\textstyle\int_0^t e^{(A_s-L_s)(t-\tau)} B_{s1}\Delta_{s1} u(\tau)\, d\tau| \\ &\leq e^{-\lambda_s t}\|e_{s0}\| + \|B_{s1}\Delta_{s1}\| \textstyle\int_0^t e^{-\lambda_s(t-\tau)} |u(\tau)|\, d\tau \\ &\leq e^{-\lambda_s t}\|e_{s0}\| + \|B_{s0}\| \textstyle\int_0^t e^{-\lambda_s(t-\tau)} |u(\tau)|\, d\tau \end{aligned}$$

By setting $r_0(t) \triangleq e^{-\lambda_s t}\|e_{s0}\|$, then we have

$$r(t) \leq r_0(t) + \|B_{s0}\| \int_0^t e^{-\lambda_s(t-\tau)} |u(\tau)|\, d\tau.$$

Prior to the actuator fault, the residual signal should satisfy $r(t) \leq r_0(t)$, and therefore, a fault is declared when the residual signal exceeds the time varying threshold $r_0(t)$. The detection time is defined as the first instance at which the residual signal exceeds the time varying threshold and is given via $t_d = \inf\{t > 0 : r(t) \geq r_0(t)\}$. Obviously one has that $t_d \geq T_a$ and the *detection delay* τ_d is the time it takes to declare a fault in the system and is $\tau_d = t_d - T_a$.

B. Fault accommodation

The fault accommodation takes the form of reconfiguration of the actuator group. The control parameter in the accommodation is the effectiveness factor associated with the actuator group that is employed by the supervisor. It is given by

$$\alpha(t) = H(t) - H(t - T_a) = \begin{cases} 1 & \text{if } t < t_d \\ 0 & \text{if } t \geq t_d \end{cases} \qquad (22)$$

Utilizing the above, it results in the slow dynamics being described by

$$\begin{aligned} \dot{x}_s(t) &= Ax_s(t) + B_{s1}\Delta_{s1}u(t), & t &\leq T_a \\ \dot{x}_s(t) &= Ax_s(t), & T_a &< t < t_d \\ \dot{x}_s(t) &= Ax_s(t) + B_{s2}\Delta_2 u(t), & t &\geq t_d \end{aligned}$$

The above shows that the system remains without any actuators in the interval $T_a < t < t_d$, i.e. no control delivered to the system. This is because $\alpha(t) = 0$ due to the fault, but $\beta(t)$ is set by the supervisor still at $\beta(t) = 1$; thus

$$B_{s1}\beta(t)\Delta_2\alpha(t) + B_{s2}(1 - \beta(t))\Delta_2(1 - \alpha(t)) = 0$$

After the fault is declared, the system behaves the same way as before the fault occurrence. This is because $B_{s1}\Delta_{s1} = B_{s2}\Delta_{s2} = B_{s0}$. This is equivalent to having a single actuator group deactivated at $t = T_a$ and reactivated at $t = T_a + \tau_d = t_d$.

V. Numerical Results

We consider the PDE with $\mathcal{A}(\xi)x(t,\xi) = \Delta x(t,\xi)$ on the spatial interval $[0,2]$. For initial conditions, we considered $x(0,\xi) = 10\cos(\frac{5.1\pi\xi}{2}) + \cos(\frac{6.1\pi\xi}{2}) + \cos(\frac{7.1\pi\xi}{2}) + \cos(\frac{8.1\pi\xi}{2})$.

Initially, we focused on identifying optimal actuator locations for this process. Towards this goal we chose the slow subsystem to be of dimension $N = 2$ and the fast subsystem to be of dimension $M = 6$, which implies that the groups would contain up to two actuators, $K = 2$. Based on these considerations, we employed the proposed algorithm to identify six groups of actuators. Specifically, we attempted to identify six actuator groups, $L = 6$, and for the optimization problem of (16) the parameters were set to $\omega_s = 1$, $\delta\xi = 0.024$, $c = 10$, and $\beta_1 = \beta_2 = 60\%$.

In Figure 1a we present the identified actuator locations for the six groups, while Figure 1b and Figure 1c present the objective values and the corresponding **H**, **S** and spatial $\mathcal{H}_2$ norms for the respective groups. We observe that the first and second groups are actually symmetrical to the process center, which is expected due to the symmetry of the modal controllabilities with respect to the process center. This implies that the two groups have exactly the same authority as shown in Figure 1c (the spatial $\mathcal{H}_2$ norms have the same value) due to the symmetry that the modal controllabilities with respect to the process center exhibit.

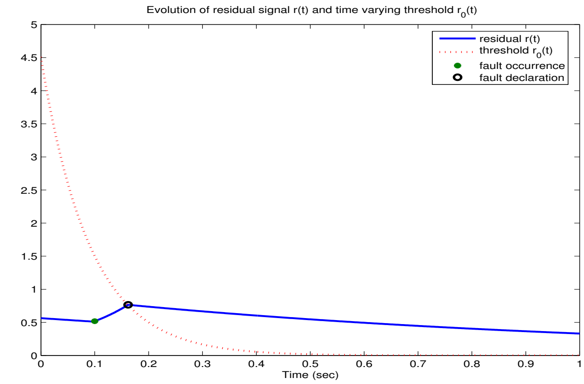

Fig. 2. Evolution of the residual for the slow subsystem and its time-varying threshold.

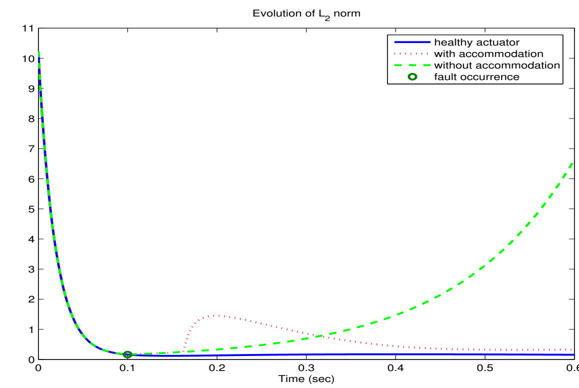

Fig. 3. Evolution of the L_2 state norm.

The effect the spillover, **S**, has on the ranking of the actuator groups can be observed in Figure 1c, where the fourth actuator group has a larger value of **H** than the third group,which implies that it has more authority on the slow system modes,however, it also has a larger value of **S** which leads to its lower ranking. We also observe that even though the algorithm is successful in identifying actuator groups, it may not properly rank the groups as can be observed in the objective value of the fourth and fifth groups in Figure 1b.

Based on the identified actuator groups, we employed the first two groups to design a fault tolerant controller for the process. The detection observer gain was chosen for simplicity as $L_s = A_s + I_s$ resulting in $\lambda_s = 1$ and $\widehat{x}_s(t) = 0$. The bound on $\|e_s(0)\|$ was taken to be $8\|e_s(0)\|$ and therefore the residual signal was $r_0(t) = 8e^{-\lambda_s t}\|e_s(0)\|$. The fault time was set at $T_a = 0.10$ and the proposed monitoring scheme was implemented. Figure 2. The detection time was $t_d = 0.162$ resulting in a detection delay of $\tau_d = 0.062$.

The effects of actuator fault accommodation can be observed in Figure 3, where the L_2 norm for the healthy case (no actuator faults), the case with fault accommodation and the case without accommodation. It can be observed that when fault accommodation is incorporated in the system, even with a certain time delay, the closed loop system eventually recovers and approaches the performance of the healthy closed loop system. Such an advantage is not observed when the actuator fault is not accommodated by the supervisor.

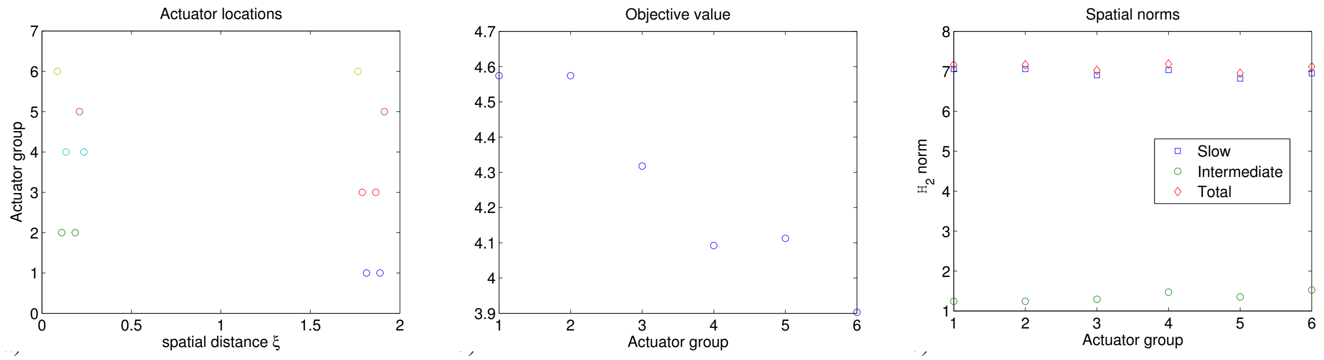

Fig. 1. a) Optimal actuator locations of first six groups, b) objective value of actuator group, c) values of **H**, **S**, and spatial $\mathscr{H}_2$ norm each group.

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