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# On the relationship between Splines, Sampling Zeros and Numerical Integration in Sampled-Data models for Linear Systems

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*Abstract*— **Real systems are usually modelled in continuous-time by differential equations but, in practice, we have to deal with them through digital devices using sampled data. For linear systems, exact sampled-data models can be obtained. These models, however, will generally have more zeros than the continuous-time model. In this paper we show that there is a specific relation between the characterization of these *sampling zeros* and the order of the B-splines used to generate the continuous-time input to the system. Moreover, we show that this connection can also be established if the smoothness of the B-spline is taken into account in the integration strategy used to discretize the continuous-time model.**

## I. Introduction

Real systems are usually modelled using the laws of physics to obtain a differential equation description of the system dynamics. In practice, however, any data collected from a given process and any action taken on its inputs will be applied by a digital device at specific time instants. Thus, the study of sampled-data models has continuously attracted the attention of researchers in control and system identification [1], [2].

Sampled-data models for continuous-time systems can be obtained in different ways and they depend on the assumptions made on the input and output signals. Usually, assumptions are made on their inter-sample behavior (e.g. piecewise constant) or their frequency content (e.g., bandlimited). For linear systems, exact sampled models can be obtained, however, extra zeros may appear in the discrete-time model. These sampling zeros have no continuous-time counterpart and they can be precisely characterized asymptotically, as the sampling period is reduced to 0, in terms of the Euler-Fröbenius polynomials and the continuous-time relative degree [3], [4]. The presence of these sampling zeros has been also characterized for stochastic systems [5] and, in the nonlinear case, *sampling zero dynamics* have been described for approximate sampled-data models [6], [7], [8].

In this paper we present a common framework to interpret the presence of sampling zeros in discrete-time models, in terms of the B-splines used to generate the continuous-time system input, and the integration strategy used to discretize the continuous-time model.

Firstly, we show the relation that exists between sampling zeros of discrete-time models and the use of B-splines for signal reconstruction. B-splines functions have been widely used for signal processing and image reconstruction [9]. This kind of splines were used in [10] to identify input-output models from an estimate of the continuous-time Fourier transform based on non-uniform sampled data. In that work, the polynomials that appear in the $\mathcal{Z}$-transform of a sampled B-spline function were shown to be related to the Euler-Fröbenius polynomials. In this paper, we show that when a B-spline is used in a Generalized Hold [11] to generate the continuous-time input to the system, there is a one-to-one relation between the order of this spline and the order of the polynomial that characterizes the asymptotic sampling zeros.

We then consider the integration strategy underlying in the sampling process of a continuous-time system. In [7], a sampled-data model was proposed for a class of nonlinear systems, by truncating a state-space Taylor series expansion based on the smoothness properties of the zero-order hold input. In this paper we proceed similarly to show that, when B-splines are used to generate the system input, a numerical integration strategy can be applied to discretize the continuous-time model, exploiting the smoothness properties of the spline. In fact, we show that this integration strategy allows one to asymptotically recover the sampling zeros, and can also be applied to non linear models.

The structure of the paper is as follows: Section II presents a review of the sampling process of deterministic linear systems and the presence of the, so called, *sampling zeros*. In Section III we show the connections between the sampling zeros polynomials and the use of B-splines to generate the continuous-time system input. Then, in Section IV, we show how this connection can be understood in terms of the integration strategy used to obtain the sampled-data model. Finally, in Section V we conclude and discuss the results in the paper.

## II. Sampled-data models for linear systems

Consider a linear system described by

$$G(s) = K\frac{(s-c_1)(s-c_2)\dots(s-c_m)}{(s-p_1)(s-p_2)\dots(s-p_n)} \quad ; n > m \qquad (1)$$

To obtain a discrete-time model for this system, we need additional assumptions. We assume that the output is sampled at equidistant time instants, with sampling period $h$, generating the sequence $y_k = y(kh)$. The input to the system, $u(t)$ is generated by a zero-order hold (ZOH), based on the input sequence, $\{u_k\}$, (see Figure 1)

$$u(t) = u_k \qquad t \in [kh, kh+h[ \qquad (2)$$

We are interested in obtaining the discrete-time transfer function $G_q(z)$, that relates the input sequence $\{u_k\}$, with

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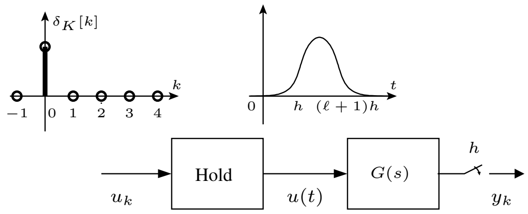


Fig. 1. Sampled continuous-time system

the sampled output sequence $\{y_k\}$, i.e.,

$$G_q(z) = \frac{\mathcal{Z}\{y_k\}}{\mathcal{Z}\{u_k\}} = \frac{\tilde{Y}(z)}{\tilde{U}(z)} \tag{3}$$

There are several ways to obtain $G_q(z)$ (see, [1], [3], [12]). For example, when the hold in Figure 1 is a ZOH, and $u_k$ is a discrete-time step function, we have that $u(t)$ will be a continuous-time step function. Thus, (3) reduces to

$$G_q(z) = \frac{\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{G(s)\frac{1}{s}\right\}\Big|_{t=kh}\right\}}{\frac{z}{z-1}} \tag{4}$$

In general, the sampled model $G_q(z)$ will have relative degree 1. This means that the discrete-time model has more zeros than the continuous-time system. These, so called *sampling zeros*, have no continuous-time counterpart and can be characterized, asymptotically as $h \to 0$, as a function of the continuous-time relative degree, $p = n - m$ [3]. In fact, they correspond to the roots of the Euler-Fröbenius polynomials [4]. These polynomials are defined as

$$B_p(z) = b_1^p z^{p-1} + b_2^p z^{p-2} + \cdots + b_p^p \qquad ; p \geq 1 \tag{5}$$

where

$$b_k^p = \sum_{l=1}^{k} (-1)^{k-l} l^p \binom{p+1}{k-l} \qquad ; k = 1, \ldots, p \tag{6}$$

*Remark 1:* The coefficients in (6) are known to satisfy several properties [4], [3], [8]:

1) The coefficients can be computed recursively

$$b_1^p = b_p^p = 1 \tag{7}$$
$$b_k^p = k b_k^{p-1} + (p-k+1) b_{k-1}^{p-1} \tag{8}$$

2) The roots of these polynomials are always negative real numbers
3) The coefficients satisfy $b_k^p = b_{p+1-k}^p$, i.e., they are symmetrical. Thus

$$B_p(z_0) = 0 \Rightarrow B_p(z_0^{-1}) = 0 \tag{9}$$
$$B_p(z) = z^{p-1} B_p(z^{-1}) \tag{10}$$

4) From (6), we have that

$$B_p(1) = p! \tag{11}$$

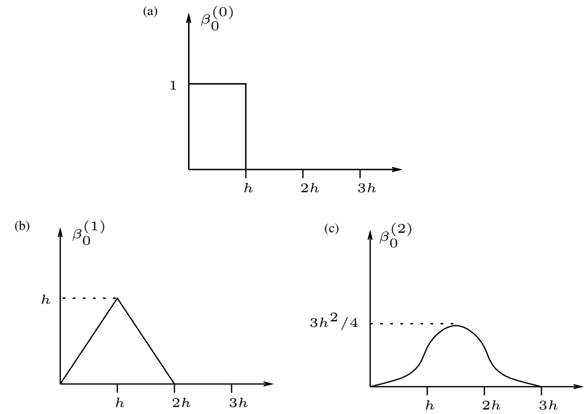


Fig. 2. Most common B-splines a)Zero-order B-spline b)First-order B-spline c)Second-order B-spline

## III. Splines and sampling zeros

In this section we show the relationship between the use of splines and the characterization of sampling zeros. More specifically we show that the order of a B-spline used in a generalized hold increases the order of the Euler-Fröbenius polynomial that appears in the discrete-time model, $G_q(z)$, with respect to the ZOH case.

Firstly, we review some important background on B-splines and splines. A *B-spline* is a spline function that has minimal support with respect to a given degree, smoothness, and domain partition. This means that they can be evaluated, differentiated and integrated easily and in finitely many steps, using the basic operations of addition, substraction and multiplication [13]. Because of these properties, they have been extensively used for signal interpolation and reconstruction [9], [14].

B-splines are defined for a *knot sequence*, i.e., a nondecreasing sequence $\{t_k\} = \{t_1, t_2, \ldots, t_m\}$. For the purpose of this paper, we will consider the knot sequence as the sequence of sampling instants $\{kh\}$, where $h$ is the sampling period. Thus, an $\ell$-order B-spline is defined as

$$\beta^{(\ell)}(t) = \sum_{p=0}^{\ell+1} \frac{(-1)^p}{\ell!} \binom{\ell+1}{p} (t - ph)^\ell \mu(t - ph) \tag{12}$$

where $\mu(t)$ is the Heaviside step function.

The base B-spline, i.e., the zero-order B-spline, is then given by:

$$\beta^{(0)}(t) = \begin{cases} 1 & t \in [0, h[ \\ 0 & t \notin [0, h[ \end{cases} \tag{13}$$

Figure 2 shows B-splines functions for $\ell = 0, 1, 2$.

A *spline* function is a linear combination of B-splines, this is

$$g^{(\ell)}(t) = \sum_{k=-\infty}^{\infty} \alpha_k \beta^{(\ell)}(t - kh) \tag{14}$$

where $\alpha_k$ is the weight of the $k$th B-spline, and $\beta^{(\ell)}(t-kh)$ is the B-spline generated at $t = kh$. It is important to notice that $g^{(\ell)}(t)$ is a polynomial with the same degree, smoothness and domain partitions as $\beta^{(\ell)}(t)$ [13]. We define,

$$\beta_k^{(\ell)}(t) = \beta^{(\ell)}(t - kh) \tag{15}$$

A fundamental property of B-splines is that they can be constructed recursively by using the convolution property [9]:

$$\beta_0^{(\ell)}(t) = \beta_0^{(\ell-1)}(t) * \beta_0^{(0)}(t) \tag{16}$$

*Remark 2:* Note that, from (16), a B-spline of order $\ell$ can be generated by convolving $\beta_0^{(0)}(t)$ with itself $\ell$ times.

B-splines were used for input-output model identification in [10] by estimating the continuous-time Fourier transform from its discrete counterpart. In that work, the polynomials that appear in the $\mathcal{Z}$-transform of a sampled B-spline were shown to be the Euler-Fröbenius polynomials. We present this result in the following lemma.

*Lemma 1:* The $\mathcal{Z}$-transform of the sampled B-spline of order $\ell$, i.e., the sequence $\beta_0^{(\ell)}(kh)$, is given by

$$\mathcal{Z}\left\{\beta_0^{(\ell)}(kh)\right\} = \frac{h^\ell}{\ell!}\frac{B_\ell(z^{-1})}{z} \tag{17}$$

where $B_\ell(\cdot)$ is the Euler-Fröbenius polynomial of order $\ell$, defined in (5)–(6).

*Proof:* The $\mathcal{Z}$-transform of a discrete B-spline is defined as

$$\mathcal{Z}\left\{\beta_0^{(\ell)}(kh)\right\} = \sum_{k=0}^{\infty}\beta_0^{(\ell)}(kh)z^{-k} \tag{18}$$

Manipulating the above equation, and introducing the variable $m = k - p$, we have

$$\mathcal{Z}\left\{\beta_0^{(\ell)}(kh)\right\} = \left(\sum_{m=0}^{\infty} m^\ell z^{-m}\right)\frac{h^\ell}{\ell!}\sum_{p=0}^{\ell+1}\binom{\ell+1}{p}(-1)^p z^{-p} \tag{19}$$

The first term, in (19), has been computed in [15], where is stated that

$$\sum_{k=0}^{\infty} k^p z^{-k} = \frac{zB_p(z)}{(z-1)^{p+1}} \qquad ;\forall p \geq 0 \tag{20}$$

The expression on the second parenthesis in (19) is a binomial function. We, thus, have that

$$\begin{aligned}\mathcal{Z}\left\{\beta_0^{(\ell)}(kh)\right\} &= \frac{zB_\ell(z)}{(z-1)^{\ell+1}}\frac{h^\ell(1-z^{-1})^{\ell+1}}{\ell!}\\ &= \frac{h^\ell}{\ell!}\frac{B_\ell(z)}{z^\ell}\end{aligned} \tag{21}$$

The result, then, follows from point 3) in Remark 1.

■

A close relationship can be established between splines and the impulse response of the hold used to generate the continuous-time system input. For example, we observe that the impulse response of a ZOH, i.e, the output of this hold when the input is a Kronecker delta, corresponds to the zero-order B-spline, $\beta_0^{(0)}(t)$ (see Figure 2). This pushes us to study the sampled model of a given system when a $\ell$-order B-spline, $\beta_0^{(\ell)}(t)$, is used as impulse response of the (generalized) hold.

We first need the following preliminary result.

*Lemma 2:* Let us consider the modified $\ell$-order B-spline

$$\tilde{\beta}_0^{(\ell)}(t) = \frac{1}{h}\tilde{\beta}_0^{(\ell-1)}(t) * \tilde{\beta}_0^{(0)}(t) \tag{22}$$

with $\beta_0^{(0)}(t) = \tilde{\beta}_0^{(0)}(t)$.

The Laplace transform of (22) is given by

$$\mathcal{B}_0^{(\ell)}(s) = \mathcal{L}\left\{\tilde{\beta}_0^{(\ell)}(t)\right\} = \frac{1}{h^\ell}\left(\mathcal{B}_0^{(0)}(s)\right)^{\ell+1} \tag{23}$$

where $\mathcal{B}_0^{(0)}(s)$, is the Laplace transform of (13).

*Proof:* Since time convolution translates to multiplication in the Laplace transform domain, we have

$$\mathcal{B}_0^{(\ell)}(s) = \frac{1}{h}\left(\mathcal{B}_0^{(\ell-1)}(s)\right)\left(\mathcal{B}_0^{(0)}(s)\right) \tag{24}$$

where, from (13), we have that

$$\mathcal{B}^{(0)}(s) = \frac{1-e^{-sh}}{s} \tag{25}$$

Note that (24) is a recursive equation, which, when applied $\ell$ times will give the result in (23). Equation (25) is straight forward from (13).

■

A scaling factor $1/h$ has been introduced in (22), in order to make the amplitude of the $\ell$-order B-spline independent of the sampling period, $h$, and thus, preventing it from going to 0 as the sampling rate is increased.

We will consider an $\ell$-order hold, for which the output is defined as a spline function (14), where the weighting factor of the $k$th B-spline is the $k$th input sample, $u_k$. Thus, the continuous-time input to the system is

$$u(t) = \sum_{k=0}^{\infty} u_k\tilde{\beta}_k^{(\ell)}(t) \tag{26}$$

and its Laplace transform is given by

$$U(s) = \sum_{k=0}^{\infty} u_k\mathcal{B}_0^{(\ell)}(s)e^{-skh} \tag{27}$$

since $\tilde{\beta}_k^{(\ell)}(t) = \tilde{\beta}_0^{(\ell)}(t - kh)$.

*Remark 3:* Note that the smoothness properties of $u(t)$ are the same as the smoothness properties of $\tilde{\beta}_k^{(\ell)}(t)$, which means that its $\ell$-th derivative will be the first one to be discontinuous.

We next show that, there is a connection between the order of the spline used in the hold and the asymptotic sampling zeros polynomial in the discrete-time model. We first consider, in the following theorem, the case of the sampled-data model for an $r$-order integrator. (Later, in Theorem 2 we consider the general linear case.)

*Theorem 1:* Consider a sampled system, as shown in Figure 1, consisting of an input sequence, $\{u_k\}$, an $\ell$-order B-spline hold (26) and a continuous-time linear integrator given by

$$G(s) = \frac{1}{s^r} \tag{28}$$

whose input is instantaneously sampled.

The discrete-time transfer function corresponding to (28) is given by

$$G_q(z) = \frac{h^r}{(r+\ell)!}\frac{B_{r+\ell}(z)}{z^\ell (z-1)^r} \tag{29}$$

where $B_{r+\ell}(\cdot)$ is the Euler-Fröbenius polynomial of order $r+\ell$ as defined in (5).

*Proof:* The computation of the pulse transfer function $G_q(z)$ is defined in (3), where $\tilde{Y}(z)$ is the $\mathcal{Z}$-transform of the sampled output $y(t)$ of the continuous-time system (28) and $\tilde{U}(z)$ the transform of the discrete input. For the sake of this proof we consider the particular sequence

$$u_k = (kh)^\ell \tag{30}$$

Note that the power of $kh$ in the input sequence, $\ell$, has been chosen to be the same as the order of the B-spline used in the hold. Using the definition of the $\mathcal{Z}$-transform and (20), $\tilde{U}(z)$ is given by

$$\tilde{U}(z) = \sum_{k=0}^{\infty}(kh)^\ell z^{-k} = \frac{h^\ell z B_\ell(z)}{(z-1)^{\ell+1}} \tag{31}$$

To compute $\tilde{Y}(z)$, some preliminary results are needed. To compute the continuous-time input to the system, we replace (30) into (27), obtaining the Laplace transform of the hold output

$$\begin{aligned} U(s) &= \sum_{k=0}^{\infty}(kh)^\ell \mathcal{B}_0^{(\ell)}(s) e^{-skh} \\ &= \mathcal{B}_0^{(\ell)}(s)\sum_{k=0}^{\infty}(kh)^\ell e^{-skh} \end{aligned} \tag{32}$$

Then, using (20) for the sum and Lemma 2, the previous equation yields

$$\begin{aligned} U(s) &= \mathcal{B}_0^{(\ell)}(s)\frac{h^\ell e^{sh} B_\ell\left(e^{sh}\right)}{\left(e^{sh}-1\right)^{\ell+1}} \\ &= \frac{\left(1-e^{-sh}\right)^{\ell+1}}{h^\ell s^{\ell+1}}\frac{h^\ell e^{sh} B_\ell\left(e^{sh}\right)}{\left(e^{sh}-1\right)^{\ell+1}} \\ &= \frac{B_\ell\left(e^{sh}\right)e^{-sh\ell}}{s^{\ell+1}} \end{aligned} \tag{33}$$

We know that $B_\ell(\cdot)$ is a symmetrical polynomial of order $(\ell-1)$ (see Remark 1). Thus, we have that

$$U(s) = \frac{B_\ell\left(e^{-sh}\right)e^{-sh}}{s^{\ell+1}} \tag{34}$$

The system output is given by

$$Y(s) = G(s)U(s) \tag{35}$$

Note that $B_\ell(e^{-sh})$ in (34) can be thought of as a polynomial of pure time delays and, therefore, $Y(s)$ is a linear combination of time delayed functions:

$$Y(s) = F(s)B_\ell\left(e^{-sh}\right) \tag{36}$$

The inverse Laplace transform, evaluated at $t=kh$, is then given by

$$y(t) = B_\ell\left(q^{-1}\right)\mathcal{L}^{-1}\{F(s)\} \tag{37}$$

where $q^{-1}f(t) = f(t-h)$ (see [16]) and

$$F(s) = G(s)\frac{e^{-sh}}{s^{\ell+1}} \tag{38}$$

Then we have that the $\mathcal{Z}$-transform of the sampled output is obtained as

$$\begin{aligned} \tilde{Y}(z) &= B_\ell\left(z^{-1}\right)\mathcal{Z}\{\mathcal{L}^{-1}\,F(s)|_{t=kh}\} \\ &= B_\ell\left(z^{-1}\right)\sum_{k=1}^{\infty}\left(z^{-k}\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{G(s)e^{-sh}}{s^{\ell+1}}e^{skh}ds\right) \\ &= B_\ell\left(z^{-1}\right)\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{G(s)}{s^{\ell+1}}\frac{1}{z-e^{sh}}ds \end{aligned} \tag{39}$$

If the integration path in (39) is closed to the right by a semi-circle, we have

$$\tilde{Y}(z) = B_\ell\left(z^{-1}\right)\left(-\sum_{k=1}^{\infty}Res_{s=s_k}\left\{\frac{G(s)}{s^{\ell+1}}\frac{1}{z-e^{sh}}\right\}\right) \tag{40}$$

Where, $s_k$ are the poles of the integral that fall inside the semi-circle, this is

$$s_k = \frac{\log z \pm 2\pi kj}{h} \qquad k\in\mathbb{Z} \tag{41}$$

Replacing in (40) yields

$$\tilde{Y}(z) = \frac{B_\ell\left(z^{-1}\right)}{hz}\left(\sum_{k=1}^{\infty}\frac{G(s_k)}{s_k^{\ell+1}}\right) \tag{42}$$

If we now substitute $G(s)$ by (28), we have that

$$\tilde{Y}(z) = \frac{h^{r+\ell}B_\ell\left(z^{-1}\right)}{z}\left(\sum_{k=1}^{\infty}\frac{1}{\left(\log z \pm 2\pi kj\right)^{r+\ell+1}}\right) \tag{43}$$

The infinite sum in (43) can be rewritten in terms of the Euler-Fröbenius polynomial [5] to obtain:

$$\tilde{Y}(z) = \frac{h^{r+\ell}B_\ell\left(z^{-1}\right)}{z}\frac{zB_{r+\ell}(z)}{(r+\ell)!(z-1)^{r+\ell+1}} \tag{44}$$

Now we can replace (31) and (44) in (3) to obtain

$$\begin{aligned} G_q(z) &= \frac{(z-1)^{\ell+1}}{h^\ell z B_\ell(z)}\frac{h^{r+\ell}B_\ell\left(z^{-1}\right)}{z}\frac{zB_{r+\ell}(z)}{(r+\ell)!(z-1)^{r+\ell+1}} \\ &= \frac{h^r}{(r+\ell)!}\frac{B_{r+\ell}(z)}{z^\ell(z-1)^r} \end{aligned} \tag{45}$$

■

The previous theorem establishes that if we consider the sampled data model for an $r$-order integrator when using an $\ell$-order B-spline generalized hold to generate the system input, then the sampling zeros are exactly given by the Euler-Fröbenius polynomial of order $r+\ell$. We show next that, in fact, the same polynomial characterizes the asymptotic sampling zeros for any system of (continuous-time) relative degree $r$, when sampled using the same B-spline hold.

*Theorem 2:* Consider a sampled system, as seen in Figure 1, with an $\ell$-order B-spline hold (26) and a continuous-time linear system defined in (1). As $h \approx 0$, its discrete-time transfer function is given by

$$G_q(z) = \frac{Kh^r}{(\ell+r)!}\frac{B_{\ell+r}(z)}{z^\ell(z-1)^r} \tag{46}$$

where, $B_{r+\ell}(\cdot)$ is the Euler-Fröbenius polynomial of order $r+\ell$ as defined in (5).

*Proof:* According to (3) and replacing with (31) and (39), the discrete-time transfer function $G_q(z)$ can be obtained as

$$G_q(z) = \frac{(z-1)^{\ell+1}}{h^\ell z B_\ell(z)} B_\ell\left(z^{-1}\right) \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{G(s)}{s^{\ell+1}}\frac{1}{z-e^{sh}} ds \tag{47}$$

Using (1) and introducing the variable $s = \omega/h$, we can rewrite (47) as follows

$$\begin{aligned} &G_q(z) \\ &= \frac{(z-1)^{\ell+1}}{h^\ell z^\ell}\frac{1}{2\pi j}\int_{ch-j\infty}^{ch+j\infty} \frac{\frac{K}{\omega^r}h^r\frac{(1-c_1 h/\omega)\ldots(1-c_m h/\omega)}{(1-p_1 h/\omega)\ldots(1-p_n h/\omega)}}{\frac{\omega^{\ell+1}(z-e^\omega)}{h^{\ell+1}}}\frac{d\omega}{h} \end{aligned} \tag{48}$$

As $h \approx 0$, we can characterize the asymptotic model by considering the following limit:

$$\begin{aligned} &\lim_{h\to 0} h^{-r} G_q(z) \\ &= K\frac{(z-1)^{\ell+1}}{z^\ell}\frac{1}{2\pi j}\int_{ch-j\infty}^{ch+j\infty}\frac{1}{\omega^{r+\ell+1}\left(z-e^\omega\right)}d\omega \\ &= K\frac{(z-1)^{\ell+1}}{z^\ell}\frac{B_{\ell+r}(z)}{(\ell+r)!(z-1)^{\ell+r+1}} \end{aligned} \tag{49}$$

This last step is computed, taking into account that the complex integral corresponds to that of the $r$-order integrator in Theorem 1.

Then, for fast sampling rates, the discrete-time system can be considered as

$$G_q(z) = \frac{Kh^r}{(\ell+r)!}\frac{B_{\ell+r}(z)}{z^\ell(z-1)^r} \tag{50}$$

■

It is well known that generalized holds can be used to change the zeros of sampled data systems [17], [11]. In the result presented here, however, we are not interested in modifying the intrinsic zeros of the system. We have shown that the (asymptotic) sampling zeros, characterized by Euler-Fröbenius polynomials, depend neatly on the order of the B-spline used in the hold. In fact, from (46) we observe that the order of the polynomial corresponding to a system of relative degree $r$, is increased exactly by the order of the hold, $\ell$. Moreover, this result is consistent with [3], given that a ZOH corresponds to $\ell = 0$.

## IV. Links to numerical integration strategies

In the literature, the presence of sampling zeros has been previously associated to the numerical integration strategy used to discretize the continuous-time model of the system. In fact, different sampling zeros characterizations have been obtained for zero-, first- and fractional order holds [3], [18], [19].

In particular, the smoothness of the continuous-time input, $u(t)$ was exploited in [7] to truncate the Taylor series expansion to different orders. In that work, the ZOH assumption was associated to a piecewise constant input, discontinuous only at the sampling instant. The approach proposed in the previous section, namely, to use an $\ell$-order spline as output of the hold, means that $u(t)$ in (26) has $\ell-1$ continuous derivatives. We can use this fact to increase the order of the Taylor series expansion, up to the $\ell$-th derivative of $u(t)$.

We next illustrate these ideas by considering the case of a first order B-spline hold applied to a linear integrator. The associated hold can be interpreted as having a causal interpolating first order hold (FOH), defined as

$$u(t) = (t-kh)\frac{u_k - u_{k-1}}{h} + u_{k-1} \qquad t \in [kh, kh+h[ \tag{51}$$

*Example 1:* Consider the continuous-time system

$$G(s) = \frac{1}{s} \tag{52}$$

We are interested in obtaining a discrete-time transfer function $G_q(z)$ corresponding to (52), by using a numerical integration strategy that *solves* that equation [20]. The solution, using the Taylor series around $t = kh$, can be expressed as follows

$$y(kh+\tau) = y(kh) + \tau u(kh) + \frac{\tau^2}{2!}\dot{u}(\xi_1) \tag{53}$$

where $kh < \xi_1 < kh+h$. It is important to note that the expansion has been truncated to the third term, because the first derivative of $u(t)$, i.e., $\dot{u}(t)$ is the first discontinuous term. This means that the expansion (53) is exact for some *unknown* time instants $kh < \xi_1 < kh+h$ [20].

From (51), we have that

$$\dot{u}(t) = \frac{u_k - u_{k-1}}{h} \qquad t \in [kh, kh+h[ \tag{54}$$

Note that, in this case, $\dot{u}(t)$ is a constant within the sampling interval and, thus, replacing $\xi_1 = kh$ gives the exact solution. This means that the exact sampled-data model is obtained. We can include (54) as part of a discrete-time state-space model if we define $x_2[k+1] = u_k$. The discrete equations that describe $G_q(z)$ are given by

$$\begin{aligned} x[k+1] &= \begin{bmatrix} 1 & h/2 \\ 0 & 0 \end{bmatrix} x[k] + \begin{bmatrix} h/2 \\ 1 \end{bmatrix} u_k \\ y[k] &= \begin{bmatrix} 1 & 0 \end{bmatrix} x[k] \end{aligned} \tag{55}$$

Which, in fact, corresponds to the exact sampled data transfer function:

$$G_q(z) = \frac{h}{2}\frac{z+1}{z(z+1)} \tag{56}$$

■

The use of an integration strategy can also be extended to non linear systems. When dealing with non linear system, the concept of sampling zeros is replaced by the concept of sampling zero dynamics [7]. A short example will help us illustrate this idea.

*Example 2:* Consider the continuous-time non linear system

$$\begin{aligned} \dot{x}_1(t) &= a(x_1(t)) + b(x_1(t))u(t) \\ y(t) &= x_1(t) \end{aligned} \tag{57}$$

We assume that the system has no continuous time zero dynamics [21], however, we will see that when using a FOH, sampling zero dynamics will appear in the sampled model. We discretize equations in (57), and follow the same procedure as in Example 1. Thus, representation of the solution for $t = kh$, can be expressed as follows

$$\begin{aligned} x_1(kh+h) &= x_1(kh) + h\frac{d}{dt}\, x_1|_{t=kh} + \frac{h^2}{2}\frac{d^2}{dt^2}\, x_1|_{t=kh} \\ x_2(kh+h) &= u_k \\ y(kh) &= x_1(kh) \end{aligned} \tag{58}$$

The Taylor series expansion has been truncated to the third term, because $\dot{u}(t)$ is the first discontinuous term and appears on the second derivative of $x_1$. The zero dynamics of this sampled model can be found considering $y \equiv x_1 \equiv 0\ \forall k$, and solving the resulting dynamic equation

$$x_2(kh+h) = -x_2(kh)\left(1 + h\left.\frac{\partial a}{\partial x_1}\right|_{t=kh} + h\left.\frac{\partial b}{\partial x_1}\right|_{t=kh}\right) \tag{59}$$

Note that when $h \to 0$, the eigenvalue of (59) approaches the asympotic sampling zero at $z = -1$ (see (56)).

■

The result obtained in Example 2 is consistent with theorem 2 in section III. In fact, when considering a linear system, i.e. $a(x) = ax$ and $b(x) = b$ in (57), then the eigenvalue of equation (59) is located at $z = -(1 + ah)$, which is an approximation of the asymptotic sampling zero at $z = -1$. This provides additional insights to the presence of sampling zeros (dynamics) in sampled-data models in terms of the integration strategy used to discretize the differential equations.

## V. Conclusions

In this paper we have presented a common framework that allows to establish relations between

1) The presence of sampling zeros in discrete-time models,
2) The use of B-splines for continuous-time signal reconstruction, and
3) The integration strategy applied to discretize a continuous-time model.

B-splines were shown to be directly linked to the asymptotic sampling zeros polynomial in discrete-time models. Generalized hold functions are known to modify the zeros of the system. Nevertheless, the results in the paper show that the order of the Euler-Fröbenius polynomial that characterize the sampling zeros is the sum of the order of the B-spline used in the hold and the continuous-time relative degree. This is, the order of the spline modifies, only, the order of the sampling zero polynomial.

We have also shown that the asymptotic sampling zeros can be explained as a consequence of the integration strategy underlying in the discretization process. This fact was highlighted by applying a truncated Taylor series expansion based on the smoothness of the plant input generated by a B-spline generalized hold. This idea can also be applied to non linear models

We think that the results allow a better understanding of the sampling process and the presence of sampling zeros in sampled-data models. In fact, the results explicitly show that the zeros are a consequence of the way the system is discretized: how the continuous-time input is generated and/or how the differential equation description is translated to the discrete-time domain.

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