2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Robust Receding - Horizon Control of Nonlinear Systems with State Dependent Uncertainties: an Input-to-State Stability Approach

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Abstract— **This paper is concerned with the robust receding horizon control of constrained discrete-time nonlinear systems affected by model uncertainty. A class of uncertainties entailing norm-bounded additive state dependent and non-state-dependent uncertainties is considered. In order to robustly enforce the constraints, a technique based on constraints tightening is formulated. Moreover, it is shown that the closed-loop system, obtained with the developed RH controller, is Regionally Input-to-State Stable with respect to the considered class of uncertainties. A simulation example shows the effectiveness of the proposed approach.**

I. Introduction

In this paper, the problem of controlling constrained discrete-time uncertain nonlinear systems is addressed by designing a robust Receding Horizon (RH) control law by exploiting some properties of "constraint tightening" techniques, originally developed for perturbed linear discrete-time systems (see [2], [4], [12]). In this respect, it is worth noting that an extension to the nonlinear case is given in [6], where the class of norm-bounded additive uncertainties is addressed. A promising RH formulation for some classes of non linear systems has also been presented in [11].

The aforementioned approaches rely on the open-loop RH paradigm, i.e., the decision variables consist in a sequence of optimal control with respect to a given cost function and a nominal prediction model. As is well known, in order to achieve the robust feasibility property, any feasible open-loop control sequence must satisfy the constraints for all the possible instances of the uncertainty. Therefore, it clearly turns out that the uncertainties have to be taken into account when computing the control law in order to guarantee robust constraint satisfaction and closed-loop stability in presence of model uncertainty [8].

Previous work is extended in this paper by addressing the case of a wider class of uncertainties, namely the case of simultaneous presence of norm-bounded state-dependent uncertanties and of disturbances not depending on the state. The robust stability analysis, carried out by resorting to Regional Input to State Stability (ISS) arguments (see [7], [3], allows to derive the ISS stability margin with respect to state-dependent uncertainty terms. Furthermore, it is shown that exploiting the robust control invariance of a constraint set imposed at the end of the control horizon, the estimation of the maximum admissible uncertainty can be made less conservative with respect to methods available in the literature.

The paper is organized as follows: first, in Section II main notations and basic definitions will be given. Then, in Section III, the structure of the algorithm will be introduced, whereas in Section IV the main stability results will be stated and proved. Finally, simulation results will be reported in Section V, showing the effectiveness of the proposed approach.

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II. Main notations and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer, and the non-negative integer sets of numbers, respectively. The Euclidean norm is denoted as $|\cdot|$. For any discrete-time sequence $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$, $\|\phi\| \triangleq \sup_{k \geq 0}\{|\phi_k|\}$ and $\|\phi_{[\tau]}\| \triangleq \sup_{0 \leq k \leq \tau}\{|\phi_k|\}$, where ϕ_k denotes the value that the sequence ϕ takes on in correspondence with the index k. The set of discrete-time sequences υ taking values in some subset $\Upsilon \subset \mathbb{R}^m$ is denoted by $\mathcal{M}_\Upsilon$, while $\Upsilon^{sup} \triangleq \sup_{\upsilon \in \Upsilon}\{|\upsilon|\}$. The symbol *id* represents the identity function from $\mathbb{R}$ to $\mathbb{R}$, while $\gamma_1 \circ \gamma_2$ is the composition of two functions γ_1 and γ_2 from $\mathbb{R}$ to $\mathbb{R}$. Given a set $A \subseteq \mathbb{R}^n$, $\text{int}(A)$ denotes the interior of A. Given a vector $x \in \mathbb{R}^n$, $\text{d}(x, A) \triangleq \inf\{|\xi - x|, \xi \in A\}$ is the point-to-set distance from $x \in \mathbb{R}^n$ to A. Given two sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, $\text{dist}(A, B) \triangleq \inf\{\text{d}(\zeta, A), \zeta \in B\}$ is the minimal set-to-set distance. The difference between two given sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, with $B \subseteq A$, is denoted as $A \backslash B \triangleq \{x : x \in A, x \notin B\}$. Given two sets $A \in \mathbb{R}^n$, $B \in \mathbb{R}^n$, then the Pontryagin difference set C is defined as $C = A \backsim B \triangleq \{x \in \mathbb{R}^n : x + \xi \in A, \forall \xi \in B\}$, while the Minkowski sum set is defined as $S = A \oplus B \triangleq \{x \in \mathbb{R}^n : \exists \xi \in A, \eta \in B, x = \xi + \eta\}$. Given a vector $\eta \in \mathbb{R}^n$ and a positive scalar $\rho \in \mathbb{R}_{>0}$, the closed ball centered in η and of radius ρ , is denoted as $\mathcal{B}(\eta, \rho) \triangleq \{\xi \in \mathbb{R}^n : |\xi - \eta| \leq \rho\}$. The shorthand $\mathcal{B}(\rho)$ is used when the ball is centered in the origin. The notions of functions of class $\mathcal{K}$, class $\mathcal{K}_\infty$, and class $\mathcal{KL}$ are used to characterize stability properties. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. It belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s \to 0} \beta(s, t) = \lim_{t \to \infty} \beta(s, t) = 0$.

Consider the following discrete-time dynamic system

$$x_{t+1} = g(x_t, \upsilon_t),\ t \in \mathbb{Z}_{\geq 0},\ x_0 = \bar{x}\,, \tag{1}$$

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where $x_t \in \mathbb{R}^n$ and $\upsilon_t \in \Upsilon \subset \mathbb{R}^r$ are the state and the bounded input of the system, respectively. The discrete-time state trajectory of the system (1), with initial state $\bar{x}$ and input sequence $\upsilon \in \mathcal{M}_\Upsilon$, $\upsilon = \{\upsilon_t, t \in \mathbb{Z}_{\geq 0}\}$, is denoted by $x(t, \bar{x}, \upsilon)$, $t \in \mathbb{Z}_{\geq 0}$. The following further definitions are given.

Definition 2.1 (RPI set): A set $\Xi \subset \mathbb{R}^n$ is a Robust Positively Invariant (RPI) set for system (1) if $g(x_t, \upsilon_t) \in \Xi$, $\forall x_t \in \Xi$ and $\forall \upsilon_t \in \Upsilon$. □

Definition 2.2 (0−AS in Ξ): Given a compact set $\Xi \subset \mathbb{R}^n$, with $\{0\} \subset \Xi$, if Ξ is RPI for (1) and if there exists a $\mathcal{KL}$-function β such that

$$|x(t, \bar{x}, 0)| \leq \beta(|\bar{x}|, t), \ \forall t \in \mathbb{Z}_{\geq 0}, \forall \bar{x} \in \Xi, \tag{2}$$

then the system (1) is said to be zero-asymptotically stable (0−AS) in Ξ. □

Definition 2.3 (Stability margin): Given a set $\Xi \subset \mathbb{R}^n$, a (nonlinear) stability margin for system (1) is any $\mathcal{K}$-function δ such that, for any feedback law, $\upsilon_t = \pi(t, x_t)$, possibly time varying, bounded by δ, $\pi(t, x_t) : |\pi(t, x_t)| \leq \delta(|x_t|)$, $\forall x_t \in \Xi, \forall t \in \mathbb{Z}_{\geq 0}$, there exists a $\mathcal{KL}$-function β such that (2) holds. □

Definition 2.4 (ISS in Ξ): Given a compact set $\Xi \subset \mathbb{R}^n$, with $\{0\} \subset \Xi$, if Ξ is RPI for (1) and if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that

$$|x(t, \bar{x}, \upsilon)| \leq \max\left\{\beta(|\bar{x}|, t), \ \gamma(\|\upsilon_{[t]}\|)\right\}, \forall t \in \mathbb{Z}_{\geq 0}, \forall \bar{x} \in \Xi,$$

then the system (1), with $\upsilon \in \mathcal{M}_\Upsilon$, is said to be Input to State Stable (ISS) in Ξ. □

In the following, the notion of Regional Input to State Stability, recently introduced in [7], is briefly discussed.

Definition 2.5 (ISS-Lyapunov Function [7], [3]): Given a pair of compact sets $\Xi \subset \mathbb{R}^n$ and $\Omega \subseteq \Xi$, with $\{0\} \subset \Omega$, a function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a (Regional) ISS-Lyapunov function for system (1), with $\upsilon \in \mathcal{M}_\Upsilon$ and $x \in \Xi$, if there exist some $\mathcal{K}_\infty$−functions α_1, α_2, α_3, and a $\mathcal{K}$-function σ such that

1) the following inequalities hold $\forall \upsilon \in \mathcal{M}_\Upsilon$:

$$V(x) \geq \alpha_1(|x|), \qquad \forall x \in \Xi, \tag{3}$$

$$V(x) \leq \alpha_2(|x|), \qquad \forall x \in \Omega, \tag{4}$$

$$V(g(x, \upsilon)) - V(x) \leq -\alpha_3(|x|) + \sigma(\|\upsilon\|), \ \forall x \in \Xi, \tag{5}$$

2) there exist a suitable $\mathcal{K}_\infty$-function ρ (with ρ such that $(id-\rho)$ is a $\mathcal{K}_\infty$-function, too) such that the following compact set $D \subset \{x : x \in \Omega, d(x, \delta\Omega) > c\}$, $\{0\} \subset D$, can be defined for some constant $c \in \mathbb{R}_{>0}$:

$$D \triangleq \{x : V(x) \leq b(\Upsilon^{sup})\}$$

where $b(s) \triangleq \alpha_4^{-1} \circ \rho^{-1} \circ \sigma(s)$, $\alpha_4 \triangleq \alpha_3 \circ \alpha_2^{-1}$. □

Theorem 2.1 (Regional ISS [7]): If system (1) admits a ISS-Lyapunov function in Ξ, and Ξ is RPI for (1), then it is Regional ISS in Ξ and $\lim_{t\to\infty} \mathrm{d}(x(t, \bar{x}, \upsilon), D) = 0$. □

III. Problem formulation

Consider the nonlinear discrete-time dynamic system

$$x_{t+1} \;=\; f(x_t, u_t, \upsilon_t) \tag{6}$$

where $x_t \in \mathbb{R}^n$ denotes the system's state, $u_t \in \mathbb{R}^m$ the control variables and $\upsilon_t \in \mathbb{R}^r$ an exogenous input which models the uncertainty not depending on the state. The state and control variables are subjected to the following constraints

$$x_t \in X, \ \ t \in \mathbb{Z}_{\geq 0} \tag{7}$$

$$u_t \in U, \ \ t \in \mathbb{Z}_{\geq 0} \tag{8}$$

where X and U are compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, containing the origin as an interior point.

The control objective consists in designing a state-feedback control law capable to robustly stabilize the system (6) and to achieve ISS stability with respect to state dependent uncertainties and non-state-dependent disturbances.

Given the system (6), let $\hat{f}(x_t, u_t)$, with $\hat{f}(0, 0) = 0$, denote the *nominal* model used for control design purposes, such that

$$x_{t+1} = \hat{f}(x_t, u_t) + d_t \tag{9}$$

where $d_t \triangleq f(x_t, u_t, \upsilon_t) - \hat{f}(x_t, u_t) \in \mathbb{R}^n$ denotes the discrete-time state transition uncertainty. Then, the following assumptions are needed.

Assumption 1 (Uncertainties): The additive transition uncertainty d_t is limited in a time varying compact ball D_t, that is $d_t \in D_t \triangleq \mathcal{B}(\delta(|x_t|) + \mu(||\upsilon||))$, $\forall x_t \in X, \forall \upsilon \in \mathcal{M}_\Upsilon$, where δ and μ are two $\mathcal{K}$-functions. It follows that d_t is given by the sum of two contributions: a state-dependent component and a non-state dependent one. □

Assumption 2: $\hat{f}$ is locally Lipschitz with respect to x for all $x \in X$, with Lipschitz constant $L_{f_x} \in \mathbb{R}_{>0}$. □

On the basis of the previous Assumptions, let us introduce the underlying controller formulation, which relies on the Finite-Horizon Optimal Control Problem.

Definition 3.1 (FHOCP): Given two positive integers $N_c \in \mathbb{Z}_{\geq 0}$ and $N_p \in \mathbb{Z}_{\geq N_c}$, at any time $t \in \mathbb{Z}_{\geq 0}$, let $\mathbf{u}_{t,t+N_p-1|t} \triangleq \mathrm{col}[u_{t|t},\ u_{t+1|t},\ \ldots,\ u_{t+N_c-1|t},\ u_{t+N_c|t},\ \ldots,\ u_{t+N_p-1|t}]$ denote a sequence of input variables over the time-horizon N_p. N_p and N_c will be called prediction and control horizon, respectively. Moreover, given x_t and $\mathbf{u}_{t,t+N_p-1|t}$, let $\hat{x}_{t+j|t}$ denote the state "predicted" by means of the nominal model, such that

$$\hat{x}_{t+j|t} = \hat{f}(\hat{x}_{t+j-1|t}, u_{t+j-1|t}),\ \hat{x}_{t|t} = x_t,\ 1 \leq j \leq N_p\,. \tag{10}$$

Then, given a stage cost function h, a terminal cost function h_f, a terminal set X_f and a series of constraint sets $X_{t+j|t} \subseteq X$, $j \in \{1, \ldots, N_p - 1\}$, to be described later on, the Finite Horizon Optimal Control Problem (FHOCP) consists in minimizing, with respect to $\mathbf{u}_{t,t+N_c-1|t}$, the cost function

$$\begin{aligned} J_{FH}(x_t, \mathbf{u}_{t,t+N_c-1|t}, N_c, N_p) &\triangleq \sum_{l=t}^{t+N_c-1} h(\hat{x}_{l|t}, u_{l|t}) + \\ &\sum_{l=t+N_c}^{t+N_p-1} h(\hat{x}_{l|t}, \kappa_f(x_{l|t})) + h_f(\hat{x}_{t+N_p|t}) \end{aligned} \tag{11}$$

subject to

1) nominal dynamics (10) with $\hat{x}_{t|t} = x_t$;
2) control and state constraints $\hat{x}_{t+j|t} \in X_{t+j|t}, j \in \{1, \ldots, N_p\}$;
3) terminal state constraints $\hat{x}_{t+N_p|t} \in X_f$. □

The usual RH control technique can now be stated as follows: given a time instant $t \in \mathbb{Z}_{\geq 0}$, let $\hat{x}_{t|t} = x_t$, and find the optimal control sequence $\mathbf{u}^\circ_{t,t+N_c-1|t}$ by solving the FHOCP. Then, according to the RH strategy, apply:

$$u_t = \kappa_{RH}(x_t)\,, \tag{12}$$

where $\kappa_{RH}(x_t) \triangleq u^\circ_{t,t}$ and $u^\circ_{t,t}$ is the first element of the optimal control sequence $\mathbf{u}^\circ_{t,t+N_c-1|t}$ (implicitly dependent on x_t).

IV. Robust RH Strategy

With particular reference to the underlined formulation of the FHOCP, in the following it will be shown that, by accurately choosing the stage cost h, the constraint sets $X_{t+j|t}$, $j \in \{1, \dots, N_p - 1\}$, the terminal cost function h_f, and by imposing a robust constraint X_{N_c} at the end of the control horizon in place of the terminal constraint X_f, it is possible to guarantee the predicted state to automatically satisfy the terminal constraint $\hat{x}_{t+N_p|t} \in X_f$, enlarging at the same time the domain of attraction of the RH controller.

In the following, X_{RH} will denote the set containing all the state vectors for which the FHOCP is feasible. In order to formulate the robust RH algorithm, let us introduce the following further assumptions.

Assumption 3: A terminal cost function h_f, an auxiliary control law κ_f, and a set X_f are given such that

1) $X_f \subset X$, X_f closed, $0 \in X_f$;
2) $\kappa_f(x) \in U$, $|\kappa_f(x)| \le L_{\kappa_f}|x|$, $L_{\kappa_f} > 0$, $\forall x \in X_f$;
3) $\left|\hat{f}(x, \kappa_f(x))\right| \le L_{f_c}|x|$, $L_{f_c} > 0$, $\forall x \in X_f$;
4) $\hat{f}(x, \kappa_f(x)) \in X_f$, $\forall x \in X_f$;
5) $h_f(x)$ is locally Lipschitz with respect to x for all $x \in X_f$, with Lipschitz constant $L_{h_f} \in \mathbb{R}_{>0}$;
6) $h_f(\hat{f}(x, \kappa_f(x))) - h_f(x) \le -h(x, \kappa_f(x))$, $\forall x \in X_f$; □

Assumption 4: The partial cost function h is such that $\underline{h}(|x|) \le h(x, u)$, $\forall x \in X, \forall u \in U$ where $\underline{h}$ is a $\mathcal{K}_\infty$-function. Moreover, h is Lipschitz with respect to x and u in $X \times U$, with Lipschitz constants $L_h \in \mathbb{R}_{\ge 0}$ and $L_{hu} \in \mathbb{R}_{\ge 0}$. □

A. Restriction of State Constraints

Throughout this section, the following notation will be used: given an optimal sequence $\mathbf{u}^\circ_{t,t+N_c-1|t}$ of control actions obtained by solving the FHOCP at time t, we will denote as $\bar{\mathbf{u}}_{t+1,t+N_c|t}$ the sequence

$$\bar{\mathbf{u}}_{t+1,t+N_c|t+1} \triangleq \text{col}\left[u^\circ_{t+1|t}, \dots, u^\circ_{t+N_c-1|t}, \bar{u}\right],$$

where $\bar{u} \in U$ is a suitably defined admissible control action depending on $\hat{x}_{t+N_c|t+1}$. The following technical result will be instrumental for the subsequent analysis.

Lemma 4.1 (Technical): Under Assumptions 1 and 2, let $L_\delta \triangleq \min\{L \in \mathbb{R}_{>0} : \delta(|x|) \le L|x|, \forall x \in X_{RH}\}$, $L_{\hat{\delta}} \triangleq (L_{f_x} + L_\delta)$ and $\bar{x}_0 \triangleq \sup_{u \in U}\left\{\left|\hat{f}(0, u)\right|\right\}$. Suppose [1], without loss of generality, $L_{f_x} \ne 1$, $L_\delta \ne 0$ and $L_{\hat{\delta}} \ne 1$. Given the state vector x_t at time t, and a feasible sequence of inputs, $\bar{\mathbf{u}}_{t,t+N_p-1|t}$, the prediction error $\hat{e}_{t+j|t} \triangleq \left|x_{t+j} - \hat{x}_{t+j|t}\right|$, with $j \in \{1, \dots, N_p\}$, and x_{t+j} obtained applying $\bar{\mathbf{u}}_{t,t+N_p-1|t}$ to the uncertain system (6), is upper bounded by

$$|\hat{e}_{t+j|t}| \le \sigma^x_j(|x_t|) + \sigma^\upsilon_j(||\upsilon||) + \sigma^{x_0}_j(\bar{x}_0),\ \forall x \in X, \forall \upsilon \in \mathcal{M}_\Upsilon$$

where $\sigma^x_j(|x_t|) \triangleq \frac{L^j_{\hat{\delta}} - L^j_{f_x}}{L_{\hat{\delta}} - L_{f_x}} L_\delta |x_t|$, $\sigma^\upsilon_j(||\upsilon||) \triangleq \left(\frac{L^j_{f_x} - 1}{L_{\hat{\delta}} - 1} + \frac{L^j_{\hat{\delta}} - L^j_{f_x}}{L_{\hat{\delta}} - L_{f_x}} \cdot \frac{L_\delta}{L_{\hat{\delta}} - 1}\right)\mu(||\upsilon||)$ and $\sigma^{x_0}_j(\bar{x}_0) \triangleq \left(\frac{L^j_{\hat{\delta}} - L^j_{f_x}}{L_{\hat{\delta}} - L_{f_x}} - \frac{L^j_{f_x} - 1}{L_{f_x} - 1}\right)\frac{L_\delta}{L_{\hat{\delta}} - 1}(\bar{x}_0)$. □

[1] The very special cases $L_{f_x} = 1$, $L_{\hat{\delta}} = 1$ and $L_\delta = 0$ can be trivially addressed by a few suitable modifications to the proof of Lemma 4.1.

Proof: For a generic $\upsilon \in \mathcal{M}_\Upsilon$, let $\bar{\mu} \triangleq \mu(||\upsilon||)$. In view of Assumptions 1 and 2, the following inequalities hold $\forall x_{t+j} \in X_{RH}$, $\forall u_{t+j} \in U$:

$$\begin{aligned}\delta(|x_{t+j+1}|) \le\ & L_\delta\Big(\left|\hat{f}(x_{t+j}, u_{t+j}) - \hat{f}(0, u_{t+j})\right| + \\ & \left|\hat{f}(0, u_{t+j}) - \hat{f}(0, 0)\right| + L_\delta(|x_{t+j}|) + \bar{\mu}\Big) \\ \le\ & L_{\hat{\delta}}\left(L_\delta|x_{t+j}| + \bar{\mu}\right) + \left(L_\delta - L_{\hat{\delta}}\right)\bar{\mu} + L_\delta \bar{x}_0 .\end{aligned}$$

Then, by adding $\bar{\mu}$ on both sides of the previous inequality and by operating inductive arguments it follows that

$$L_\delta|x_{t+j}| + \bar{\mu} \le L^j_{\hat{\delta}}\left(L_\delta|x_t| + \bar{\mu}\right) + \frac{L^j_{\hat{\delta}} - 1}{L_{\hat{\delta}} - 1}\left[(1 - L_{f_x})\bar{\mu} + L_\delta \bar{x}_0\right].$$

In view of this intermediate result, an upper bound on the norm of the prediction error can be given by recursion

$$|\hat{e}_{t+j+1|t}| \le L_{f_x}|\hat{e}_{t+j|t}| + L^j_{\hat{\delta}}(L_\delta|x_t| + \bar{\mu}) + \frac{L^j_{\hat{\delta}} - 1}{L_{\hat{\delta}} - 1}\left[(1 - L_{f_x})\bar{\mu} + L_\delta \bar{x}_0\right],$$

and, by induction, we obtain

$$|\hat{e}_{t+j|t}| \le L^{j-1}_{f_x}(L_\delta|x_t| + \bar{\mu}) + \sum_{k=1}^{j-1} L^{j-k-1}_{f_x}\Big\{L^k_{\hat{\delta}}(L_\delta|x_t| + \bar{\mu}) + \frac{L^k_{\hat{\delta}} - 1}{L_{\hat{\delta}} - 1}\left[(1 - L_{f_x})\bar{\mu} + L_\delta \bar{x}_0\right]\Big\}.$$

Finally, the statement follows by a little algebra. ■

Under Assumption 1 and in view of Lemma 4.1, a norm-bound on the state prediction error can be evaluated. The satisfaction of the original state constraints under the worst case uncertainty can be ensured by imposing restricted constraints to the predicted open-loop trajectories. In this connection, the following result will be useful in the sequel.

Lemma 4.2 (State Constraints Tightening): Denoting the restricted state constraints at the $j-$th prediction step of the FHOCP as $X_{t+j|t}$, with $X_{t+j|t} \triangleq X \backsim \mathcal{B}(\rho_{t+j|t})$ and

$$\rho_{t+j|t} \triangleq \sigma^x_j(|x_t|) + \sigma^\upsilon_j(||\upsilon||) + \sigma^{x_0}_j(\bar{x}_0), \forall x_t \in X, \forall \upsilon \in \mathcal{M}_\Upsilon, \quad (13)$$

then each feasible input sequence evaluated solving the FHOCP by means of the nominal model under the restricted state constraints guarantees that the true state will satisfy $x_{t+j} \in X$, $\forall j \in \{1, \dots, N_p\}$, $\forall x_t \in X_{RH}$, $\forall \upsilon \in \mathcal{M}_\Upsilon$. □

Proof: The proof trivially follows from Lemma 4.1, considering the inclusion: $\mathcal{B}(\rho_{t+j|t}) \supset \mathcal{B}(|\hat{e}_{t+j|t}|)$. ■

B. Terminal State Constraint and Bound on Uncertainties

In the proposed FHOCP formulation, in order to enforce the robust feasibility, the terminal state constraint $\hat{x}_{t+N_p|t} \in X_f$ is replaced by a fixed constraint $\hat{x}_{t+N_c|t} \in X_{N_c}$ at the end of the control horizon. In order to design the robust constraint X_{N_c}, the following assumptions are formulated.

Assumption 5: Let $X_{\kappa_f} \subseteq X$, with $\{0\} \in X_{\kappa_f}$, denote a compact set for which $\tilde{\mathbf{u}}_{t,t+N_p-1|t} \triangleq \text{col}[\kappa_f(\hat{x}_{t|t}), \kappa_f(\hat{x}_{t+1|t}), \dots, \kappa_f(\hat{x}_{t+N_p-1|t})]$, with $\hat{x}_{t|t} = x_t$, is an admissible control sequence for the FHOCP and for which Points 2) and 3) of Assumption 3 are satisfied. Moreover, let $h_f(\hat{f}(x, \kappa_f(x))) - h_f(x) < 0$, $\forall x \in X_{\kappa_f} \backslash \{0\}$. □

Assumption 6 (Robust constraint X_{N_c}): The robust constraint set of the FHOCP, X_{N_c}, is chosen such that

1) $X_{N_c} \subset X_{\kappa_f}$, X_{N_c} closed, $\{0\} \in X_{N_c}$;
2) for all $x \in X_{N_c}$ the state can be steered to X_f in $N_p - N_c$ steps or less under the nominal dynamics in closed-loop with the auxiliary control law κ_f;

3) there exists a positive scalar $\epsilon \in \mathbb{R}_{>0}$ such that $\hat{f}(x_t, \kappa_f(x_t)) \in X_{N_c} \backsim \mathcal{B}(\epsilon), \ \forall x_t \in X_{N_c}$. □

The following definition will be useful.

Definition 4.1 ($\mathcal{P}(\Xi)$): Given a set $\Xi \subset X$, the (one-step) Predecessor set, $\mathcal{P}(\Xi)$, is defined as

$$\mathcal{P}(\Xi) \triangleq \left\{ x_t \in \mathbb{R}^n \middle| \exists u_t \in U : \hat{f}(x_t, u_t) \in \Xi \right\},$$

i.e., $\mathcal{P}(\Xi)$ is the set of states which can be steered to Ξ by a control action under $\hat{f}(x_t, u_t)$, subject to (8). □

Lemma 4.3 (Technical): Given X_{N_c} such that Assumption 6 holds, let us define $\bar{d}_{\kappa_f} \triangleq \epsilon/L_{f_x}$ and $\bar{d} \triangleq \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{P}(X_{N_c}), X_{N_c})$. Under Assumption 2, it holds that

1) $X_{N_c} \subset X_{N_c} \oplus \mathcal{B}(\bar{d}_{\kappa_f}) \subseteq \mathcal{P}(X_{N_c})$;

2) $\bar{d} \geq \bar{d}_{\kappa_f}$. □

Proof: Notice that, given a vector $x \in X_{N_c} \oplus \mathcal{B}(\bar{d}_{\kappa_f})$, there exist at least one vector $x' \in X_{N_c}$ such that $|x - x'| \leq \epsilon/L_{f_x}$. Since $\hat{f}(x', \kappa_f(x')) \in X_{N_c} \backsim \mathcal{B}(\epsilon)$, with $\kappa_f(x') \in U$, then, by Assumption 2, it follows that $\hat{f}(x, \kappa_f(x')) \in \mathcal{B}(\hat{f}(x', \kappa_f(x')), \epsilon) \subseteq X_{N_c}$, and hence $x \in \mathcal{P}(X_{N_c}), \forall x \in X_{N_c} \oplus \mathcal{B}(\bar{d}_{\kappa_f})$, thus proving the statement. ■

A procedure for the computation of pre-images and predecessor sets, for some classes of nonlinear system, is described in [10]. For a general nonlinear system the exact determination of the predecessor set is a very difficult task, hence in [1] an algorithm for its numerical approximation is presented. Now, the following further assumption is posed.

Assumption 7 (Bound on uncertainties): The $\mathcal{K}$-functions δ and μ are such that following inequality holds

$\delta(x_t) + \mu(||\upsilon||) \leq L_{f_x}^{1-N_c} \bar{d}, \ \forall x_t \in X, \forall \upsilon \in \mathcal{M}_\Upsilon$. □

With respect to previous literature, Assumption 7, for several classes of systems, relaxes the constraint on the maximal admissible uncertainty which the controller can cope with.

Lemma 4.4 (Technical): Given x_t and $x_{t+1} = \hat{f}(x_t, u_t) + d_t$, with u_t given by (12) and $d_t \in D_t$, consider the predictions $\hat{x}_{t+N_c|t}$ and $\hat{x}_{t+N_c+1|t+1}$, obtained respectively using the input sequences $\mathbf{u}^\circ_{t,t+N_c-1|t}$ and $\bar{\mathbf{u}}_{t+1,t+N_c|t}$, and initialized with $\hat{x}_{t|t} = x_t$ and $\hat{x}_{t+1|t+1} = x_{t+1}$. Under Assumption 6, suppose that $\hat{x}_{t+N_c|t} \in X_{N_c}$. If $\delta(|x_t|) + \mu(||\upsilon||) \leq \bar{d}$, , then $\hat{x}_{t+N_c|t+1} \in \mathcal{P}(X_{N_c})$. Moreover, if $\delta(|x_t|) + \mu(||\upsilon||) \leq \bar{d}_{\kappa_f}$, then $\hat{x}_{t+N_c|t+1} \in X_{N_c} \oplus \mathcal{B}(\bar{d}_{\kappa_f})$. □

Proof: Given $x_t \in X_{RH}$, let $\xi \triangleq \hat{x}_{t+N_c|t+1} - \hat{x}_{t+N_c|t}$; then, the following inequality holds [13]

$$|\xi| \leq |\hat{x}_{t+N_c|t+1} - \hat{x}_{t+N_c|t}| \leq L_{f_x}^{N_c-1}(\delta(|x_t|) + \mu(||\upsilon||)) \,.$$

Hence, $\xi \in \mathcal{B}\left(L_{f_x}^{N_c-1}(\delta(|x_t|) + \mu(||\upsilon||))\right)$. Since $\hat{x}_{t+N_c|t} \in X_{N_c}$, in view of Assumption 7, it follows that $\hat{x}_{t+N_c|t} + \xi = \hat{x}_{t+N_c|t+1} \in \mathcal{P}(X_{N_c})$. ■

The following important result can now be proved.

Theorem 4.1 (Feasibility): Let a system be described by equation (6) and subject to (7) and (8). Under Assumptions 2, 3, 6 and 7, the set in which the FHOCP is feasible, X_{RH}, is also RPI for the closed-loop system under the action of the control law given by (12). □

Proof: It will be shown that the region X_{RH} is RPI for the closed-loop system, proving that, for all $x_t \in X_{RH}$, there exists a feasible solution of the FHOCP at time instant $t+1$, based on the optimal solution in t, $\mathbf{u}^\circ_{t,t+N_c-1|t}$, and a possible choice is $\bar{\mathbf{u}}_{t+1,t+N_c|t} = \mathrm{col}\left[u^\circ_{t+1|t}, \ \ldots, u^\circ_{t+N_c-1|t}, \bar{u}\right]$, where $\bar{u} = \bar{u}(\hat{x}_{t+N_c|t+1}) \in U$ is a feasible control action, suitably chosen to satisfy the robust constraint $\hat{x}_{t+N_c+1|t+1} \in X_{N_c}$. The proof is divided in two steps:

1) $\hat{x}_{t+j|t+1} \in X_{t+j|t+1}$: Consider the predictions $\hat{x}_{t+j|t}$ and $\hat{x}_{t+j|t+1}$, with $j \in \{1, \ldots, N_c\}$, made respectively using the input sequences $\mathbf{u}^\circ_{t,t+N_c-1|t}$ and $\bar{\mathbf{u}}_{t+1,t+N_c-1|t}$, and initialized with $\hat{x}_{t|t} = x_t$ and $\hat{x}_{t+1|t+1} = \hat{f}(x_t, \kappa_{RH}(x_t))$. Assuming that $\hat{x}_{t+j|t} \in X_{t+j|t} \triangleq X \backsim \mathcal{B}(\rho_{t+j|t})$, with $\rho_{t+j|t}$ given by (13), let us introduce $\eta \in \mathcal{B}(\rho_{t+j|t+1})$. Let $\xi \triangleq \hat{x}_{t+j|t+1} - \hat{x}_{t+j|t} + \eta$, then under Assumption 2 it follows that

$$\begin{aligned}
|\xi| &\leq |\hat{x}_{t+j|t+1} - \hat{x}_{t+j|t}| + |\rho_{t+j|t+1}| \\
&\leq L_{f_x}^{j-1}(\delta(|x_t|) + \bar{\mu}) + \sigma_j^x(|x_{t+1}|) + \sigma_j^\upsilon(||\upsilon||) + \sigma_j^{x_0}(\bar{x}_0) \\
&\leq L_{f_x}^{j-1}(L_\delta |x_t| + \bar{\mu}) + \frac{L_{\hat{\delta}}^{j-1} - L_{f_x}^{j-1}}{L_{\hat{\delta}} - L_{f_x}} L_\delta (L_{\hat{\delta}}|x_t| + \bar{\mu} + \bar{x}_0) \\
&\quad + \left(\frac{L_{f_x}^{j-1} - 1}{L_{\hat{\delta}} - 1} + \frac{L_{\hat{\delta}}^{j-1} - L_{f_x}^{j-1}}{L_{\hat{\delta}} - L_{f_x}} \frac{L_\delta}{L_{\hat{\delta}} - 1} \right) \bar{\mu} \\
&\quad + \left(\frac{L_{\hat{\delta}}^{j-1} - L_{f_x}^{j-1}}{L_{\hat{\delta}} - L_{f_x}} \frac{L_\delta}{L_{\hat{\delta}} - 1} - \frac{L_{f_x}^{j-1} - 1}{L_{f_x} - 1} \frac{L_\delta}{L_{\hat{\delta}} - 1} \right) \bar{x}_0 \,.
\end{aligned}$$

After some algebra, we have

$$\begin{aligned}
|\xi| &\leq \frac{L_{\hat{\delta}}^{j} - L_{f_x}^{j}}{L_{\hat{\delta}} - L_{f_x}} L_\delta |x_t| + \left(\frac{L_{f_x}^{j} - 1}{L_{\hat{\delta}} - 1} + \frac{L_{\hat{\delta}}^{j} - L_{f_x}^{j}}{L_{\hat{\delta}} - L_{f_x}} \frac{L_\delta}{L_{\hat{\delta}} - 1} \right) \bar{\mu} \\
&\quad + \left(\frac{L_{\hat{\delta}}^{j} - L_{f_x}^{j}}{L_{\hat{\delta}} - L_{f_x}} - \frac{L_{f_x}^{j} - 1}{L_{f_x} - 1} \right) \frac{L_\delta}{L_{\hat{\delta}} - 1} \bar{x}_0 \\
&= \sigma_j^x(|x_t|) + \sigma_j^\upsilon(||\upsilon||) + \sigma_j^{x_0}(\bar{x}_0) = \rho_{t+j|t}
\end{aligned}$$

and hence, $\xi \in \mathcal{B}(\rho_{t+j|t})$. Since $\hat{x}_{t+j|t} \in X_{t+j|t}$, it follows that $\hat{x}_{t+j|t} + \xi = \hat{x}_{t+j|t+1} + \eta \in X, \ \forall \eta \in \mathcal{B}(\rho_{t+j|t+1})$, yielding to $\hat{x}_{t+j|t+1} \in X_{t+j|t+1}$.

2) $\hat{x}_{t+N_c+1|t+1} \in X_{N_c}$: if $L_{f_x}^{N_c-1}(\delta(|x_t|) + \mu(||\upsilon||)) \leq \bar{d}_{\kappa_f}$, in view of Lemma 4.3 there exists a feasible control action such that the statement holds. If $\bar{d}_{\kappa_f} < L_{f_x}^{N_c-1}(\delta(|x_t|) + \mu(||\upsilon||)) \leq \bar{d}$, thanks to Lemma 4.4, it follows that $\hat{x}_{t+N_c|t+1} \in \mathcal{P}(X_{N_c})$. Hence, there exists a feasible control action, namely $\bar{u} \in U$, such that $\hat{x}_{t+N_c+1|t+1} \in X_{N_c}$, thus ending the proof. ■

The uncertainty dependent dichotomy, introduced at Point 2 in the proof of Theorem 4.1, serves to derive, in the following section, the nonlinear stability margin for the closed-loop system with respect of the state-dependent uncertainty.

C. Regional Input to State Stability

In the following, the stability properties of system (6) in closed-loop with (12) are analyzed.

Theorem 4.2 (Regional Input to State Stability): In view of the described RH policy, under Assumptions 2-7, if the stage cost h is such that $\alpha_3(|x_t|) \triangleq \underline{h}(|x_t|) - \varphi_x(|x_t|)$ is a $\mathcal{K}_\infty$-function for all $x_t \in X_{RH}$, with $\varphi_x(|x_t|) \triangleq \left[L_h \frac{L_{f_x}^{N_c} - 1}{L_{f_x} - 1} + (L_h + L_{h_u} L_{\kappa_f}) \frac{L_{f_c}^{N_p - (N_c+1)} - 1}{L_{f_c} - 1} L_{f_x}^{N_c} + L_{h_f} L_{f_c}^{N_p - (N_c+1)} L_{f_x}^{N_c} \right] \delta(|x_t|)$, then system (10) under the action of the RH control law (12) is Regionally ISS in X_{RH}, with respect of $d_t \in D_t$. □

Proof: The proof consists in showing that the optimal RH-cost, $V(x_t)$, is an ISS-Lyapunov function for the closed-loop system in X_{RH}. First, by Assumption 5, an admissible control sequence for FHOCP for any $x_t \in X_{\kappa_f}$ is given by $\tilde{\mathbf{u}}_{t,t+N_p-1|t} \triangleq \mathrm{col}[\kappa_f(\hat{x}_{t|t}),\ \kappa_f(\hat{x}_{t+1|t}),\ \ldots,\ \kappa_f(\hat{x}_{t+N_c-1|t})]$. Then $X_{RH} \supseteq X_{\kappa_f} \supseteq X_f$. In this respect, it will be shown that $V(x_t) = J_{FH}(x_t, \mathbf{u}^\circ_{t,t+N_c-1}, N_c, N_p)$ is an ISS-Lyapunov function in X_{RH}. Suppose[2] that $L_{f_c} \neq 1$, then, in view of Point 5) of Assumption 3, it holds

$$\begin{aligned} V(x_t) &\leq J_{FH}(x_t, \tilde{\mathbf{u}}_{t,t+N_c-1|t}, N_c, N_p) \\ &= \sum_{l=t}^{t+N_p-1}\left[h(\hat{x}_{l|t}, \tilde{u}_{l|t})\right] + h_f(\hat{x}_{t+N_p|t}) \\ &\leq \quad (L_h + L_{h_u}L_{\kappa_f})\frac{L_{f_c}^{N_p}-1}{L_{f_c}-1}|x_t| + L_{h_f}L_{f_c}^{N_p}|x_t|. \end{aligned}$$

Hence, there exists a $\mathcal{K}$-functions $\alpha_2(|x_t|)$ such that

$$V(x_t) \leq \alpha_2(|x_t|),\ \forall x_t \in X_{\kappa_f}. \tag{14}$$

Inequality (4) holds with $\Omega = X_{\kappa_f}$. The lower bound on $V(x_t)$ can be easily obtained using Assumption 4:

$$V(x_t) \geq \underline{h}(|x_t|),\ \forall x_t \in X_{RH}, \tag{15}$$

Now, in view of Assumption 3 and Theorem 4.1, given the optimal control sequence at time t, $\mathbf{u}^\circ_{t,t+N_c-1|t}$, the sequence $\bar{\mathbf{u}}_{t+1,t+N_p|t} \triangleq \mathrm{col}[u^\circ_{t+1|t}, \ldots, u^\circ_{t+N_c-1|t}, \bar{u}, \kappa_f(\hat{x}_{t+N_c+1|t+1}), \ldots, \kappa_f(\hat{x}_{t+N_p|t+1})]$ with

$$\bar{u} = \begin{cases} \kappa_f(\hat{x}_{t+N_c|t+1}), & \text{if } \delta(|x_t|) + \mu(||v||) \leq \bar{d}_{\kappa_f} \\ \bar{u} \in U : \hat{f}(\hat{x}_{t+N_c|t+1}, \bar{u}) \in X_{N_c}, & \\ \qquad\qquad \text{if } \bar{d}_{\kappa_f} < \delta(|x_t|) + \mu(||v||) \leq \bar{d} \end{cases}$$

is an admissible (in general, suboptimal) control sequence for the FHOCP at time $t+1$, with cost

$$\begin{aligned} &J_{FH}(x_{t+1}, \bar{\mathbf{u}}_{t+1,t+N_c|t}, N_c, N_p) \\ &= V(x_t) - h(x_t, u^\circ_{t,t}) + \sum_{l=t+1}^{t+N_c-1}\left[h(\hat{x}_{l|t+1}, u^\circ_{l|t}) - h(\hat{x}_{l|t}, u^\circ_{l|t})\right] \\ &\quad + h(\hat{x}_{t+N_c|t+1}, \bar{u}) - h(\hat{x}_{t+N_c|t}, \kappa_f(\hat{x}_{t+N_c|t})) \\ &\quad + \sum_{l=t+(N_c+1)}^{t+N_p-1}\left[h(\hat{x}_{l|t+1}, \kappa_f(\hat{x}_{l|t+1})) - h(\hat{x}_{l|t}, \kappa_f(\hat{x}_{l|t}))\right] \\ &\quad + h(\hat{x}_{t+N_p|t+1}, \kappa_f(\hat{x}_{t+N_p|t+1})) \\ &\quad + h_f(\hat{f}(\hat{x}_{t+N_p|t+1}, \kappa_f(\hat{x}_{t+N_p|t+1}))) - h_f(\hat{x}_{t+N_p|t}) \end{aligned}$$

Using Assumption 4 and Point 2) of Assumption 3, it follows that, for all $j \in \{1, \ldots, N_c - 1\}$

$$\begin{aligned} &\left|h(\hat{x}_{t+j|t+1}, u^\circ_{t+j|t}) - h(\hat{x}_{t+j|t}, u^\circ_{t+j|t})\right| \\ &\qquad \leq L_h L_{f_x}^{j-1}(\delta(|x_t|) + \mu(||v||)). \end{aligned} \tag{16}$$

For $j = N_c$

$$\begin{aligned} &\left|h(\hat{x}_{t+N_c|t+1}, \bar{u} - h(\hat{x}_{t+N_c|t}, \kappa_f(\hat{x}_{t+N_c|t}))\right| \\ &\leq L_h L_{f_x}^{N_c-1}(\delta(|x_t|) + \mu(||v||)) + L_{h_u}\Delta_u(\delta(|x_t|) + \mu(||v||)), \end{aligned} \tag{17}$$

where

$$\Delta_u(s) \triangleq \begin{cases} 0, & \text{if } s < \bar{d}_{\kappa_f} \\ \max\{|u-w|, \forall (u,w) \in U \times U\}, & \\ & \text{if } \bar{d}_{\kappa_f} \leq s \leq \bar{d}. \end{cases} \tag{18}$$

Finally, for all $j \in \{N_c + 1, \ldots, N_p - 1\}$, the following intermediate result holds

$$\begin{aligned} &\left|h(\hat{x}_{t+j|t+1}, \kappa_f(\hat{x}_{t+j|t+1})) - h(\hat{x}_{t+j|t}, \kappa_f(\hat{x}_{t+j|t}))\right| \\ &\qquad \leq (L_h + L_{h_u}L_{\kappa_f})L_{f_c}^{j-(N_c+1)}\Delta_x(\delta(|x_t|) + \mu(||v||)), \end{aligned} \tag{19}$$

where

$$\Delta_x(s) \triangleq \begin{cases} L_{f_x}^{N_c}(s), & \text{if } s < \bar{d}_{\kappa_f} \\ \sup\{|x-\xi|, \forall (x,\xi) \in \mathcal{P}(X_{N_c}) \times \mathcal{P}(X_{N_c})\}, & \\ & \text{if } \bar{d}_{\kappa_f} \leq s \leq \bar{d}. \end{cases} \tag{20}$$

Considering, for the sake of simplicity, $\delta(|x_t|) + \mu(||v||) \leq \bar{d}_{\kappa_f}, \forall x_t \in X, \forall v \in \mathcal{M}_\Upsilon$, then, in view of Points 3), 5) and 6) of Assumption 3 and by using (16), (17), (18), (19) and (20), the following inequalities hold

$$\begin{aligned} &J_{FH}(x_{t+1}, \bar{\mathbf{u}}_{t+1,t+N_c|t}, N_c, N_p) \\ &\leq V(x_t) - h(x_t, u^\circ_{t,t}) + \sum_{j=1}^{N_c} L_h L_{f_x}^{j-1}(\delta(|x_t|) + \mu(||v||)) \\ &\quad + \sum_{j=N_c+1}^{N_p}(L_h + L_{h_u}L_{\kappa_f})L_{f_c}^{j-(N_c+1)}L_{f_x}^{N_c}(\delta(|x_t|) + \mu(||v||)) \\ &\quad + h(\hat{x}_{t+Np|t+1}, \kappa_f(\hat{x}_{t+Np|t+1})) + h_f(\hat{x}_{t+N_p+1|t+1}) \\ &\quad - h_f(\hat{x}_{t+N_p|t+1}) + L_{h_f}L_{f_c}^{N_p-(N_c+1)}L_{f_x}^{N_c}(\delta(|x_t|) + \mu(||v||)) \\ &\leq V(x_t) - h(x_t, u^\circ_{t,t}) + \varphi_x(|x_t|) + \varphi_v(||v||), \end{aligned}$$

where $\varphi_v(||v||) \triangleq \Big[L_h \frac{L_{f_x}^{N_c}-1}{L_{f_x}-1} + (L_h + L_{h_u}L_{\kappa_f})\frac{L_{f_c}^{N_p-N_c-1}-1}{L_{f_c}-1}L_{f_x}^{N_c} + L_{h_f}L_{f_c}^{N_p-(N_c+1)}L_{f_x}^{N_c}\Big]\mu(||v||)$ is a $\mathcal{K}$-function $\forall x_t \in X_{RH}$, $\forall v \in \mathcal{M}_\Upsilon$. Now, from inequality $V(x_{t+1}) \leq J_{FH}(x_{t+1}, \bar{\mathbf{u}}_{t+1,t+N_c}, N_c, N_p)$ it follows that

$$V(x_{t+1}) - V(x_t) \leq -\alpha_3(|x_t|) + \sigma(||v||), \tag{21}$$

where $\alpha_3(|x_t|) \triangleq \underline{h}(|x_t|) - \varphi_x(|x_t|)$ and $\sigma(||v||) \triangleq \varphi_v(||v||)$, $\forall x_t \in X_{RH}$, with $v \in \mathcal{M}_\Upsilon$, N_c and N_p fixed.

In view of Point 6) of Assumption 3, Assumption 5 and Assumption 7, it follows from Theorem 4.1 that X_{RH} is a RPI set for system (10) under the action of the control law (12) and $d_t \in D_t$. Further, by (14), (15), (21), the optimal cost $J_{FH}(x_t, \mathbf{u}^\circ_{t,t+N_p-1|t}, N_c, N_p)$ is an ISS-Lyapunov function for the closed-loop system, and hence the closed-loop system is Regionally ISS in X_{RH}, with respect of $d_t \in D_t$. ■

Remark 4.1 (Nonlinear stability margin): Disregarding at this stage the non-state-dependent uncertainties, it must be noticed that any $\mathcal{K}$-function $\delta(|x_t|)$ which satisfies $\delta(|x_t|) \leq \bar{d}_{\kappa_f}, \forall x_t \in X$, and such that $\underline{h}(|x_t|) - \varphi_x(|x_t|)$ is a $\mathcal{K}_\infty$-function in turn is a nonlinear stability margin for the closed-loop system with respect of state-dependent uncertainties in the transition function. In fact, for all state-dependent uncertainties, $d_t = d(t, x_t)$, bounded by δ (i.e. $d(t, x_t)$: $|d(t, x_t)| \leq \delta(|x_t|)$), the ISS inequality (21) holds with $\alpha_3(|x_t|) \triangleq \underline{h}(|x_t|) - \varphi_x(|x_t|)$ a $\mathcal{K}_\infty$-function. From ISS inequalities (14), (15) and (21), following the constructive proof of [5], it is possible to derive a $\mathcal{KL}$-function β for the closed-loop system, hence it is 0-AG in X_{RH}. Notice that the stability margin depends on $\bar{d}_{\kappa_f}$, i.e. it is related to the particular choice of the set X_{N_c} and on the auxiliary control law κ_f. Hence, in general, it can be used only to qualitatively analyze the asymptotic behavior of the closed-loop system under state dependent uncertainty. □

V. EXAMPLE

Consider the following discrete-time model of an undamped nonlinear oscillator

$$\begin{cases} x_{(1)_{t+1}} = x_{(1)_t} + 0.05\left[-x_{(2)_t} + 0.5\left(1 + x_{(1)_t}\right)u_t\right] \\ x_{(2)_{t+1}} = x_{(2)_t} + 0.05\left[x_{(1)_t} + 0.5\left(1 - 4x_{(1)_t}\right)u_t\right], \end{cases} \tag{22}$$

where the subscript (i) denotes the i-th component of a vector. System (22) is subject to state and input constraints

[2]The very special case $L_{f_c} = 1$ can be trivially addressed by a few suitable modifications to the proof of Theorem 4.2.

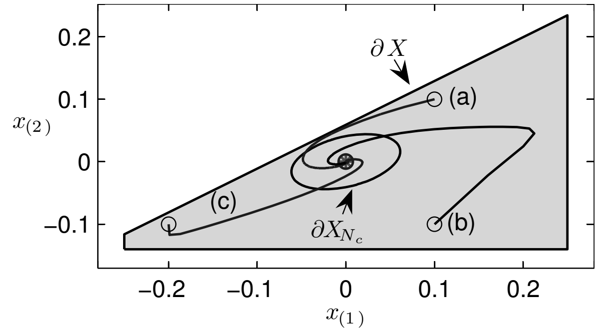

Fig. 1. Sample closed-loop trajectories with initial points: (a)=(0.1,0.1)^T, (b)=(0.1,-0.1)^T, (c)=(-0.2,-0.1)^T. In evidence the state constraint set X and the robust constraint set X_{N_c}.

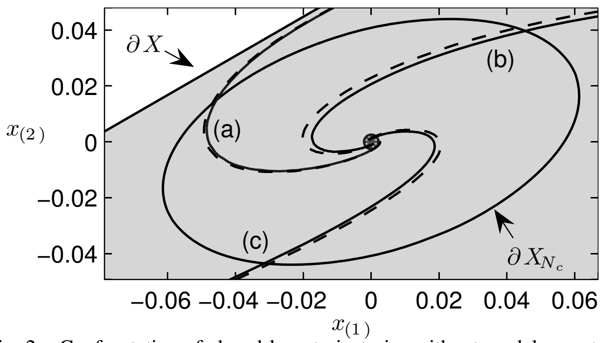

Fig. 2. Confrontation of closed-loop trajectories without model uncertainty (solid) and with state-dependent uncertainty (dashed).

(7) and (8). The set X is depicted in Figure 1, while $U \triangleq \{u \in \mathbb{R} : |u| \leq 2\}$. Given X and U, the Lipschitz constant of the system is $L_{f_x} = 1.1390$. Since affordable algorithms exist for the numerical computation of the Pontryagin difference set of polytopes, for implementation purposes the balls to be subtracted (in the Potryagin sense) from the constraint X to obtain $X_{t+j|t}$, $\forall j \in \{1, \ldots, N_c\}$ are outer approximated by convex parallelotopes.

A linear state feedback control law $u_t = \kappa_f(x_t) = k^T \cdot x_t$, with $k \in \mathbb{R}^2$, stabilizing (22) in a neighborhood of the origin, can be designed as described in [9]. Choosing $k = [0.74 \;\; 1.80]^T$, the following ellipsoidal set, $X_f \in X$, is RPI under the dynamics (22) in closed-loop with $\kappa_f(x_t)$

$$X_f \triangleq \left\{ x_t \in \mathbb{R}^n \, : \, x_t^T \cdot \begin{bmatrix} 309.21 & -162.53 \\ -162.53 & 602.72 \end{bmatrix} \cdot x_t \leq 1 \right\}.$$

In X_f, the auxiliary control law satisfies Points 2) and 3) of Assumption 3, with L_{κ_f}=1.95 and L_{f_c}=1.10. Let the stage cost h be given by $h(x, u) \triangleq x^T \cdot \mathrm{Q} \cdot x + u^T \cdot \mathrm{R} \cdot u$, and the final cost h_f by $h_f(x) \triangleq x^T \cdot P \cdot x$ with

$$\mathrm{Q} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \mathrm{R} = \begin{bmatrix} 1 \end{bmatrix}, \mathrm{P} = \begin{bmatrix} 171.46 & -90.12 \\ -90.12 & 334.21 \end{bmatrix},$$

then h_f satisfies Points 2) and 6) of Assumption 3 with $L_{h_f} = 19.35$. In addition, let $N_p = N_c = 8$. Hence, in view of Assumption 6, it is possible to choose $X_{N_c} = X_f$, for which the following values for $\bar{d}_{\kappa_f}$ and $\bar{d}$ are given: $\bar{d}_{\kappa_f} = 3.38 \cdot 10^{-4}$ and $\bar{d} = 1.01 \cdot 10^{-3}$. It follows that the admissible uncertainties, for which the feasibility set X_{RH} is RPI under the closed-loop dynamics, are bounded by

$$\delta(|x_t|) + \mu(||\upsilon||) \leq 4.06 \cdot 10^{-4}, \;\; \forall x_t \in X, \forall \upsilon \in \mathcal{M}_\Upsilon.$$

Figure 2 shows the closed-loop trajectories of the system without uncertainties (solid) and with state-dependent uncertainty given by d_t=$8.22 \cdot 10^{-4} x_t$. Notice that also in presence of model mismatch the system is asymptotically stable in closed-loop with the proposed controller.

VI. Conclusion

In this paper, a robust RH controller for constrained discrete-time nonlinear systems with state-dependent and non-state dependent uncertainties is presented. In particular, under suitable assumptions, the robust constraints satisfaction is guaranteed for the considered class of uncertainties, employing a constraint tightening technique. Further, the closed-loop system under the action of the RH control law is shown to be Input to State state stable under the considered class of uncertainties. Finally, a nonlinear stability margin with respect of state dependent uncertainties is given.

Future research efforts will be devoted to further increase the degree of robustness of the RH control law, to enlarge the class of uncertainties, to allow for less conservative results and addressing the unavoidable approximation errors involved in the computation of the optimal control actions.

Acknowledgment

The authors wish to thank Prof. Franco Blanchini for many helpful comments and suggestions.

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