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Stable Controller Interpolation for LPV Systems

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Abstract— **This paper examines the gain-scheduling problem with a particular focus on controller interpolation with guaranteed nonlinear stability. For Linear Parameter Varying model representations, a method of interpolating between controllers utilizing the Youla parameterization is proposed. Quadratic stability despite fast scheduling is guaranteed by construction, while the characteristics of individual controllers designed a priori are recovered at the design points.**

I. Introduction

Many physical systems exhibit dynamics sufficiently nonlinear that a single linear controller may fail to achieve acceptable performance throughout the envelope of conditions. Gain-scheduling is one of the most popular approaches in industry for controlling a nonlinear system, often by interpolating a family of local controllers, thus dividing the nonlinear control design problem into several smaller problems where linear design tools are employed [1].

The principal challenge facing gain-scheduling research is guaranteeing stability of the nonlinear closed loop system. The simplicity in *design*, where linear controllers and ad hoc interpolation methods are used, is contrasted with difficulties in *analysis*, where guaranteeing the stability of the resulting nonlinear closed loop system can be extremely challenging. Moreover, the presence of "hidden coupling" terms or "scheduling dynamics" due to the interpolation functions [2] can create unanticipated stability problems. This paper proposes a method of controller blending that guarantees quadratic stability of the closed loop system, while recovering the local controller designs at specified operating conditions. This approach utilizes the the Youla Parameterization [3]. For polytopic LPV systems the framework for interpolation is constructed utilizing a finite set of Linear Matrix Inequalities (LMIs) [4].

The proposed method is demonstrated using the Quadruple Tank System [5]. Quasi-LPV models for the plant are presented, and a variety of local controllers are constructed. Utilizing the techniques proposed in this paper, a gain scheduled controller is constructed that guarantees stable interpolation, despite local controllers that are open loop unstable and differ in structure.

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The remainder of the paper is outlined as follows. Section 2 presents necessary background on gain scheduled control and the Youla parameterization. In section 3, the interconnection of an LPV plant model with a Local Controller Network is examined, and necessary conditions for stability are identified. Section 4 shows an alternative formulation is proposed wherein quadratic stability is guaranteed, and recovery of the local controllers at design points is assured. Section 5 provides an illustrative example where dissimilar controllers are blended using the proposed framework and simulated on the quadruple tank system.

II. Background

A rough categorization of gain scheduling would include 1) gain scheduling "the LPV way" [6] and 2) local controller interpolation. The former method provides some guarantees of closed loop stability, albeit often with slowly varying scheduling parameter assumptions. However, a solution to the controller synthesis problem may not exist or be computationally feasible. The latter uses local linear models, either linearized first principles models or data-driven identified models, and local linear controllers constructed using any linear control design tools. Thus the controllers at critical design points can be tuned to achieve high performance, and the gain scheduling problem is reduced to one of interpolation to ensure stable transitions between critical design points. Another advantage of this method is the ability to gain-schedule between controllers with different sizes and structures by virtue of output blending.

Despite the simplicity of the design methodology, guaranteeing stability of the closed loop system with an interpolated controller can pose a challenging problem. Consider the following example: Let a plant and two stabilizing controllers be defined as in Eq. 1. An interpolated controller could be defined as in Eq. 2 where $\alpha \in [0,1]$. Although both K_1 and K_2 stabilize the plant, the blended controller K_b destabilizes the plant for the majority of the intermediate values: $\alpha \in [0.25,\ 1]$.

$$P(s) = \frac{1}{s+1},\ K_1(s) = \frac{-1}{s+0.5},\ K_2(s) = \frac{-1}{s-0.5} \quad (1)$$

$$K_b(s) = \alpha K_1 + (1-\alpha) K_2 \quad (2)$$

Interpolation methods that guarantee stability for any fixed value of the scheduling parameter, known as *frozen parameter stability*, have been termed "stability-preserving" interpolation methods [7], [8]. Recent research has focused on guaranteeing this level of stability by design [9],[10].

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However, in this analysis, no scheduling dynamics are considered and global stability can only be inferred by assuming slowly varying scheduling variables.

The interpolation method used in the previous example, has been termed a Local Controller Network (LCN) [11] [12]. Under the LCN framework (Figure 1), the nonlinear controller is formed by the weighted sum of the outputs of individual linear controllers. These weighting or blending functions are a function of a scheduling variable ρ, as $\alpha = h(\rho)$. While selection of the scheduling variables is based on a physical understanding of which variables most accurately capture the system nonlinearities, and is thus situation dependent, selection of the weighting functions is by design. In general the weighting functions are designed such that $\alpha_i \in [0,1]$ and $\sum \alpha_i = 1$, with the magnitude based on the relative distance to the respective design point in the scheduling space. We will denote the LCN as $K_\alpha(s) = \sum \alpha_i K_i(s)$.

However, as demonstrated by the previous example, this method may fail for simple cases. A generalization of this approach involves is based on the dual Youla parameterization [13], [14] and is formed as follows. First, each local controller K_i and each local plant P_i may be decomposed into coprime factors $K_i(s) = U_i V_i^{-1} = \tilde{V}_i^{-1}\tilde{U}_i$ and $P_i(s) = N_i M_i^{-1} = \tilde{M}_i^{-1}\tilde{N}_i$, such that $U_i, \tilde{U}_i, V_i, \tilde{V}_i \in RH_\infty$ and $N_i, \tilde{N}_i, M_i, \tilde{M}_i \in RH_\infty$. Then selecting a nominal plant model $P_0 (= N_0 M_0^{-1})$ and controller $K_0 (= U_0 V_0^{-1})$ such that K_0 stabilizes P_0, the set of all controllers that stabilize P_0 can be parameterized in terms of a Youla parameter Q as $K(Q) = (U_0 + M_0 Q)(V_0 + N_0 Q)^{-1}$, where $Q \in RH_\infty$. Thus if P_0 is stabilized by each local controller K_i, then there exist Youla parameters given by Q_i such that $K_i = K(Q_i)$. These Youla parameters are given as $Q_i = \tilde{U}_i V_0 - \tilde{V}_i U_0$ where the coprime factors are selected such that they satisfy the Bezout identity (Eq. 3). The blending of local controllers is then replaced by the blending of Youla parameters Q_i (Fig. 1) and can be represented as the lower fractional transformation (LFT) of fixed dynamic system J_K (Eq. 4) and Local Q-Network (LQN).

$$\tilde{V}_i M_0 - \tilde{U}_i N_0 = I \tag{3}$$

$$J_K(s) = \begin{bmatrix} U_0 V_0^{-1} & \tilde{V}_0^{-1} \\ V_0^{-1} & -V_0^{-1} N_0 \end{bmatrix} \tag{4}$$

$$K(Q_\alpha) = \left[U_0 + M_0\left(\sum \alpha_i Q_i\right)\right]\left[V_0 + N_0\left(\sum \alpha_i Q_i\right)\right]^{-1} \tag{5}$$

Variations of this approach have been termed *J-Q* interpolation [9], or blending of the Youla parameters [15]. By virtue of the Youla parameterization, this framework permits the scheduling of unstable controllers [16]. Moreover, this framework has the intuitive appeal of isolating common controller elements in the function J_K and blending only the differences between the individual controllers. Note that at $\alpha_i = 1$ the original local controller K_i is recovered. Moreover, because $K(Q)$ stabilizes P_0 for any $Q \in RH_\infty$, then $K(Q_\alpha)$ also stabilizes P_0 for every frozen value of α, since each $Q_i \in RH_\infty$ and thus $\sum \alpha_i Q_i \in RH_\infty$, something not necessarily guaranteed with the LCN framework. As needed, the nonlinear plant can be similarly characterized by a network of dual Youla parameters, S_i [17].

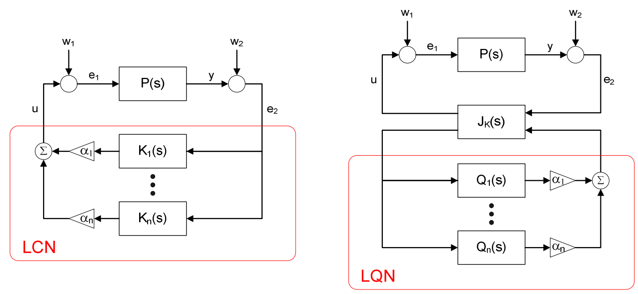

Figure 1: Output Blending of Local Controller Network (LCN) and Local Q-Network (LQN)

Gain scheduling using a Youla parameter based framework offers greater stability than the typical LCN approach. In LQN framework, a local controller can be recovered by LFT $F_l(J_K, Q_i) = K_i$, and a simple quadratic Lyapunov function can be found to guarantee arbitrarily fast transitions between the two Youla parameters. Thus while a simple blending of controllers would not stabilize the plant at fixed intermediate points, the blending of equivalent Youla parameters guarantees frozen parameter stability by construction, and stability under fast transitions can be verified with a common Lyapunov function, while previous works employ the rate limits of scheduling parameters [9].

III. Classical Controller Interpolation for LPV Systems

When nonlinear system models are constructed using first principles, the state variables generally remain tied to system physics. This naturally leads to Linear Parameter Varying (LPV) models where linear models at different operating points "share" the same state variables, and the state space system matrices are parameterized in terms of the scheduling variable, θ:

$$\begin{aligned} \dot{x}_p &= A_p(\theta)x_p + B_p(\theta)u \\ y &= C_p(\theta)x_p \end{aligned} \tag{6}$$

This is in contrast with a set of controllers, defined *a priori,* where there is no physical relationship between state variables. In this case, the Local Control Network (LCN) representation is more appropriate. Assuming individual controllers are represented in state space form as:

$$\begin{aligned}\dot{x}_{ki} &= A_{ki}x_{ki} + B_{ki}e \\ u_{ki} &= C_{ki}x_{ki}\end{aligned} \tag{7}$$

The full LCN can be constructed as:

$$\begin{bmatrix}\dot{x}_{k1}\\ \vdots \\ \dot{x}_{kn}\end{bmatrix} = \underbrace{\begin{bmatrix}A_{k1} & & 0\\ & \ddots & \\ 0 & & A_{kn}\end{bmatrix}}_{A_K}\underbrace{\begin{bmatrix}x_{k1}\\ \vdots \\ x_{kn}\end{bmatrix}}_{x_K} + \underbrace{\begin{bmatrix}B_{k1}\\ \vdots \\ B_{kn}\end{bmatrix}}_{B_K}e$$

$$u = \underbrace{\begin{bmatrix}\alpha_1 C_{k1} & \cdots & \alpha_n C_{kn}\end{bmatrix}}_{C_K(\alpha)}\begin{bmatrix}x_{k1}\\ \vdots \\ x_{kn}\end{bmatrix} + \underbrace{\begin{bmatrix}\sum_{i=1}^{n}\alpha_i D_{ki}\end{bmatrix}}_{D_K(\alpha)}e \tag{8}$$

Assuming a polytopic representation of the LPV plant and blending functions designed such that when $\theta=\theta_i$:

$$\alpha_i = 1 \quad \text{and} \quad \alpha_j = 0 \quad \forall \quad j \neq i, \tag{9}$$

a sufficient condition for stability of the closed loop system is the existence of a common quadratic Lyapunov function. This can be checked using the finite set of Linear Matrix Inequalities (LMIs):

$$PA_{CL,LPV}(\theta_i) + A_{CL,LPV}(\theta_i)^T P < 0 \quad \text{where} \tag{10}$$

$$A_{CL,LPV} = \begin{bmatrix}A_P(\theta_i) + B_P(\theta_i)D_K(\alpha)C_P(\theta_i) & B_P(\theta_i)C_K(\alpha)\\ B_K C_P(\theta_i) & A_K(\alpha)\end{bmatrix} \tag{11}$$

When this polytopic system is evaluated at its vertices, the state matrix in Eq. 11 assumes an upper block triangular structure. By inspection, we may conclude that a necessary precondition for stability is that each controller must be open loop stable. Moreover, we note that the existence of a common Lyapunov function may be computationally elusive, particularly for a large set of controllers. However, using the Youla parameterization as an alternate framework for controller interpolation is possible with less restrictive conditions and with guaranteed stability.

IV. Proposed Controller Interpolation for LPV Systems using the Youla Parameterization

To guarantee stable interpolation between controllers we adopt a similar approach presented in [18] that employs the LPV controller interpolation by optimizing Q-parameter in L_2 – gain performance. However, instead of focusing on LPV controller synthesis, we will exploit particular properties to guarantee stable controller interpolation under LQN framework. First, an LPV observer-based controller, $K_0(\theta)$, for the LPV plant is constructed as:

$$\begin{aligned}\dot{x}_k &= \left[A_p(\theta) + B_p(\theta)F_p(\theta) + H(\theta)C_p(\theta)\right]x_K - H_p(\theta)y \\ u &= F_p(\theta)x_K\end{aligned}, \tag{12}$$

where the feedback and observer gains are calculated as

$$F_p(\theta) = X(\theta)P_f^{-1} \quad \text{and} \quad H_p(\theta) = P_h^{-1}W(\theta), \tag{13}$$

such that the following Linear Matrix Inequalities (LMIs) are satisfied:

$$A_P(\theta)P_f + P_f A_P(\theta)^T + B(\theta)X(\theta) + X(\theta)^T B(\theta)^T < 0, \tag{14}$$

$$P_h A_P(\theta) + A_P(\theta)^T P_h + W(\theta)C(\theta) + C(\theta)^T W(\theta)^T < 0. \tag{15}$$

For polytopic LPV systems, these conditions can be written as a finite set of LMIs.

A doubly coprime factorization satisfying the Bezout identities for the LPV plant and nominal LPV controller, $K_0(\theta)$, can be constructed as:

$$\begin{bmatrix}M(\theta) & U_0(\theta)\\ N(\theta) & V_0(\theta)\end{bmatrix} = \left[\begin{array}{c|cc}A_P(\theta) + B_P(\theta)F_P(\theta) & B_P(\theta) & -H_P(\theta)\\ \hline F_P(\theta) & I & 0\\ C_P(\theta) & 0 & I\end{array}\right] \tag{16}$$

$$\begin{bmatrix}\tilde{V}_0(\theta) & -\tilde{U}_0(\theta)\\ -\tilde{N}(\theta) & \tilde{M}(\theta)\end{bmatrix} = \left[\begin{array}{c|cc}A_P(\theta) + H_P(\theta)C_P(\theta) & B_P(\theta) & -H_P(\theta)\\ \hline F_P(\theta) & I & 0\\ C_P(\theta) & 0 & I\end{array}\right] \tag{17}$$

The resulting closed loop LPV system is guaranteed to be quadratically stable by construction, but to recover the local controller behavior at each operating condition, we create a Local Q-Network (LQN) formed by Youla parameters, constructed as follows. Dropping the notation $A_P(\theta_i)$ for the more compact $A_{P,i}$, the coprime factorization for the LPV plant at the "i[th]" operating condition and the *a priori* controller, K_i, is:

$$\begin{bmatrix}M(\theta_i) & U_{Ki}\\ N(\theta_i) & V_{Ki}\end{bmatrix} = \left[\begin{array}{cc|cc}A_{Pi} + B_{Pi}F_{Pj} & 0 & B_{Pj} & 0\\ 0 & A_{Ki} + B_{Ki}F_{Ki} & 0 & B_{Ki}\\ \hline F_{Pj} & C_{Ki} + D_{Ki}F_{Ki} & I & D_{Ki}\\ C_{Pj} & F_{Ki} & 0 & I\end{array}\right] \tag{18}$$

$$\begin{bmatrix}\tilde{V}_{Ki} & -\tilde{U}_{Ki}\\ -\tilde{N}(\theta_i) & \tilde{M}(\theta_i)\end{bmatrix} = \left[\begin{array}{cc|cc}A_{Pi} + B_{Pi}D_{Ki}C_{Pi} & B_{Pi}C_{Ki} & -B_{Pi} & B_{Pi}D_{Ki}\\ B_{Ki}C_{Pi} & A_{Ki} & 0 & B_{Ki}\\ \hline F_{Pi} - D_{Ki}C_{Pi} & -C_{Ki} & I & -D_{Ki}\\ C_{Pi} & -F_{Ki} & 0 & I\end{array}\right] \tag{19}$$

We define the individual Youla parameters as:

$$Q_i = -\begin{bmatrix}\tilde{V}_{Ki} & -\tilde{U}_{Ki}\end{bmatrix}\begin{bmatrix}U_0(\theta_i)\\ V_0(\theta_i)\end{bmatrix}$$

$$= \left[\begin{array}{ccc|c}A_{Pi} + B_{Pi}D_{Ki}C_{Pi} & B_{Pi}C_{Ki} & -B_{Pi}(F_{Pi} + D_{Ki}C_{Pi}) & B_{Pi}D_{Ki}\\ B_{Ki}C_{Pi} & A_{Ki} & B_{Ki}C_{Pi} & B_{Ki}\\ 0 & 0 & A_{Pi} + B_{Pi}F_{Pi} & -H_{Pi}\\ \hline D_{Ki}C_{Pi} - F_{Pi} & C_{Ki} & D_{Ki}C_{Pi} - F_{Pi} & D_{Ki}\end{array}\right] \tag{20}$$

The Local Q-Network is defined similar to Eq. 8, and is implemented as illustrated in Fig. 2, with the system $J_K(\theta)$ constructed from the coprime factors of the LPV plant and nominal LPV controller (Eq. 21). Quadratic stability of the closed loop system can be established without any restrictions on the rate of change in the scheduling variable. We observe that quadratic stability of each Youla parameter as given in Eq. 20 is guaranteed by construction.

$$J_K(\theta)=\begin{bmatrix} U_0(\theta)V_0(\theta)^{-1} & \tilde{V}_0(\theta)^{-1} \\ V_0(\theta)^{-1} & -V_0(\theta)^{-1}N_0(\theta)\end{bmatrix}$$
$$=\left[\begin{array}{c|cc} A_P(\theta)+B_P(\theta)F_P(\theta)+H_P(\theta)C_P(\theta) & -H_P(\theta) & B_P(\theta) \\ \hline F_P(\theta) & 0 & I \\ -C_P(\theta) & I & 0 \end{array}\right] \quad (21)$$

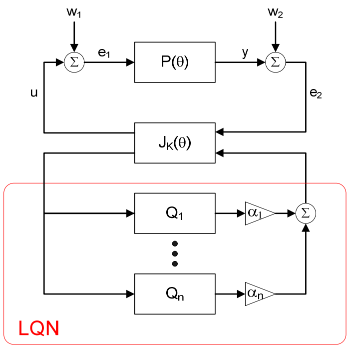

Figure 2: Output Blending of Local Q-Network (LQN)

Note that the Youla parameter's state matrix is upper block-triangular (Eq. 20) and state matrix of the resulting closed loop system is also upper block-triangular (Eq. 23). The upper left portion of the state matrix is the closed loop dynamics between the plant model at the i^th condition and the i^th controller, guaranteed quadratically stable by construction. The lower portion consists of the dynamics of the i^th plant under state feedback control, and is also guaranteed stable by construction via Eq. 14. Thus we conclude that for each Youla parameter Q_i, there must exist a positive definite P_{Qi} such that:

$$A_{Qi}P_{Qi}+P_{Qi}A_{Qi}^{T}<0 \quad (22)$$

where A_{Qi} is defined consistent with Eq. 20.

The state matrix of the resulting closed loop system is given by:

$$A_{CL,LQN}=\begin{bmatrix} A_P(\theta)+B_P(\theta)F_P(\theta) & B_P(\theta)\left(F_P(\theta)-D_{LQN}C_P(\theta)\right) & B_P(\theta)C_{LQN} \\ 0 & A_P(\theta)+H_P(\theta)C_P(\theta) & 0 \\ 0 & -B_{LQN}C_P(\theta) & A_{LQN}+B_{Pi}F_{Pi}\end{bmatrix} \quad (23)$$

It can be shown [18] that there exists constants λ_f and λ_h such that the matrix:

$$\begin{bmatrix} \lambda_f P_f & 0 & 0 \\ 0 & \lambda_h P_h & 0 \\ 0 & 0 & P_Q \end{bmatrix} \quad (24)$$

is a quadratically stabilizing common Lyapunov function for the closed loop system defined by Eq. 23.

To show that we do in fact recover the controller K_i at the i^th operating condition, we explore the lower fractional transformation of $J_K(\theta)$ and Q_i, while once again dropping the notation $U_0(\theta_i)$ for the simpler $U_{0,i}$:

$$F_{LFT}(J_K(\theta_i),Q_i)=U_{0,i}V_{0,i}^{-1}+\tilde{V}_{0,i}^{-1}Q_i\left(I+V_{0,i}^{-1}N_{0,i}Q_i\right)^{-1}V_{0,i}^{-1} \quad (25)$$

An equivalent expression is given by:

$$F_{LFT}(J_K(\theta_i),Q_i)=\left(U_{0,i}+M_{0,i}Q_i\right)\left(V_{0,i}+N_{0,i}Q_i\right)^{-1} \quad (26)$$

Substituting the definition of the Youla parameters Q_i and simplifying yields:

$$F_{LFT}(J_K(\theta_i),Q_i)=\left(U_{0,i}+M_{0,i}\tilde{U}_{Ki}V_{0,i}-M_{0,i}\tilde{V}_{Ki}U_{0,i}\right)\cdot \left(V_{0,i}+N_{0,i}\tilde{U}_{Ki}V_{0,i}-N_{0,i}\tilde{V}_{Ki}U_{0,i}\right)^{-1} \quad (27)$$

Using the Bezout identities to replace selected sections

$$F_{LFT}(J_K(\theta_i),Q_i)=\left(U_{0,i}+\left(U_{Ki}\tilde{M}_{0,i}\right)V_{0,i}-\left(I+U_{Ki}\tilde{N}_{0,i}\right)U_{0,i}\right)\cdot \left(V_{0,i}+\left(V_{Ki}\tilde{M}_{0,i}-I\right)V_{0,i}-\left(V_{Ki}\tilde{N}_{0,i}\right)U_{0,i}\right)^{-1} \quad (28)$$

and collecting terms and simplifying, results in recovery of the *a priori* designed controller.

$$F_{LFT}(J_K(\theta_i),Q_i)=\left(U_{Ki}\left(\tilde{M}_{0,i}V_{0,i}-\tilde{N}_{0,i}U_{0,i}\right)\right)\left(V_{Ki}\left(\tilde{M}_{0,i}V_{0,i}-\tilde{N}_{0,i}U_{0,i}\right)\right)^{-1}$$
$$=U_{Ki}V_{Ki}^{-1}=K_i \quad (29)$$

V. Gain-Scheduled Control of a Quadruple Tank System

To demonstrate efficacy of the gain-scheduling framework, a simulated quadruple tank system is selected. This system is a well known multivariable controls example and has been discussed in detail in [5]. A schematic diagram of the system is shown in Fig. 3. The two inputs to the system are the input voltages to pumps 1 and 2, and two outputs of interest are the fluid levels in tanks 1 and 2. Two valves divide the flow from each of the pumps to the upper and lower tanks. The upper tanks (tanks 3 and 4) drain into the lower tanks (tanks 1 and 2) which drain into a reservoir. The cross flow from pump 1 to tank 4 and from pump 2 to tank 3 creates interesting dynamic phenomena.

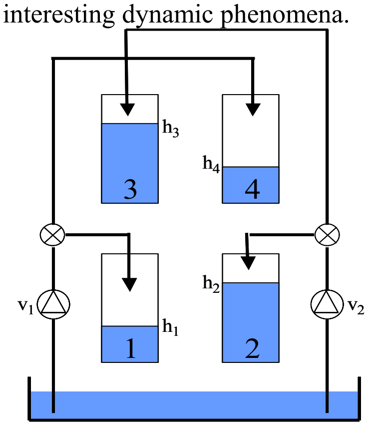

Figure 3: Diagram of a Quadruple Tank System

This system can be modeled using mass balances and Bernoulli's law. The resulting nonlinear model is given in Eq. 30 where A is the tank cross-sectional area, a is the orifice cross-sectional area, h is the fluid level, u is the pump input with a scalar gain k_u. The valve parameters $\gamma\in[0,1]$ determine the flow to each tank. The selected outputs are the fluid levels of tanks 1 and 2 and are measured with a scalar gain k_y. The linearized version of this model is determined by Jacobian linearization given in [5]. This system was chosen for several notable reasons.

$$
\begin{aligned}
A_1\dot{h}_1 &= -a_1\sqrt{2gh_1} + a_3\sqrt{2gh_3} + \gamma_1 k_{u1}u_1 \\
A_2\dot{h}_2 &= -a_2\sqrt{2gh_2} + a_4\sqrt{2gh_4} + \gamma_2 k_{u2}u_2 \\
A_3\dot{h}_3 &= -a_3\sqrt{2gh_3} + (1-\gamma_2)k_{u2}u_2 \\
A_4\dot{h}_4 &= -a_4\sqrt{2gh_4} + (1-\gamma_1)k_{u1}u_1
\end{aligned}
\qquad (30)
$$

First, this is a well known multivariable controls example with a validated modeling approach available in the literature. Second, the poles of the system depend strongly on the nominal fluid height in the tanks; thus as the fluid heights, the system dynamics change significantly. Third, the system has a multivariable zero that can be arbitrarily placed in the right or left half plane.

A slight variation of the parameter values published in [5] is used for the simulations presented here. The values of tank and orifice areas, and input/output scaling depends on the units used are given in Table 1. The gravity is given as 9.81 [m/s^2], and the steady state values at the chosen operating conditions are given in Table 2.

Table 1: Tank and Orifice Areas [m^2] / Input/Output Scaling

A_1	2.8E-03	a_1	7.1E-06	k_u	V	3.33E-06
A_2	3.2E-03	a_2	5.7E-06		m^3/s	1.0
A_3	2.8E-03	a_3	7.1E-06	k_y	m	1.0
A_4	3.2E-03	a_4	5.7E-06		Pa	9.81E+03

Table 2: Chosen Operating Condition (minimum phase, nonminimum phase)

h_1^0	(0.12, 0.12) [m]	u_1^0	(2.44, 3.80) [V]
h_2^0	(0.12, 0.12) [m]	u_2^0	(3.80, 2.44) [V]
h_3^0	(0.081, 0.037) [m]	γ_1	(0.7, 0.4) [-]
h_4^0	(0.052, 0.014) [m]	γ_2	(0.6, 0.3) [-]

The valve parameters γ_1 and γ_2 determine the flow ratio of lower to upper tank. Low values of γ signify a significant amount of cross-flow, thus resulting in nonminimum phase behavior. In this case a multivariable right half plane zero will be present when $\gamma_1 + \gamma_2 < 1$, as depicted in Fig. 4. For this example γ_1 and γ_2 are selected as the scheduling variables. External changes to these variables will change the underlying system dynamics, as well as a disturbance to the closed loop system attempting to regulate the fluid height of the lower tanks. Two operating points in a quasi-LPV model [18] are selected: one is in the minimum phase region and the other in the nonminimum phase region (Fig. 4). As advocated in [5], a decoupled PID controller is designed for the minimum phase condition. A steady state decoupling matrix $W(s) = G^{-1}(0)$ is used, and PID controllers are designed (Eq. 31-32).

For the nonminimum phase operating point, [5] suggests the use of an $\mathcal{H}_\infty$ controller. Using standard design and model reduction procedures, a 4th order $\mathcal{H}_\infty$ controller is designed (Eq. 33). Both controllers perform adequately around their respective design points, and are easily able to track reference changes in the desired fluid height (Fig. 5).

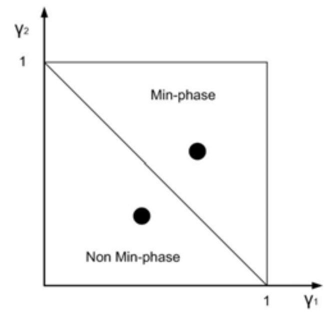

Figure 4: Controller design points in minimum and non-minimum phase region due to gamma values

Although the second closed loop system displays significant undershoot, this is to be expected given the strong nonminimum phase nature of the plant.

$$
K_{PID}(s) = \begin{bmatrix} 1.26 + \dfrac{0.042}{s} + 0.019s & 0 \\ 0 & 1.29 + \dfrac{0.029}{s} + 0.028s \end{bmatrix} \qquad (31)
$$

$$
W(s) = \begin{bmatrix} 0.54064 & -0.28962 \\ -0.27032 & 0.50204 \end{bmatrix} \qquad (32)
$$

$$
H_\infty = \left[\begin{array}{c|c} A_{H_\infty} & B_{H_\infty} \\ \hline C_{H_\infty} & D_{H_\infty} \end{array}\right] \qquad (33)
$$

$$
= \left[\begin{array}{cccc|cc}
-0.0013 & 0.0275 & 0.0027 & -0.0002 & 0.1200 & -0.0120 \\
-0.0037 & -0.1872 & -0.2249 & 0.0048 & -0.8053 & 1.0630 \\
-0.0027 & 0.2248 & -0.0001 & -0.0013 & 0.0231 & -0.0234 \\
0.0003 & -0.0209 & 0.0013 & -0.0009 & -0.0004 & 0.0783 \\
\hline
0.0262 & -0.8776 & -0.0216 & -0.0761 & 0.0793 & -0.8876 \\
-0.1177 & 1.0040 & 0.0248 & -0.0184 & -0.8823 & -0.0404
\end{array}\right]
$$

These controllers, however, are not effective at controlling the system at off-design conditions, and in fact are destabilizing. The closed loop system poles for the four possible combinations of plant/controller are given below.

Table 3: Closed Loop System Poles

PID controller with minimum phase plant: -0.06,-0.03±0.04j, -0.02±0.01j, -0.03
PID controller with nonminimum phase plant: -0.03±0.01j, -0.02, **+0.05**, **+0.01±0.02j**
$\mathcal{H}_\infty$ controller with nonminimum phase plant: -0.10, -0.03±0.07j, -0.02±0.02j, -0.008, -0.02±0.004j
$\mathcal{H}_\infty$ controller with minimum phase plant: -0.008, -0.05, -0.11±0.33j, -0.015±0.003j, **+0.05**, **+0.024**

A principal advantage to the LQN framework for controller interpolation is to interpolate controllers of different size and structure, or open loop unstable. A nominal LPV observer-based controller is designed for the system (Eq.

12-15), and Q parameters are calculated such that the original PID and $\mathcal{H}_\infty$ controller are recovered near the design point, blended by exponential weighting function.

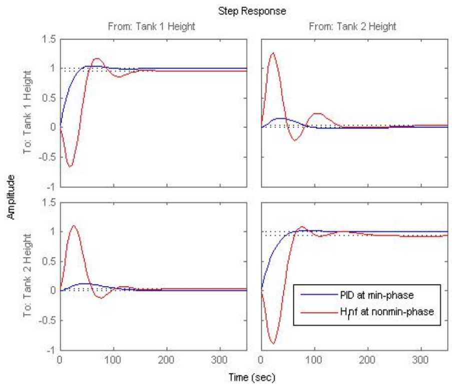

Figure 5: Step response of PID controller at minimum phase design point

The resulting controller is effective at regulating the fluid heights in spite of rapidly changing values of γ_1 and γ_2 (Fig. 6), which both induce disturbances on the system and change the underlying system dynamics from minimum phase to nonminimum phase. As the scheduling variables change, the exponential weighting factors (Fig. 7) allow smooth transitioning between the two Q functions. The control input voltages remain within reasonable bounds (Fig. 8), and fluid heights in the two lower tanks are effectively regulated (Fig. 9).

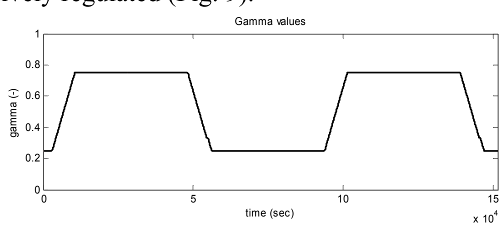

Figure 6: Scheduling parameters γ_1 and γ_2

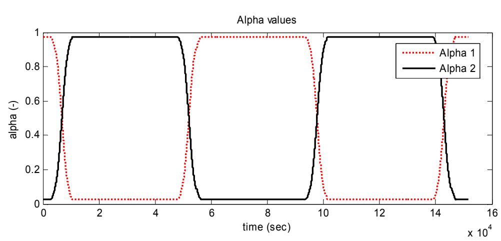

Figure 7: Variation of weighting factors for LQN

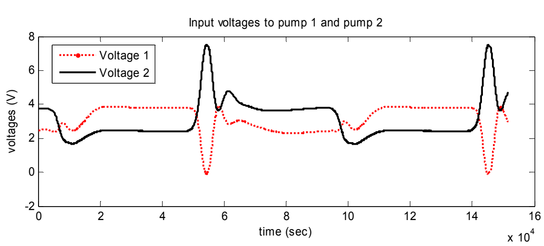

Figure 8: Control input voltages to pump 1 and 2

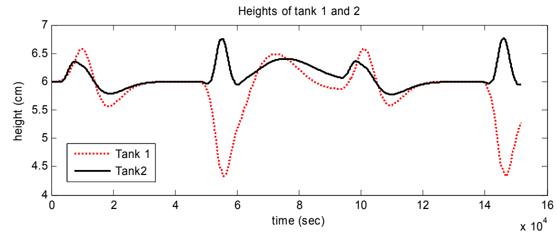

Figure 9: Fluid heights in lower tanks for gain scheduled controller (disturbance rejection due to changes in γ_1 / γ_2)

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