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Stability of Congestion Control Schemes with Delay Sensitive Traffic

Ying Li   Antonis Papachristodoulou   Mung Chiang

***Abstract*— Network congestion control schemes for the Internet aim to achieve efficient sharing of the available bandwidth while at the same time avoiding congestion collapse. Previous work has concentrated on users who are not delay-sensitive in the benefit they see when allocated a specific bandwidth share. In this paper, we consider a congestion control scheme that takes into account delay sensitive traffic and derive stability conditions for arbitrary network sizes that include propagation delay. The nature of the algorithm allows both the development of delay-independent and delay-dependent conditions, something unique to 'dual' algorithms proposed thus far.**

I. INTRODUCTION

Internet congestion control [20] is a distributed algorithm to allocate available resources to competing sources so as not to exceed link capacities and hence avoid congestion collapse. Congestion signals are generated at the Active Queue Management (AQM) part of the algorithm implemented at the links; the congestion measure is usually based on either delay or packet loss. The source rates are then adapted at the Transmission Control Protocol (TCP) part of the algorithm according to the size of the aggregate price signal that the user sees on the links he uses. The challenge is to use these feedback signals in order to stabilize the system around a 'fair' research allocation equilibrium for arbitrary networks sizes in a robust way.

What is interesting and extensively researched in the last few years is that the problem of Internet congestion control can be cast as a fully centralized optimization program [11], [10], [5], which aims to maximize the aggregate Utility of all the users, subject to the constraint that the total flow on each link should be no greater than the link capacity. Here, the Utility of each user i with a transmission rate x_i is a strictly concave, continuously differentiable non-decreasing function $U_i(x_i)$, measuring the user's 'happiness' when allocated transmission rate x_i. Essentially, it is a measure of the Quality of Service (QoS). For best-effort data traffic, a function of the transmission rate alone is good enough to reflect QoS, but for delay-sensitive traffic, a Utility function that is only a function of the transmission rate of user i may not reflect user's perception of QoS. For example, for a Voice over IP (VoIP) application, the R-factor which is a measure of user's satisfactory of the QoS, has a linear term of the end-to-end average delay in it [1].

When the Utility functions are only functions of the transmission rate, the resource allocation problem can be decomposed into a primal and a dual problem by introducing duality-based price signals [11]. In this way, the congestion measures are the 'dual' variables, while the transmission rates are the 'primal' variables. The aim of the designed AQM and TCP algorithms is to drive the congestion signals and the source rates exactly at or approximately close to the optimum of the distributed resource allocation optimization problem. In the simplest case, the structure of the dynamics that are chosen for TCP and AQM are usually based on a sub-gradient descent algorithm on the dual decomposition [5], [3], [2], whose nonlinear dynamics can be shown to be asymptotically stable.

Congestion control for delay-sensitive traffic is underexplored and was investigated in e.g., [8], [9], [18], [19] (see also the references therein). In this work, the Utility function was extended and made a function of the transmission rate as well as the end-to-end average delay. If queueing delay is considered, the end-to-end average delay of user i is a function of all the flows of the links on its path, where the link flow itself is the aggregate transmission rate of all the users sharing it. Such a Utility function couples the rate of the user and the flow on the links. It explicitly reflects how the transmission rate and delay affect users' satisfaction of the service, and provides a richer framework on how to accomplish rate control for delay sensitive traffic.

In this paper, we focus on an important feature that was ignored in the stability analysis in [8]. This is the presence of communication delays during packet/acknowledgement transfer. Delays should not be ignored as in general their presence results in degradation of performance or even instabilities. Indeed stability is an important measure of the functionality of the system;

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Y. Li and M. Chiang are with the Department of Electrical Engineering, Olden Street, Princeton University, Princeton, NJ, 08544, USA. Email: {yingli, chiangm}@princeton.edu.

A. Papachristodoulou is with the Department of Engineering Science, University of Oxford, Parks Road, Oxford, OX1 3PJ, U.K. Email: antonis@eng.ox.ac.uk.

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without stability transmission rates oscillate which could result in a reduction in the link utilization, 'mice' packets, i.e., short-lived small packets on which congestion control is difficult get dropped, and predictability of the behaviour of the system is lost. The problem of analyzing system behaviour at the linear and nonlinear level with heterogeneous delays is difficult [17] but several procedures have been developed for that purpose: for example the methodology developed in [21] analyzes the linearizations of the nonlinear equations and the methods developed in [15] and [22] deal with nonlinear system descriptions.

In this paper we analyze stability of a congestion control scheme with delay sensitive traffic that was proposed in [8] when propagation delays are taken into account. For this, we use the linearization of the scheme around the optimum and derive two conditions, one based on the delay and one that is delay-independent.

The rest of this paper is organized as follows. In Section II we present the problem and the congestion control algorithm we wish to analyze. In Section III we present the stability analysis of the linearization including the effect of delays. We conclude the paper in Section IV.

II. PROBLEM FORMULATION

A. Basic review

Consider a network of L communication links shared by S sources. The routing matrix R is given by:

$$R_{li} = \begin{cases} 1 & \text{if source } i \text{ uses link } l \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Associated with each source i is a transmission rate x_i. All sources whose flow passes through resource l contribute to the *aggregate rate* y_l, the rates being added with some forward time delay $\tau^f_{i,l}$:

$$y_l(t) = \sum_{i=1}^{S} R_{li} x_i(t - \tau^f_{i,l}) \triangleq r_f(x_i, \tau^f_{i,l}). \quad (2)$$

The resources l react to the aggregate rate y_l by setting a *price* p_l. This is the Active Queue Management (AQM) part of the algorithm. The prices of all the links that source i uses are added to form q_i, the *aggregate price* for source i, again through a delay $\tau^b_{i,l}$:

$$q_i(t) = \sum_{l=1}^{L} R_{li} p_l(t - \tau^b_{i,l}) \triangleq r_b(p_l, \tau^b_{i,l}). \quad (3)$$

The prices q_i can then be used to set the rate x_i of source i. This is the Transmission Control Protocol (TCP) part of the algorithm, which completes the picture shown in Figure 1. The capacity of link l is denoted by c_l. The forward and backward delays can be combined to yield

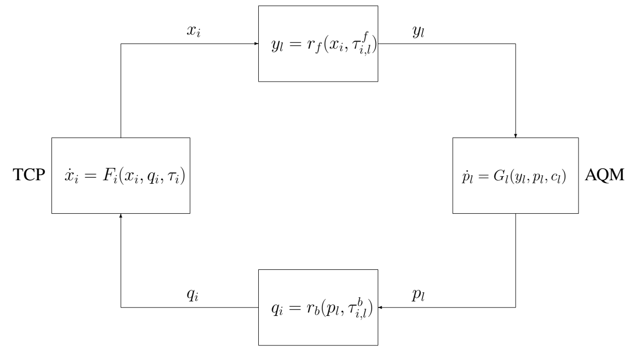

Fig. 1. The Internet as an interconnection of sources and links through delays.

the Round Trip Time (RTT) for source i, τ_i:

$$\tau_i = \tau^f_{i,l} + \tau^b_{i,l}. \quad (4)$$

This setting is *universal*, and what needs to be specified are two control laws that describe how the ith source reacts to the price signal q_i that it sees

$$\dot{x}_i = F_i(x_i, q_i, \tau_i), \quad (5)$$

and how the lth router reacts to the signal y_l it observes

$$\dot{p}_l = G_l(y_l, p_l, c_l). \quad (6)$$

Here F_i models TCP algorithms (e.g. Reno, Vegas) and G_l models AQM algorithms (e.g. RED, REM).

It is well known that the resource allocation algorithm can be reverse-engineered as the solution of an optimization problem [11], [20]. We associate with each user i a strictly concave, continuously differentiable non-decreasing Utility function $U_i(x_i)$ when being allowed to have a transmission rate x_i. Then the optimization of the whole system can be cast as follows:

$$\begin{array}{ll} \max_{x_i \geq 0} & \sum_{i=1}^{S} U_i(x_i) \\ \text{s.t.} & \sum_{i=1}^{S} R_{li} x_i \leq c_l, \quad \forall\ l = 1, \ldots, L \end{array} \quad (7)$$

where the inequality constraint is the natural limitation that the sum of all transmission rates through link l has to be less than or equal to its capacity. The uniqueness of solution to the above problem is guaranteed since U_i are strictly concave functions.

The above optimization problem cannot be solved in a decentralized way, as the source rates are coupled in the shared links through the inequality constraints and solving for x^* would require cooperation among possibly all sources. However it can be decomposed into a primal problem that the sources are trying to solve and a dual that the links are trying to solve, regarding the sources x_i as primal variables and the prices set by the links p_l as dual variables. Under specific assumptions the optimal point of the two sub-problems coincides with the optimal point of the original problem, which is unique. More details can be found in [11]. The dynamical system defined by (5–6) with delays ignored aims to drive the system close to or exactly at the optimal point (x^*, p^*), using well-known sub-gradient algorithms.

B. Delay-sensitive utility maximization

In this paper, we consider users with delay-sensitive Utility functions. In particular, we set the Utility function as in [8]:

$$U_i = f_i(x_i) - b_i \sum_{l=1}^{L} R_{li} d_l(y_l)$$

where f_i is a function in rate x_i, $y_l = \sum_i R_{li} x_i$ is the link load on link l, $d_l(y_l)$ is the average delay experienced by a packet on link l, and $b_i > 0$ is some constant incorporating the normalization and the relative importance of the delay versus rate of user i.

We have the following assumptions:

Assumption 1: The function $f_i(x_i)$ is increasing and strictly concave in $x_i \geq 0$, for all i.

Assumption 2: The function $d_l(y_l)$ is positive, increasing and strictly convex in $y_l \geq 0$, and $d_l(c_l) = \infty$ for all l.

Note that Assumption 2 implies that $y_l \leq c_l$. The function $d_l(y_l)$ is a delay function. For instance, if d_l is the average queueing delay of an $M/M/1$ queue [6], then $d_l(y_l) = \frac{a}{c_l - y_l}$ where $a > 0$ is some constant. With this Utility function, problem (7) still decentralizes and has similar properties to the one with $b_i = 0$.

The corresponding optimization problem becomes

$$\begin{array}{ll} \max_{x_i \geq 0} & \sum_{i=1}^{S} [f_i(x_i) - b_i \sum_{l=1}^{L} R_{li} d_l(y_l)] \\ \text{s.t.} & \sum_{i=1}^{S} R_{li} x_i \leq y_l, \quad \forall l = 1, \dots, L, \end{array}$$

where $y_l \leq c_l, \forall l$. Note that the inequality sign is equivalent to the equality sign in the constraint $\sum_{i=1}^{S} R_{li} x_i \leq y_l$ because the objective function is monotonically decreasing in y_l.

In this paper, we will concentrate on a 'dual' congestion control scheme using the above Utility function with dynamics at the links but a static source law. The stability properties of the undelayed system can be obtained directly, as the gradient algorithm results in a weighted *potential system* [20].

When the communication delays during packet/acknowledgement transfer are introduced, say, when τ_i are brought in the dynamics, the dynamic analysis becomes more complicated, and a scalable analysis methodology is difficult. Here we will obtain two results that hold for arbitrary network topologies - one that is delay-independent and one that is delay-dependent. The tools we use are based on [21].

Note that the delay $d_l(y_l)$ in the Utility function for the delay-sensitive traffic considered here is mainly the queueing delay, which provides a richer profile for rate control; the delay τ_i in our dynamic analysis is the communication delay which includes the queueing delay, propagation delay, processing delay, and so on. For the simplicity of the analysis, $\tau_i' s$ are assumed to be constants. In future work, we will consider the case where $\tau_i' s$ are time varying.

The routing matrix R is assumed fixed and full row rank. This means that there are no algebraic constraints between link flows, i.e., they can vary independently by choice of source flows x_i. As a consequence, equilibrium prices are uniquely determined.

C. The dual congestion control scheme

The congestion control algorithm we will be looking at has the following laws at the sources and links:

$$\begin{aligned} \dot{p}_l(t) &= \kappa_l \left[\sum_{i=1}^{S} R_{li} x_i - d_l'^{-1} \left(\frac{p_l}{B_l} \right) \right]_{p_l}^{+} && (8) \\ x_i(t) &= f_i'^{-1}(q_i), && (9) \end{aligned}$$

where $B_l = \sum_{i=1}^{S} R_{li} b_i > 0$, $\kappa_l > 0$ and $[f(x)]_x^+$ means

$$[f(x)]_x^+ = \begin{cases} f(x) & x > 0 \\ \max\{f(x), 0\}, & x = 0 \end{cases}.$$

The equilibrium of interest for the above system is at:

$$\begin{aligned} y_l^* &= d_l'^{-1} \left(\frac{p_l^*}{B_l} \right) \\ x_i^* &= f_i'^{-1}(q_i^*), \end{aligned}$$

given the relations $y_l^* = \sum_{i=1}^{S} R_{li} x_i^*$ and $q_i^* = \sum_{l=1}^{L} R_{li} p_l^*$.

We proceed to linearize the above system about this equilibrium. In this case we have

$$\begin{aligned} \dot{\delta p_l} &= \kappa_l \left(\sum_{i=1}^{S} R_{li} \delta x_i - \frac{1}{B_l} \frac{\delta p_l}{d_l''(y_l^*)} \right) \\ \delta x_i &= \frac{\delta q_i}{f_i''(x_i^*)} \end{aligned}$$

where all δ's indicate small changes about the equilibrium position. Define the following matrices,

$$F = \text{diag}\{f_i''(x_i^*)\} < 0, \quad D = \text{diag}\{d_l''(y_l^*)\} > 0,$$
$$K = \text{diag}\{\kappa_l\} > 0, \quad B = \text{diag}\{B_l\} > 0.$$

We also have the relation

$$\delta q = R^T \delta p.$$

Then the above equations become:

$$\dot{\delta p} = K(RF^{-1}R^T - B^{-1}D^{-1})\delta p \triangleq A\delta p.$$

The matrix $A < 0$ by construction, therefore the equilibrium is locally asymptotically stable. In fact, it is globally asymptotically stable as shown in [8].

III. A MODEL WITH DELAY AND ITS STABILITY ANALYSIS

We now introduce heterogeneous delays to the system in order to describe the transmission and propagation time needed for the packets to reach the destination and acknowledgements to be received by the source.

The presence of delays is most of the times destabilizing and may affect greatly the performance of the system. Stability analysis of linear time delay systems has been investigated greatly in the past years [12]. Just as in the stability analysis of system described by linear *Ordinary* differential equations, there are in general two methodologies for investigating stability: using time-domain (Lyapunov) or frequency domain arguments. Using a frequency domain methodology more accurate descriptions of the stability boundaries can be obtained and this method is scalable for the special case of Internet Congestion Control [21]. On the other hand, Lyapunov-based arguments are more conservative; they are however useful for the exact investigation of the stability of nonlinear systems [7]. In this paper we will analyze the system described in the previous section using a generalized Nyquist criterion.

The nonlinear delayed model when heterogeneous time-delays are taken into account becomes

$$\begin{aligned} \dot{p}_l(t) &= \kappa_l \left[\sum_{i=1}^{S} R_{li} x_i(t-\tau_{i,l}^f) - d_l'^{-1}\left(\frac{p_l(t)}{B_l}\right)\right]_{p_l}^{+} \\ x_i(t) &= f_i'^{-1}(q_i(t)) \\ q_i(t) &= \sum_{l=1}^{L} R_{li} p_l(t-\tau_{i,l}^b). \end{aligned}$$

We define the following matrices, that will simplify the notation:

$$[R_f(s)]_{li} = \begin{cases} e^{-s\tau_{i,l}^f} & \text{if user } i \text{ uses link } l \\ 0 & \text{otherwise} \end{cases}$$

and

$$[R_b(s)]_{li} = \begin{cases} e^{-s\tau_{i,l}^b} & \text{if user } i \text{ uses link } l \\ 0 & \text{otherwise} \end{cases}.$$

Note that

$$R_b(s) = R_f(-s)\text{diag}\{e^{-s\tau_i}\}.$$

Linearizing the nonlinear time-delayed system about the same equilibrium as before, we get:

$$\begin{aligned} \dot{\delta p_l}(t) &= \kappa_l \left[\sum_{i=1}^{S} R_{li}\delta x_i(t-\tau_{i,l}^f) - \frac{\delta p_l(t)}{B_l d_l''(y_l^*)}\right] && (10) \\ \delta x_i(t) &= \frac{1}{f_i''(x_i^*)}\delta q_i(t) && (11) \\ \delta q_i(t) &= \sum_{l=1}^{L} R_{li}\delta p_l(t-\tau_{i,l}^b). && (12) \end{aligned}$$

In the following two subsections, we will present two stability results for the linearization, one that is delay-independent and one that is delay-dependent.

A. Delay-dependent sufficient condition for stability

Here is the result concerning delay-dependent stability.

Theorem 1: Given Equations (10)-(12), for $\kappa_l = \frac{1}{y_l^*}$ and $f_i''(x_i^*) = -\frac{M_i\tau_i}{x_i^*\alpha_i}$ where $\alpha_i < \pi/2$ and $M_i = \sum_{l=1}^{L} R_{li}$ the equilibrium is asymptotically stable.

Proof: Taking Laplace transforms and dropping all δ's:

$$\begin{aligned} sp(s) - p(0) &= K\left(R_f(s)x(s) - B^{-1}D^{-1}p(s)\right) \\ x(s) &= F^{-1}q(s) \\ q(s) &= R_b^T(s)p(s). \end{aligned}$$

Combining we get:

$$sp(s) - p(0) = K\left(R_f(s)F^{-1}R_b^T(s) - B^{-1}D^{-1}\right)p(s).$$

Therefore,

$$\begin{aligned} p(0) = & \left(sI + KB^{-1}D^{-1}\right)p(s) \\ & + KR_f(s)(-F^{-1})T(s)R_f^T(-s)p(s). \end{aligned}$$

This system is stable if its poles lie in the left half plane, i.e., if the solution to

$$\det\left(\left(s + \frac{\kappa_l}{B_l\hat{d}_l}\right)I + \left(s + \frac{\kappa_l}{B_l\hat{d}_l}\right)G\right) = 0$$

has only negative real parts, where the return ratio is

$$\begin{aligned} G = & \text{ diag}\left\{\frac{\kappa_l}{s + \frac{\kappa_l}{B_l\hat{d}_l}}\right\} R_f(s) \\ & \text{ diag}\left\{\frac{\tau_i}{-\hat{f}_i}\right\} \text{diag}\left\{\frac{e^{-s\tau_i}}{\tau_i}\right\} R_f^T(-s) \end{aligned},$$

where $\hat{f}_i = f_i''(x_i^*)$ and $\hat{d}_l = d_l''(y_l^*)$. We then show if $\hat{f}_i = -\frac{M_i\tau_i}{x_i^*\alpha_i}$ and $\kappa_l = \frac{1}{y_l^*}$, then for $\alpha_i < \pi/2$ the equilibrium is asymptotically stable for arbitrary topologies.

Since the open-loop system is stable, we need to ensure that the eigenvalues of the above function G, for $s = j\omega$ do not encircle the -1 point. These eigenvalues are the same as the eigenvalues of

$$\text{diag}\left(\frac{j\omega}{j\omega+\frac{\kappa_l}{B_l\hat{d}_l}}\right)\hat{R}(j\omega)\text{diag}\left(\frac{\alpha_i e^{-j\omega\tau_i}}{j\omega\tau_i}\right)\hat{R}^T(-j\omega)$$

where

$$\hat{R}(j\omega) = \text{diag}\left\{\sqrt{\frac{1}{y_l^*}}\right\} R_f(j\omega)\text{diag}\left\{\sqrt{\frac{x_i^*}{-M_i}}\right\}.$$

Note that

$$\overline{\sigma}^2(\hat{R}(j\omega)) \leq 1$$

where $\overline{\sigma}^2(\hat{R}(j\omega)) = \rho(\hat{R}(j\omega)\hat{R}^T(-j\omega))$, where $\rho(Z)$ denotes the spectral radius of a matrix Z. The argument is similar as the one in [21].

Now if λ is an eigenvalue of the above, then there exists a v for which $\|v\| = 1$ such that

$$\text{diag}\left(\frac{j\omega}{j\omega+\frac{\kappa_l}{B_l\hat{d}_l}}\right)\hat{R}(j\omega)\text{diag}\left(\frac{\alpha_i e^{-j\omega\tau_i}}{j\omega\tau_i}\right)\hat{R}^T(-j\omega)v = \lambda v$$

and hence

$$\lambda = \frac{v^*\hat{R}(j\omega)\text{diag}\left(\alpha_i\frac{e^{-j\omega\tau_i}}{j\omega\tau_i}\right)\hat{R}^T(-j\omega)v}{v^*\text{diag}\left(1-\frac{k_l}{B_l\hat{d}_l j\omega}\right)v}$$

Since $\|\hat{R}^T(-j\omega)v\| \leq 1$ this implies that

$$\lambda \in \frac{\text{Co}\left(0 \cup \left\{\alpha_i\frac{e^{-j\omega\tau_i}}{j\omega\tau_i}\right\}\right)}{\text{Co}\left\{\left(1-\frac{k_l}{B_l\hat{d}_l j\omega}\right)\right\}}$$

and since the eigen-loci cannot cross the real axis at or to the left of the point -1 for $\alpha_i < \pi/2$, the closed loop system is stable by the generalized Nyquist stability criterion [21]. ■

We note that the condition is similar to the one that is given in [13] - however, the dynamics that we are considering are far more interesting, as they take into account delay-sensitive traffic.

Remark 3: In general, it may be difficult for the resources to be able to estimate y_l^* and for the sources to know M_i. However it is possible to relax the condition that $\kappa_l = \frac{1}{y_l^*}$ to the condition that $\kappa_l = \frac{1}{c_l}$, which can be implemented in a decentralized way. Note that due to the nature of the algorithm, $y_l^* < c_l$, i.e., the links may not be fully utilized.

B. Delay-independent sufficient condition for stability

Here we present a delay-independent stability condition.

Theorem 2: Given Equations (10)-(12), if $\kappa_l = \frac{1}{y_l^*}$, $f_i''(x_i^*) = -\frac{M_i}{\alpha_i x_i^*}$, $\kappa_l > B_l d_l''(y_l^*)$ and $\alpha_i < 1$ then the equilibrium is asymptotically stable.

Proof: Again, we write $\hat{f}_i = f_i''(x_i^*)$ and $\hat{d}_l = d_l''(y_l^*)$. The return ratio in this case is the same as in the proof of Theorem 1, i.e.,

$$L(s) = \text{diag}\left\{\frac{\kappa_l}{s+\frac{\kappa_l}{B_l\hat{d}_l}}\right\} R_f(s) \text{diag}\left\{\frac{1}{-\hat{f}_i}\right\}\text{diag}\left\{e^{-s\tau_i}\right\} R_f^T(-s).$$

The eigenvalues of this are the same as the ones of

$$\text{diag}\left\{\frac{1}{s+\frac{\kappa_l}{B_l\hat{d}_l}}\right\} R_f(s) \text{diag}\left\{\frac{1}{-\hat{f}_i}\right\}\text{diag}\left\{e^{-s\tau_i}\right\} R_f^T(-s)\text{diag}\left\{\kappa_l\right\}$$

In the same way as before, and imposing that $\hat{f}_i = -\frac{M_i}{x_i^*\alpha_i}$ and that $\kappa_l = \frac{1}{y_l^*}$ we get:

$$\lambda = \frac{v^*\hat{R}(j\omega)\text{diag}\left(\alpha_i e^{-j\omega\tau_i}\right)\hat{R}^T(-j\omega)v}{v^*\text{diag}\left(\frac{\kappa_l}{B_l\hat{d}_l}+j\omega\right)v}$$

In this case, for asymptotic stability we require that $\kappa_l > B_l\hat{d}_l$ and $\alpha_i < 1$. This condition is delay-independent. ■

Remark 3 also holds in this case. Note that the delay-independent condition allows us to have a fixed gain at the sources which is irrespective of the size of the delay, i.e., the sources don't have to compensate their gain for long delays. The price, of course, is performance degradation.

The delay-independent condition above ensures that the Nyquist plot not only does not encircle the -1 point (which is what we ensured in the delay-dependent condition), but rather that the whole Nyquist plot never leaves the unit disc.

C. A simple network

In order to put the above results in perspective, we will consider the stability conditions we get by looking at a simple, single-link single-source network with the ones given by Theorems 1 and 2 when they are reduced to this simple case.

The model for a simple such network reads:

$$\delta\dot{p} = \kappa\left(\frac{1}{\hat{f}}\delta p(t-\tau) - \frac{\delta p(t)}{B\hat{d}}\right).$$

where $\hat{f} = f''(x^*)$ and $\hat{d} = d''(x^*)$. This system is [4]

- Delay-dependent stable if $-\frac{1}{\hat{f}} > \frac{1}{B\hat{d}}$ and

$$\tau < \frac{\arccos\left(\frac{\hat{f}}{B\hat{d}}\right)}{\kappa\sqrt{\frac{1}{\hat{f}^2}-\frac{1}{B^2\hat{d}^2}}} \tag{13}$$

- Delay-independent stable if

$$\frac{1}{B\hat{d}} > \frac{1}{-\hat{f}}. \tag{14}$$

The delay-dependent condition that is given in Theorem 1 makes the following assumptions:

$$\kappa = \frac{1}{x^*}, \quad \hat{f} = -\frac{\tau}{x^*\alpha}, \quad \alpha < \pi/2.$$

Under these conditions, condition (13) simplifies to

$$\tau < \frac{\arccos\left(-\frac{\tau}{\alpha B\hat{d}x^*}\right)}{\frac{1}{x^*}\sqrt{\frac{\alpha^2 x^{*2}}{\tau^2} - \frac{1}{B^2\hat{d}^2}}}$$

and this condition is valid for $\alpha B\hat{d}x^* > \tau$. Let us introduce a variable $\mu = \frac{\tau}{x^*\alpha B\hat{d}}$, $\mu \in [0,1]$. Then the above condition reads

$$\alpha\sqrt{1-\mu^2} < \arccos(-\mu)$$

Indeed this condition is only valid for $\alpha < \frac{\pi}{2}$. Hence the two conditions, the one given in Theorem 1 for arbitrary network sizes and the one above are equivalent; therefore the delay-dependent condition is necessary and sufficient for the single-source single-link case.

The delay-independent condition that we have in Theorem 2 reduces in the single-source single-link case to

$$\kappa = \frac{1}{x^*}, \quad \hat{f} = -\frac{1}{\alpha x^*}, \quad B\hat{d} < \kappa, \quad \alpha < 1.$$

Combining these we get $B\hat{d} < -\alpha\hat{f}$, and since we required $\alpha < 1$ in the conditions of Theorem 2, the two conditions are again equivalent in the simplest network case.

IV. CONCLUSIONS

In this paper we have considered an Internet congestion control scheme which takes into account both delay in utility function for delay-sensitive traffic and delay in the algorithm's dynamics. Stability conditions in the presence of heterogeneous propagation delays have been developed. Two sets of such conditions emerged: a delay-independent and delay-dependent one, which makes it the first dual algorithm to have this feature.

We note, however, that the conditions in Theorems 1 and 2 involve knowledge of y_l^* at the links. This may not be possible, but as remarked in Remark 3, the two conditions can be relaxed to $\kappa_l = \frac{1}{c_l}$, where c_l is the capacity of the link. This relaxation would make the two conditions conservative, but is more easily implementable. Other approaches can also be considered.

Another feature of the algorithm that we will be considering in the future is the choice of the delay-free Utility function f_i at the users. Right now, the conditions that impose $f_i''(x_i^*) = \frac{\beta}{x_i^*}$ for some $\beta < 0$ imply a particular shape of Utility function. Future research will concentrate on primal-dual algorithms that would allow the sources to adapt to a Utility function of their choice at a much slower time-scale, like the one in [14].

Nonlinear stability analysis, as in [16] is another next step in research on this topic.

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