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Delay-Dependent Stability for Vector Nonlinear Stochastic Systems with Multiple State Delays

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***Abstract*— Global asymptotic stability conditions for vector nonlinear stochastic systems with multiple state delays are obtained based on the convergence theorem for semimartingale inequalities, without assuming the Lipschitz conditions for nonlinear drift functions. The Lyapunov-Krasovskii and degenerate functionals techniques are used. The derived stability conditions are directly expressed in terms of the system coefficients. Nontrivial examples of nonlinear systems satisfying the obtained stability conditions are given.**

I. INTRODUCTION

The stability and stabilizabilty problems for time-delay systems have been extensively studied in recent years due to direct applicability of the obtained results to various technical problems ([1], [2], [3]). Initiated in the background works [4], [5], [6], the stability theory for linear time-delay systems is now actively being developed. To prove stability results for a selected class of linear time-delay systems, the Lyapunov-Krasovskii or Lyapunov-Razumikhin functionals are applied in the framework of the Lyapunov direct method. Two types of stability conditions can be obtained: delay-independent, establishing stability for all possible delay values, or delay-dependent, corresponding to some restricted values of delay shifts. While the first type of conditions is comprehensive but conservative, the second one is more selective, flexible, and, as a consequence, preferable. Some examples of delay-dependent stability conditions can be found in ([7], [8], [9], [10], [11], [12], [13], [14]) for various deterministic linear time-delay systems and in [15], [16], [17], [18], [19] for stochastic ones. Note that it is frequently needed to make a special transformation of an original time-delay system to obtain such stability conditions. Virtually all known results involving delay-dependent stability conditions have been obtained for linear time-delay systems, with certain or even uncertain coefficients.

This paper concentrates on design of the stability conditions for vector nonlinear stochastic time-delay systems governed by multidimensional nonlinear Ito differential equations with multiple state delays and a nontrivial diffusion term. To obtain the results, a modified Lyapunov-Krasovskii functional, known as *degenerate functional*, is employed, which was introduced and described in details in [4], [5]. Applications of degenerate functionals for various classes of deterministic functional-differential equations can be found in [4], [5], [6]. In [20], the degenerate functionals are used for obtaining delay-dependent stability conditions for deterministic scalar delay-differential equations. This paper generalizes the result of [20] to vector stochastic nonlinear time-delay systems. The convergence theorem for semimartingale inequalities [21] serves as a key tool for obtaining stability conditions in terms of stochastic system coefficients, without any transformation of the original system itself. However, some conditions in this paper are more restrictive than those in [20], because of the nature of the solutions of stochastic Ito equations. Even small initial values cannot guarantee that solutions to stochastic Ito equation with nontrivial diffusion will be almost surely (*a.s.*) bounded on a finite interval. Therefore, a global linear growth condition for nonlinear drift functions is used in this paper instead of a local one in [20]. Nonetheless, a significant advance reached in this paper in comparison to [20] is elimination of the Lipschitz condition for nonlinear drift functions. Similar delay-dependent stability conditions for discrete-time systems can be found in [22]. Finally note that design of a stabilizing controller for a class of nonlinear stochastic systems, based on the stability conditions given in this paper, would be a direct application of the obtained results (see [10] for a similar scheme of stabilizing controller design for linear systems).

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II. BASIC DEFINITIONS AND RESULTS

In this section, some basic definitions and results from the theory of stochastic processes are briefly reviewed (see ([23], [24] for details). All stochastic variables or processes are allowed to be multi-dimensional (stochastic vector or vector processes), for which equalities and inequalities are regarded component-wisely. The following notation is used: x^T means the transpose of a vector x, $|x| = \sqrt{(x^T x)}$ denotes the Euclidean 2-norm of a vector x, and $|\sigma|$ denotes the Euclidean 2-norm of a matrix σ, i.e., the sum of squares of all matrix entries.

Let (Ω, F, P) be a complete probability space with a non-decreasing right-continuous family of σ-algebras (filtrations) $F = \{\mathscr{F}_t\}_{t \geq t_0}$. A stochastic process M_t is said to be an $\mathscr{F}_t$-*martingale*, if $\mathbf{E}|M_t| < \infty$ and $\mathbf{E}\big(M_t \big| \mathscr{F}_s\big) = M_s$ for all $t > s \geq t_0$. A stochastic process is called a *semimartingale* if it admits the representation $X_t = X_{t_0} + M_t + A_t$, where M_t is a martingale, $M_{t_0} = 0$, and A_t is a process with almost surely (*a.s.*) bounded variation, $A_{t_0} = 0$, X_{t_0} is a random variable.

The following lemma, originally proved in [21], presents a modification of the martingale convergence theorems (cf. [23]) in terms of inequalities, which plays a key role in establishing the asymptotic stability conditions.

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Lemma 1. Let A^1, A^2, B^1, B^2 be almost sure (*a.s.*) non-decreasing processes with $B^1 \leq A^1$, $B^2 \geq A^2$ and $A = B^1 - B^2$. Let also $Z = Z_{t_0} + A + M$ be a non-negative semimartingale. Then, a.s.

$$\left\{\omega : A^1_\infty < \infty\right\} \subseteq \{Z \rightarrow\} \cap \left\{\omega : A^2_\infty < \infty\right\}, \quad (1)$$

where $\{Z \rightarrow\}$ denotes the set of all $\omega \in \Omega$ for which $Z_\infty = \lim_{t\to\infty} Z_t$ exists and is finite.

Remark 1. In this paper, the notation $x(t)$ is used for a value of any stochastic process (other than a Wiener process w_t) at a moment t, although the notation x_t is commonly accepted for this purpose in the literature on stochastic processes. This enables one to distinguish between the pointwise value $x(t)$ of a stochastic process and its history up to the moment t, which is denoted by x_t, $x_t(s) = x(s)$, $s \leq t$, as commonly accepted in the literature on time delay systems.

The space $C_{[a,b]}$ of continuous vector functions $u(s) \in \mathbf{R}^n$ in the interval $[a,b]$ is defined as

$$C_{[a,b]} = \{u : [a,b] \to \mathbf{R}^n \mid u(s) \text{ is continuous for any } s \in [a,b], \text{ with the norm } \|u\|_{C_{[a,b]}} = \sup_{1\le k\le n} \sup_{a\le s\le b} |u_k(s)| < \infty\}.$$

III. Stability conditions

In this section, the asymptotic stability problem is considered for a state $x(t) \in \mathbf{R}^n$ satisfying a system of Ito stochastic nonlinear differential equations with several nonlinear functions $N_i(u) \in \mathbf{R}^n$, $i = 1, \ldots, m$, and multiple discrete delays $h_i > 0$

$$dx(t) = -\sum_{i=1}^{m} a_i N_i(x(t))dt - \sum_{i=1}^{m} b_i N_i(x(t-h_i))dt + \sigma(t,x_t)dw_t,\ t \geq t_0, \quad x(t) = \phi(t),\ t \in [t_0 - h, t_0], \quad (2)$$

where $\phi(t) \in \mathbf{R}^n$ is an initial continuous function defined on $[t_0 - h, t_0]$, $h = \max(h_1, \ldots, h_m)$, and $w_t \in \mathbf{R}^p$ is a vector Wiener process with independent components, i.e., a stochastic process whose components are scalar independent standard Wiener processes. The diffusion matrix $\sigma(t,x_t)$ is of dimension $n \times p$, and coefficients a_i and b_i, $i = 1, \ldots, m$, are scalar constants.

The equation (2) is equivalent to the following equation

$$d[x(t) - \sum_{i=1}^{m} b_i \int_{t-h_i}^{t} N_i(x(\theta))d\theta] = -\sum_{i=1}^{m}(a_i + b_i)N_i(x(t))dt + \sigma(t,x_t)dw_t, \quad t \geq t_0,$$
$$x(t) = \phi(t), \quad t \in [t_0 - h, t_0]. \quad (3)$$

Sometimes, the equation (3) is called a neutral type equation, since it contains the unknown function evaluated in some points $\theta \leq t$ under the sign of differential.

The functions $N_i(u)$, $i = 1, \ldots, m$, in the equations (2),(3) are assumed to satisfy the sector-like condition

$$u^T N_i(u) > 0, \quad \text{for any} \quad u \in \mathbf{R^n}, u \neq 0,\ N_i(0) = 0. \quad (4)$$

The first part of this condition, which should hold for any real nonzero value $u = x(t)$ or $u = x(t - h_i)$, $i = 1, \ldots, m$, rules out some unstable (for a nonzero initial function) systems, such as $dx(t) = x(t)dt$, $dx(t) = x(t-h)x^T(t-h)x(t-h)dt$, or not asymptotically stable systems, such as $dx(t) = \sin(x(t)x^T(t)x(t))dt$. Nonetheless, a modified *sin* function would satisfy the condition (4), as shown further in Example 5.

Next, assume that there exist such constants $K_i, \gamma_1^{(i)}, \gamma_2^{(i)} > 0, i = 1, \ldots, m$, and a function $\gamma : [t_0, \infty) \to \mathbf{R}^+$ that the following conditions are satisfied

$$|N_i(u)|^2 \leq K_i u^T N_i(u), \quad \text{for any} \quad u \in \mathbf{R}^n; \quad (5)$$

$$tr(\sigma(t,x_t)\sigma^T(t,x_t)) \leq \sum_{k=1}^{m} \gamma_1^{(k)} |N_k(x(t))|^2 + \sum_{k=1}^{m} \gamma_2^{(k)} |N_k(x(t-h_k))|^2 + \gamma(t) \quad (6)$$

for any $x_t \in C_{[t_0-h,t]}$;

$$a_i + b_i > 0; \quad (7)$$

$$\alpha_2 = \sum_{i=1}^{m} |b_i| h_i K_i < 1; \quad (8)$$

$$\beta_k = \left(\frac{|b_k|h_k}{2(a_k+b_k)}\left(\sum_{i=1}^{m}(a_i+b_i)\right) + \frac{1}{2}\sum_{i=1}^{m}|b_i|h_i + \frac{\gamma_1^{(k)}+\gamma_2^{(k)}}{2(a_k+b_k)}\right)K_k < 1; \quad (9)$$

$$\int_{t_0}^{\infty} \gamma(s)ds < \infty. \quad (10)$$

Discussing the conditions (5)-(10), note that the deterministic system obtained by setting $\sigma(t,x_t) = 0$ would be asymptotically stable in view of Theorem 1. The conditions (6) and (10) imply that the diffusion term $\sigma(t,x_t)$ converges to zero as time tends to infinity, i.e., the system (2) becomes deterministic at the infinity, although remains stochastic for any fixed large t. The condition (7) indicates that the "total" summarized coefficient of the current and delayed values for each function N_i should have a strictly negative value, which assures asymptotic stability of the corresponding system mode. Note that a_i and b_i cannot be simultaneously equal to zeros, since the corresponding mode could be stable but not asymptotically stable, as for the system $dx(t) = 0$, or even unstable, as for the system $dx(t) = K(t)dw(t)$, where $K(t) = 1$ for $t_0 \leq t \leq T$, T is a sufficiently large time moment, and $K(t) = 0$ for $t > T$.

Remark 2. If $K_i \equiv K$, $\gamma_1^{(i)} = \gamma_2^{(i)} = 0$ for all $i = 1, \ldots, m$, it is not difficult to show that the condition (9) implies (8). Indeed, the condition (9) gives

$$\frac{|b_k|h_k K}{2(a_k+b_k)}\left(\sum_{i=1}^{m}(a_i+b_i)\right) + \frac{1}{2}\sum_{i=1}^{m}|b_i|h_i K < 1,$$

$$|b_k|h_k K\left(\sum_{i=1}^{m}(a_i+b_i)\right) + \left(\sum_{i=1}^{m}|b_i|h_i K\right)(a_k+b_k) < 2(a_k+b_k), \text{ and}$$

$$\sum_{k=1}^{m}|b_k|h_k K\left(\sum_{l=1}^{m}(a_i+b_i)\right) + \left(\sum_{i=1}^{m}|b_i|h_i K\right)\sum_{k=1}^{m}(a_k+b_k)$$

$$< 2\sum_{k=1}^{m}(a_k+b_k).$$

Dividing the latter by $\sum_{i=1}^{m}(a_i+b_i)$ yields

$$\sum_{k=1}^{m}|b_k|h_kK+\sum_{i=1}^{m}|b_i|h_iK<2.$$

For all $t\geq t_0$ and $x_t\in C_{[t_0-h,t]}$, $x(s)=\phi(s)$, $s\in[t_0-h,t_0]$, the degenerate functional is defined

$$\bar{Y}(t,x_t)=x(t)-\sum_{i=1}^{m}b_i\int_{t-h_i}^{t}N_i(x(\theta))d\theta. \qquad (11)$$

The functional $Y(t,x_t)$ is not negative but also is not positive definite. However, for every $t\geq t_0$, the norm $\|Y(\bullet,x_\bullet)\|_{C_{[t_0,t]}}$ can be bounded from the below by the norm $\|x\|_{C_{[t_0-h,t]}}$, as shown in the following Lemma 2.

The next lemma and following theorem establish asymptotic stability conditions for solutions of the equation (3).

Lemma 2. Let conditions (5), (9) be satisfied and $\alpha_2=\sum_{i=1}^{m}|b_i|h_iK_i<1$. Then, for $t\geq t_0$ and $x_t\in C_{[t_0-h,t]}$, $x(s)=\phi(s)$, $s\in[t_0-h,t_0]$, the state trajectory norm satisfies the inequality

$$\|x\|_{C_{[t_0-h,t]}}\leq\frac{1}{1-\alpha_2}\left[\|\bar{Y}(\bullet,x_\bullet)\|_{C_{[t_0,t]}}+\|\phi\|_{C_{[t_0-h,t_0]}}\right]. \qquad (12)$$

Proof.

$$\|x(s)\|_{C_{[t_0,t]}}=\|x(s)-\sum_{i=1}^{m}b_i\int_{t-h_i}^{t}N_i(x(\theta))d\theta$$
$$+\sum_{i=1}^{m}b_i\int_{t-h_i}^{t}N_i(x(\theta))d\theta\|_{C_{[t_0,t]}}$$
$$\leq\|\bar{Y}(s,x_s)\|_{C_{[t_0,t]}}+\sum_{i=1}^{m}|b_i|\|\int_{t-h_i}^{t}N(x(\theta))d\theta\|_{C_{[t_0,t]}}$$
$$\leq\|\bar{Y}(\bullet,x_\bullet)\|_{C_{[t_0,t]}}+\sum_{i=1}^{m}|b_i|h_iK_i\|x\|_{C_{[t_0-h_i,t]}}.$$

Then,

$$\|x\|_{C_{[t_0-h,t]}}\leq\|x\|_{C_{[t_0,t]}}+\|\phi\|_{C_{[t_0-h,t_0]}}$$
$$\leq\|\bar{Y}(\bullet,x_\bullet)\|_{C_{[t_0,t]}}+\alpha_2\|x\|_{C_{[t_0-h,t]}}+\|\phi\|_{C_{[t_0-h,t_0]}},$$

which implies (12).

Theorem 1. Let conditions (4)-(10) be satisfied. Then, $\lim_{t\to+\infty}x(t)=0$ holds a.s. for all solutions x of the equation (2).

Proof. Define functional $V=V(x(t),t)$ by the formula

$$V=|\bar{Y}|^2+V_1+V_2,$$
$$V_1=\left(\sum_{i=1}^{m}(a_i+b_i)\right)\sum_{k=1}^{m}|b_k|\int_{t-h_k}^{t}ds\int_{s}^{t}|N_k(x(\theta))|^2d\theta,$$
$$V_2=\sum_{k=1}^{m}\gamma_2^{(k)}\int_{t-h_k}^{t}|N_k(x(\theta))|^2d\theta,$$

where $\bar{Y}$ is given by (11).

Applying Ito formula to the $V(x(t),t)$ along the trajectories of solutions of the equation (3) yields

$$V(x(t),t)=V(0,t_0)+\int_{t_0}^{t}\bar{F}_sds+\bar{M}_t,$$

with $V(0,t_0)=0$,

$$\bar{F}_t=-2[x^T(t)-\sum_{i=1}^{m}b_i\int_{t-h_i}^{t}N_i^T(x(\theta))d\theta]$$
$$\times\sum_{k=1}^{m}(a_k+b_k)N_k(x(t))+tr(\sigma(t,x_t)\sigma^T(t,x_t))$$
$$+\sum_{i=1}^{m}(a_i+b_i)(\sum_{k=1}^{m}|b_k|[h_k|N_k(x(t))|^2$$
$$-\int_{t-h_k}^{t}|N_k(x(\theta))|^2d\theta])$$
$$+\sum_{k=1}^{m}\gamma_2^{(k)}[|N_k(x(t))|^2-|N_k(x(t-h_k))|^2] \qquad (13)$$

and

$$d\bar{M}(t)=2\left[x^T(t)-\sum_{i=1}^{m}b_i\int_{t-h_i}^{t}N_i^T(x(\theta))d\theta\right]\sigma(t,x_t)dw_t.$$

Applying the Minkowski's inequality yields

$$2(\sum_{i=1}^{m}|b_i|\int_{t-h_i}^{t}N_i^T(x(\theta))d\theta)(\sum_{k=1}^{m}(a_k+b_k)N_k(x(t)))$$
$$\leq\sum_{i=1}^{m}\sum_{k=1}^{m}(a_k+b_k)|b_i|[h_i|N_k(x(t))|^2$$
$$+\int_{t-h_i}^{t}|N_i(x(\theta))|^2d\theta]. \qquad (14)$$

Using (14) in (13) leads to

$$F_t\leq-2\sum_{k=1}^{m}(a_k+b_k)x^T(t)N_k(x(t))$$
$$+\sum_{i=1}^{m}|b_i|h_i\sum_{k=1}^{m}(a_k+b_k)|N_k(x(t))|^2$$
$$+\sum_{i=1}^{m}\sum_{k=1}^{m}(a_k+b_k)|b_i|\int_{t-h_i}^{t}|N_i(x(\theta))|^2d\theta$$
$$+(\sum_{i=1}^{m}(a_i+b_i))\sum_{k=1}^{m}|b_k|h_k|N_k(x(t))|^2$$
$$-\sum_{i=1}^{m}(a_i+b_i)\sum_{k=1}^{m}|b_k|\int_{t-h_k}^{t}|N_k(x(\theta))|^2d\theta$$
$$+\sum_{k=1}^{m}\gamma_1^{(k)}|N_k(x(t))|^2+\sum_{k=1}^{m}\gamma_2^{(k)}|N_k(x(t-h_k))|^2$$
$$+\gamma(t)+\sum_{k=1}^{m}\gamma_2^{(k)}[|N_k(x(t))|^2-|N_k(x(t-h_k))|^2]$$
$$\leq-2\sum_{k=1}^{m}(a_k+b_k)x^T(t)N_k(x(t))$$
$$+(\sum_{i=1}^{m}(a_i+b_i))\sum_{k=1}^{m}|b_k|h_k|N_k(x(t))|^2$$
$$+\sum_{i=1}^{m}|b_i|h_i\sum_{k=1}^{m}(a_k+b_k)|N_k(x(t))|^2$$
$$+\sum_{k=1}^{m}(\gamma_1^{(k)}+\gamma_2^{(k)})|N_k(x(t))|^2+\gamma(t)$$
$$\leq-2\sum_{k=1}^{m}(a_k+b_k)[1-(\frac{|b_k|h_k}{2(a_k+b_k)}\sum_{i=1}^{m}(a_i+b_i)$$

$$+\frac{1}{2}\sum_{i=1}^{m}|b_i|h_i+\frac{\gamma_1^{(k)}+\gamma_2^{(k)}}{2(a_k+b_k)})]K_k x^T(t)N_k(x(t))+\gamma(t)$$
$$\leq -2\sum_{k=1}^{m}(a_k+b_k)[1-(\frac{|b_k|h_k}{2(a_k+b_k)}(\sum_{i=1}^{m}(a_i+b_i))$$
$$+\frac{1}{2}\sum_{i=1}^{m}|b_i|h_i+\frac{\gamma_1^{(k)}+\gamma_2^{(k)}}{2(a_k+b_k)})]K_k x^T(t)N_k(x(t))+\gamma(t)$$
$$\leq -2\sum_{k=1}^{m}(a_k+b_k)[1-\beta_k]x^T(t)N_k(x(t))+\gamma(t). \quad (15)$$

Then,

$$V(x(t),t)=V(0,t_0)+\int_{t_0}^{t}\big[\bar{F}_s+2\sum_{k=1}^{m}(a_k+b_k)$$
$$\times[1-\beta_k]x^T(s)N_k(x(s))\big]ds-2\sum_{k=1}^{m}(a_k+b_k)[1-\beta_k]$$
$$\times\int_{t_0}^{t}x^T(s)N_k(x(s))ds+\bar{M}_t=V(0,t_0)+\bar{B}_t^1-\bar{B}_t^2+\bar{M}_t,$$

where

$$\bar{B}_t^1=\int_{t_0}^{t}\max\{0,\bar{F}_s+2\sum_{k=1}^{m}(a_k+b_k)[1-\beta_k]x^T(s)N_k(x(s))\}ds,$$

$$\bar{B}_t^2 = \int_{t_0}^{t}[2\sum_{k=1}^{m}(a_k+b_k)[1-\beta_k]x^T(s)N_k(x(s)) \quad (16)$$
$$+\max\{0,-\bar{F}_s-2\sum_{k=1}^{m}(a_k+b_k)[1-\beta_k]x^T(s)N_k(x(s))\}]ds.$$

It is easy to see from (15) and (16) that

$$\bar{B}_t^1\leq\bar{A}_t^1=\int_{t_0}^{t}\gamma(s)ds,$$

$$\bar{B}_t^2\geq\bar{A}_t^2=2\sum_{k=1}^{m}(a_k+b_k)[1-\beta_k]\int_{t_0}^{t}x^T(t)N_k(x(t))dt.$$

Now, applying Lemma 1 and the inequalities (1) and (10) implies that $P\{V\rightarrow\}=1$ and, therefore, $P\{\sup_{t\geq t_0}V_t<H\}=1$ almost surely for a certain random variable $H=H(\omega)<\infty$. Next, the definition (13) of the functional V yields $P\{\sup_{t\geq t_0}|Y_t|^2<H\}=1$, $P\{\sup_{t\geq t_0}(V_1)_t<H\}=1$ and $P\{\sup_{t\geq t_0}(V_2)_t<H\}=1$. Finally, applying Lemma 2 (see (12)) and the inequality (5) implies that a.s.

$$\|x_t\|\leq\frac{1}{1-\alpha_2}[\|Y(\bullet,x_\bullet)\|_{C_{[t_0,t]}}+\|\phi\|_{C_{[t_0-h,t_0]}}]$$
$$\leq\frac{1}{1-\alpha_2}[H+\|\phi\|_{C_{[t_0-h,t_0]}}]=H_1,$$
$$\big|N_i(x(t))\big|^2\leq K_i x^T(t)N_i(x(t))\leq K_i\|x(t)\|\|N_i(x(t))\|,$$
therefore, $\|N_i(x(t))\|\leq K_i\|x(t)\|\leq K_i H_1$.

Thus,

$$P\{\sup_{t\geq t_0}|x|^2(t)\leq H_1^2\}=1,\quad P\{\sup_{t\geq t_0}\|N_i(x(t))\|\leq K_i H_1\}=1.$$

From (13), $V=|Y|^2+V^*$, where $V^*=V_1+V_2$. Almost sure convergence of $V_t=V(x_t,t)$, as $t\rightarrow\infty$, implies, in particular, that V_t is a.s. uniformly continuous on $[t_0,\infty)$. Let us show now that V_t^* is also a.s. uniformly continuous on $[t_0,\infty)$. Indeed,

$$(V_1)_t-(V_1)_\theta=(V_1)'_\kappa(t-\theta),$$

where κ is a point between t and θ, and

$$\big|(V_1)'_\kappa\big|=(\sum_{i=1}^{m}(a_i+b_i))\sum_{k=1}^{m}|b_k||h_k||N_k(x(\kappa))|^2$$
$$-\int_{\kappa-h_k}^{\kappa}|N_k(x(\theta))|^2d\theta\leq(\sum_{i=1}^{m}(a_i+b_i))$$
$$\times\sum_{k=1}^{m}|b_k||\int_{\kappa-h_k}^{\kappa}(|N_k(x(\kappa))|^2-|N_k(x(\theta))|^2)d\theta|$$
$$\leq(\sum_{i=1}^{m}(a_i+b_i))\sum_{k=1}^{m}|b_k||\int_{\kappa-h_k}^{\kappa}K_k^2\big|x(\kappa)-x(\theta)\big|^2d\theta|$$
$$\leq 4H_1^2(\sum_{i=1}^{m}(a_i+b_i))(\sum_{k=1}^{m}|b_k|h_k K_k^2).$$

Then, a.s.

$$\big|(V_1)_t-(V_1)_\theta\big|\leq 4H_1^2\left(\sum_{i=1}^{m}(a_i+b_i)\right)\left(\sum_{k=1}^{m}|b_k|h_k K_k^2\right)|t-\theta|.$$

It means that $(V_1)_t$ is a.s. uniformly continuous on $[t_0,\infty)$. To prove that $(V_2)_t$ is also a.s. uniformly continuous on $[t_0,\infty)$, note that

$$|(V_2)_t-(V_2)_\theta|=\sum_{k=1}^{m}\gamma_2^{(k)}$$
$$\times|\int_{t-h_k}^{t}|N_k(x(\theta))|^2d\theta-\int_{\theta-h_k}^{\theta}|N_k(x(\tau))|^2d\tau|$$
$$\leq\sum_{k=1}^{m}\gamma_2^{(k)}|\int_{t_0}^{t}|N_k(x(\theta))|^2d\theta-\int_{t_0}^{\theta}|N_k(x(\tau))|^2d\tau$$
$$-\int_{t_0}^{t-h_k}|N_k(x(\theta))|^2d\theta+\int_{t_0}^{\theta-h_k}|N_k(x(\tau))|^2d\tau|$$
$$=\sum_{k=1}^{m}\gamma_2^{(k)}[\int_{t}^{\theta}|N_k(x(\tau))|^2d\tau+\int_{t-h_k}^{\theta-h_k}|N_k(x(\tau))|^2d\tau]$$
$$\leq 2H_1^2\sum_{k=1}^{m}\gamma_2^{(k)}K_k^2|t-\theta|.$$

Thus, $Y_t^2=V_t-V_t^*$ has also to be a.s. uniformly continuous on $[t_0,\infty)$. The a.s. uniform continuity of $x(t)$ on $[t_0,\infty)$ can be obtained from the following inequalities

$$|x(t)-x(s)|=\Big|x(t)-\sum_{k=1}^{m}[b_k\int_{t-h_k}^{t}N_k(x(\tau))d\tau]-x(s)$$
$$+\sum_{k=1}^{m}[b_k\int_{s-h_k}^{s}N_k(x(\theta))d\theta\ ]+\sum_{k=1}^{m}[b_k\int_{t-h_k}^{t}N_k(x(\tau))d\tau]$$
$$-\sum_{k=1}^{m}[b_k\int_{s-h_k}^{s}N_k(x(\theta))d\theta]\Big|\leq|Y(x(t),t)-Y(x(s),s)|$$
$$+\sum_{k=1}^{m}|b_k||\int_{s}^{t}N_k(x(\tau))d\tau|+\sum_{k=1}^{m}|b_k||\int_{s-h_k}^{t-h_k}N(x(\tau))d\tau|$$
$$\leq|Y(x(t),t)-Y(x(s),s)|+2H_1\sum_{k=1}^{m}[|b_k|K_k]|t-s|.$$

Moreover, $x^T(t)N_k(x(t))$ is also a.s. uniformly continuous on $[t_0,\infty)$ for any $k=1,\dots,m$. Indeed, for any $t,\theta\ge t_0$, a.s.

$$|x^T(t)N_k(x(t))-x^T(\theta)N_k(x(\theta))|\le|N_k(x(t))||x(t)-x(\theta)|$$
$$+|x(\theta)||N_k(x(t))-N_k(x(\theta))|\le 2H_1K_k|x(t)-x(\theta)|.$$

Suppose now that $P\left\{\limsup_{t\to\infty}|x(t)|=\zeta_0(\omega)>0\right\}=p_0>0$. In view of continuity of a function N_i, $i=1,\dots,m$, and the condition (4), there exist a.s. such a finite random variable $\zeta_1(\omega)>0$ and a sequence of random moments $t_k=t_k(\omega)\to\infty$, as $k\to\infty$, that $P(\Omega_1)=p_0>0$, where $\Omega_1=\{\omega\colon|x^T(t_k)N_i(x(t_k))|(\omega)>\zeta_1(\omega)>0\}$. Since $x^T(t)N_i(x(t))$ is a.s. uniformly continuous on $[t_0,\infty)$, for $\varepsilon=\varepsilon(\omega)=\zeta_1(\omega)/2$, there exists $\delta=\delta(\omega)$ such that

$$|x^T(t_k)N_i(x(t_k))-x^T(s)N_i(x(s))|\le\varepsilon=\zeta_1(\omega)/2,$$

for $\omega\in\Omega_1$ and $|s-t_k|\le\delta$. Then, for $\omega\in\Omega_1$ and $s\in[t_k-\delta,t_k+\delta]$, the following inequality is obtained

$$|x^T(s)N_i(x(s))|\ge|x^T(t_k)N_i(x(t_k))|$$
$$-|x^T(t_k)N_i(x(t_k))-x^T(s)N_i(x(s))|\ge\zeta_1(\omega)/2.$$

Without loss of generality, suppose that $t_{k+1}(\omega)-t_k(\omega)>2\delta(\omega)$ for any $\omega\in\Omega_1$. Let $k(n)$ be the number of elements in the sequence $\{t_k\}$ belonging to the interval $[t_0,n]$. Applying inequality (17) implies, for $\omega\in\Omega_1$, that

$$\int_{t_0}^{n}x^T(s)N_i(x(s))ds\ge\sum_{k:t_0\le t_k+\delta\le n}\int_{t_k-\delta}^{t_k+\delta}x^T(s)N_i(x(s))ds$$
$$\ge\frac{\zeta_1}{2}\sum_{k:t_0\le t_k+\delta\le n}\int_{t_k-\delta}^{t_k+\delta}ds=\delta\zeta_1\sum_{k\le k(n)}1=k(n)\delta\zeta_1\to\infty,$$

as $n\to\infty$, since $k(n)\to\infty$ as $n\to\infty$. Hence, $P\{A_\infty^2=\infty\}\ge p_0>0$, which contradicts (1). Theorem 1 is proved.

The following examples illustrate applicability of the Theorem 1. The state equation in Example 1 contains two different nonlinear functions and two delays, as well as the linear term with a positive coefficient. Examples 2–4 present various nonlinear vector functions $N(x)$, satisfying the condition (5). This illustrates viability of the condition (5) in the multi-dimensional case.

Example 1. Consider the nonlinear system ($t_0=0$)

$$dx(t)=-b_1\frac{x^5(t-h_1)}{1+x^4(t-h_1)}dt-b_2x(t-h_2)dt$$
$$+0.3x(t)dt+\frac{\sqrt{(1+x^2(t))}}{1+t}dw_t. \qquad (17)$$

For a sufficiently large T and all $t>T$, the conditions of Theorem 1 can be satisfied. Indeed, the conditions (4)-(5) and (7) are satisfied for $N_1(u)=\frac{u^5}{1+u^4}$, $N_2(u)=u$, $K_1=K_2=1$, $b_1=b_2=0.5$, $a_1=0$, $a_2=-0.3$, $h_1=h_2=0.4$. The condition (8) is satisfied, since $K_1b_1h_1+K_2b_2h_2=0.4$. Finally, taking into account that

$$\beta_1=\left(\frac{|b_1|h_1}{2(a_1+b_1)}\left(\sum_{i=1}^{2}(a_i+b_i)\right)+\frac{1}{2}\sum_{i=1}^{2}|b_i|h_i\right)K_1=0.55$$

and

$$\beta_2=\left(\frac{|b_2|h_2}{2(a_2+b_2)}\left(\sum_{i=1}^{2}(a_i+b_i)\right)+\frac{1}{2}\sum_{i=1}^{2}|b_i|h_i\right)K_2=0.34,$$

the conditions (6) and (9) are also satisfied for $\gamma_2^{(1)}=\gamma_2^{(2)}=\gamma_1^{(1)}=0$, $\gamma(t)=\frac{1}{(1+t)^2}$ and $\gamma_1^{(2)}=\frac{1}{(1+t)^2}$. Since the equation (17) has a unique solution on $[-h,\infty)$, all the conditions, in particular, (6) and (9), can be verified only for sufficiently large $t>T$. Then, the coefficient $\gamma_1^{(2)}$ can be taken arbitrarily small.

Example 2. Let constants $C,A>0$ be such that for all $x\in\mathbf{R}^n$, $N(x)\in\mathbf{R}^n$, $n\ge2$,

$$|N(x)|^2\le C|x|^2 \text{ and } x^TN(x)\ge A|x|^2. \qquad (18)$$

Then, the condition (5) holds for $K=\frac{C}{A}$.

Example 3. Let $(N_i)_k^2(x)\le K_{ik}x_k^2$, and $x_kN_k(x)>0$ for all $x_k\ne0$, $x\in\mathbf{R}^n$, $k=1,2,\dots,n$, for any $i=1,2,\dots,m$, where $K_i>0$ are positive constants. Then, for $k=1,2\dots n$ and $i=1,2,\dots,m$,

$$(N_i)_k^2(x)=(N_i)_k(N_i)_k\le\sqrt{K_{ik}}x_k(N_i)_k(x) \text{ and}$$

$$|N_i(x)|^2\le\sum_{k=1}^{n}\sqrt{K_{ik}}x_k(N_i)_k(x)\le K_i\sum_{k=1}^{n}x_k(N_i)_k(x)=Kx^TN_i(x),$$

where $K_i=\max_{k=1,2,\dots,n}\{\sqrt{K_{ik}}\}$. Thus, the condition (5) holds.

Example 4. Let $f_i:\mathbf{R}\to\mathbf{R}$ be scalar continuous functions, $i=1,2$, and

$$\frac{1}{2}|u|\le|f_i(u)|\le|u|,\quad uf_i(u)>0, \qquad (19)$$
$$\text{for all}\quad u\ne0,\quad u\in\mathbf{R},\quad i=1,2.$$

Let $K=3$ and $-0.36<c<0.224$. For all $x=(x_1,x_2)\in\mathbf{R}^2$, define

$$N(x)=\begin{pmatrix} f_1(x_1)+cx_2\\ cx_1+f_2(x_2)\end{pmatrix}. \qquad (20)$$

Since

$$\begin{aligned}|N(x)|^2 &= f_1^2(x_1)+c^2x_2^2+2cx_2f_1(x_1)\\ &+ f_2^2(x_2)+c^2x_1^2+2cx_1f_2(x_2)\\ &\le \left(1+c^2\right)\left(x_1^2+x_2^2\right)+2c\left[x_2f_1(x_1)+x_1f_2(x_2)\right]\end{aligned}$$

and $x^TN(x)=x_1f_1(x_1)+2cx_1x_2+x_2f_2(x_2)$
$\ge\frac{1}{2}\left(x_1^2+x_2^2\right)+2cx_1x_2$, then

$$Kx^TN(x)-|N(x)|^2\ge\left(\frac{K}{2}-1-c^2\right)\left(x_1^2+x_2^2\right)$$
$$+2c\left[Kx_1x_2-x_2f_1(x_1)-x_1f_2(x_2)\right]$$
$$\ge\begin{cases}(\frac{K}{2}-1-c^2)\left(x_1^2+x_2^2\right)+2cx_1x_2(K-2), & \text{if } x_1x_2>0;\\ (\frac{K}{2}-1-c^2)\left(x_1^2+x_2^2\right)+2cx_1x_2(K-1), & \text{if } x_1x_2\le0.\end{cases}$$

For $K=3$, the last inequality takes the form

$$Kx^TN(x)-|N(x)|^2$$
$$\ge\begin{cases}(\frac{1}{2}-c^2)\left(x_1^2+x_2^2\right)+2cx_1x_2, & \text{if } x_1x_2>0;\\ (\frac{1}{2}-c^2)\left(x_1^2+x_2^2\right)+4cx_1x_2, & \text{if } x_1x_2\le0.\end{cases} \qquad (21)$$

It can be readily verified from (21) that the condition (5) holds for the function $N(x)$ defined by (20) and (19), if $K = 3$ and $-0.36 < c < 0.224$.

The following two examples are related to the case of one state delay and one nonlinear function in the state equation. In the second example, the proper dynamics of the system does not depend on the current value of the state variable but on its value at a certain previous point $t-h$, i.e., $a = 0$. In the first example, the term $N(x(t))$ is present with positive coefficient $a = \frac{b}{2}$.

Example 5.

$$dx(t) = -bN(x(t-h))dt + \frac{b}{2}N(x(t))dt + \frac{1}{1+t}dw_t, \ t_0 = 0,$$

where the function N is defined as follows

$$N(u) = \begin{cases} \sin u, & \text{if } |u| \le \frac{\pi}{2}, \\ u - \frac{\pi}{2} + 1, & \text{when } u \ge \frac{\pi}{2}, \\ u + \frac{\pi}{2} - 1, & \text{when } u \le -\frac{\pi}{2}. \end{cases}$$

It is easy to see that $y = N(x)$ satisfies the conditions (4)-(5) with $K = 1$. The conditions (6)-(10) are also satisfied with $b > 0$, $h \ge 0$, such that $bh < 1$, $\gamma_1 = \gamma_2 = 0$, $\gamma(t) = \frac{1}{(1+t)^2}$.

Example 6.

$$dx(t) = -b\frac{x^5(t-h)}{1+x^4(t-h)}dt + \frac{1}{1+t}dw_t, \quad t_0 = 0.$$

The function $N(u) = \frac{u^5}{1+u^4}$ satisfies conditions (4)-(5) with $K = 1$. The conditions (6)-(7) are also satisfied with $b > 0$, $h \ge 0$, such that $bh < 1$, $\gamma_1 = \gamma_2 = 0$, $\gamma(t) = \frac{1}{(1+t)^2}$.

Remark 3. In particular, Example 2 demonstrates that the case of an essentially nonlinear function $N(u)$ for the small u is also covered by Theorem 1. In Example 3, $N(u)$ behaviors like u^5 as $u \to 0$, $\lim\limits_{u\to 0} \frac{N(u)}{u^5} = 1$.

IV. Conclusions

The asymptotic stability problem has been considered for a vector nonlinear stochastic system governed by a multidimensional Ito differential equation with multiple drift terms, with and without state delays, and a nontrivial diffusion. No Lipshitz condition has been assumed for the nonlinear drift terms in the system. The global almost sure asymptotic stability conditions have been obtained and directly expressed in terms of the system coefficients. The Lyapunov-Krasovskii and degenerate functionals techniques have been used for establishing asymptotic stability in the framework of the Lyapunov direct method. The convergence theorem for semimartingale inequalities has served as a key tool for obtaining stability conditions in terms of stochastic system coefficients, without any transformation of the original system itself. The paper has introduced a systematic approach which would be applicable to design of the stability conditions for other classes of vector nonlinear stochastic systems with state delays.

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