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Stability Conditions for Decentralized Model Predictive Control under Packet Drop Communication

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Abstract— **We propose a decentralized model predictive control (MPC) design approach for possibly large-scale processes whose structure may not be dynamically decoupled. The decoupling assumption only appears in the prediction models used by the different MPC control agents. In [1] we presented a sufficient criterion for analyzing a posteriori the asymptotic stability of the process model in closed-loop with the set of decentralized MPC controllers. The communication model among neighboring MPC controllers was supposed faultless, so that each MPC could successfully receive the information about the states of its corresponding submodel. Here we present a sufficient condition for ensuring closed-loop stability of the overall closed-loop system when a certain number of packets containing state measurements may be lost.**

I. INTRODUCTION

Recently much attention has been directed towards the study of control methodologies for large-scale systems that can be often characterized by a set of multiple interconnected subsystems with constraints on information flows between them. The desirable goals of structuring a distributed information and decision framework for large scale systems do not "mesh" with the available centralized methodologies and procedures associated with classical and modern control theory, thus providing impetus for a decentralized control scheme. An important issue that arises in decentralized control is to determine under which conditions there exists a set of appropriate local feedback control laws that will stabilize the entire system. The main contributions in this research topic are given by [2]–[6].

The typical structure of such systems is composed by several local control stations. At each station the controller observes only local system outputs and controls only local inputs. All the controllers contribute, however, in controlling the overall large-scale system. Several different versions of the problem of coordinating local controllers acting on a spatially distributed system have been formulated and examined in [7]–[10].

The last two decades have seen the widespread diffusion of model predictive control (MPC) techniques, which are now recognized as a very useful approach to deal with control problems with several inputs and outputs and under constraints on such variables, as it is typically the case in the process industry. However, centralized MPC is largely viewed as impractical, for control of large-scale systems due to (i) the need of converging all the measurements in one single location, where the optimization is solved, and (ii) the computation time needed to solve the (large) optimization problem within a sampling step. Decentralized MPC (DMPC) is a decomposition of a single centralized MPC problem into a set of M subproblems, and each subproblem is assigned to a different model predictive controller. The goal of the decomposition is twofold: First, each subproblem is much smaller than the overall problem (that is, fewer decision variables and constraints), and second, each subproblem is coupled only to a few other subproblems. DMPC methods have already been studied in [11], [12], and in a number of papers cited there. Along with the benefits of a decentralized design, inherent issues in ensuring stability and feasibility of the system have to be faced.

In a previous paper [1], we presented a sufficient criterion for analyzing a posteriori the asymptotic stability of the process model in closed-loop with the set of decentralized MPC controllers. When information is propagated from sensors to controllers through wireless networks, data packet dropout can be a potential source of instability and poor performance of the overall system. The goal of this paper is to extend the results of [1] to the case where communication between neighbors is not guaranteed at every sampling time.

II. PROBLEM SETUP

We recall the MPC problem setup of [1].

A. Centralized MPC

Consider the standard MPC problem based on the linear discrete-time prediction model

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the vector of command variables at time step t, and the following finite-time optimal control problem

$$V(x(t)) = \min_{U} \quad x_N^\top P x_N + \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k \tag{2a}$$

$$\text{s.t.} \quad x_{k+1} = Ax_k + Bu_k,\ k = 0,\dots,N-1 \tag{2b}$$

$$x_0 = x(t) \tag{2c}$$

$$u_{\min} \le u_k \le u_{\max},\ k = 0,\dots,N_u-1 \tag{2d}$$

$$u_k = 0,\ k = N_u,\dots,N-1 \tag{2e}$$

where N is the prediction horizon, $N_u \le N$ is the input horizon, and $u_{\min} < 0 < u_{\max} \in \mathbb{R}^m$ define saturation constraints on input variables, and "$\le$" denotes componentwise inequalities.

Problem (2) can be recast as a Quadratic Programming (QP) problem (see e.g. [13], [14]), whose solution

$$U^*(x(t)) \triangleq [u_0^{*\top}(x(t)) \ \dots \ u_{N-1}^{*\top}(x(t))]^\top \in \mathbb{R}^{Nm}$$

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is a sequence of optimal control inputs. In (2) we assume that $Q = Q^\top \geq 0$, $R = R^\top > 0$ are square weight matrices defining the performance index, and that the terminal weight $P = P^\top \geq 0$ is a square matrix satisfies the Lyapunov equation

$$A^\top PA - P = -Q \tag{3}$$

so that the cost (2a) is equal to $\sum_{k=0}^{\infty} x_k^\top Q x_k + u_k^\top R u_k$. The existence of matrix P is ensured by the following assumption

Assumption 1: Matrix A is strictly Hurwitz.

Another restriction taken in this paper is that problem (2) only tackles input constraints (2d), which makes problem (2) feasible for any value of the state vector $x(t) \in \mathbb{R}^n$

At each sampling time t, problem (2) is solved for the given measured (or estimated) current state $x(t)$. Only the first optimal move $u_0^*(x(t))$ of the optimal sequence $U^*(x(t))$ is applied to the process,

$$u(t) = u_0^*(x(t)), \tag{4}$$

the remaining optimal moves are discarded and the optimization is repeated at time $t+1$.

Theorem 1 ([15]): Under Assumption 1, system (1) in closed-loop with the MPC algorithm (2), (4) is asymptotically stable.

B. Decentralized MPC

Let the system to be controlled be described again by the process model (1). Matrices A, B will have a certain number of negligible components corresponding to partially dynamically decoupled subsystems, or even be block diagonal in case of total dynamical decoupling (this is the case for instance of independent moving agents each one having its own dynamics, as in [6], [7]).

Let M be the number of decentralized control actions that we want to design, for example $M = m$ in case each individual actuator is governed by its own controller. For all $i = 1, \ldots, M$, we define

$$x^i = W_i^\top x = \begin{bmatrix} x^{i1} \\ \vdots \\ x^{in_i} \end{bmatrix}, \; u^i = Z_i^\top u = \begin{bmatrix} u^{i1} \\ \vdots \\ u^{im_i} \end{bmatrix}$$

where $x^i \in \mathbb{R}^{n_i}$ as the vector collecting a subset $I_{xi} \subseteq \{1, \ldots, n\}$ of the state components, $W_i \in \mathbb{R}^{n \times n_i}$ collects the n_i columns of the identity matrix of order n corresponding to the indices in I_{xi}, and, similarly, u^i is the vector of input signals tackled by the i-th controller, where $Z_i \in \mathbb{R}^{m \times m_i}$ collects m_i columns of the identity matrix of order m corresponding to the set of indices $I_{ui} \subseteq \{1, \ldots, m\}$. Note that $W_i^\top W_i = I_{n_i}$, $Z_i^\top Z_i = I_{m_i}$, $\forall i = 1, \ldots, M$.

An *approximation* of (1) is obtained by getting M new prediction models of reduced order

$$x^i(t+1) = A_i x^i(t) + B_i u^i(t) \tag{5}$$

where matrices $A_i = W_i^\top A W_i \in \mathbb{R}^{n_i \times n_i}$ and $B_i = W_i^\top B Z_i \in \mathbb{R}^{m_i \times m_i}$ are submatrices of the original A and B matrices, respectively, describing in a possibly approximate way the evolution of the states of subsystem #i.

Assumption 2: Matrix A_i is strictly Hurwitz, for all $i = 1, \ldots, M$.

Note that each model (5) has in general a smaller size than the original process model (1). The choice of the dimensions n_i, m_i and of matrices W_i, Z_i are a tuning knob of the decentralized procedure and should be inspired by the inspection of zero or negligible entries in A, B (or in other words, by physical insight on the process dynamics) and by taking into account the requirement stated in Assumption 2.

We design a controller for each set of moves $u^i \in \mathbb{R}^{m_i}$ according to the prediction model (5) and based on feedback on x^i, for all $i = 1, \ldots, M$. Note that in general different states x^i, x^j and different u^i, u^j may share common components. For the sake on simplicity of notation, since now on we will assume that $M = m$ and that $I_{ui}^{\#} = i$, $i = 1, \ldots, m$, i.e., that each controller #i only controls the ith input signal. In general $I_{xi} \cap I_{xj} \neq \emptyset$, meaning that controller #i may partially share the same feedback information with controller #j, and $I_{ui} \cap I_{uj} \neq \emptyset$, meaning that controller #i may take into account the effect of control actions that are actually decided by another controller #j, $i \neq j$, $i, j = 1, \ldots, M$.

For all $i = 1, \ldots, M$ consider the following infinite-time constrained optimal control problem

$$\begin{aligned} V_i(x(t)) = \min_{u_0^i} \sum_{k=0}^{\infty} & x_k^{i\top} W_i^\top Q W_i x_k^i + u_k^{i\top} Z_i^\top R Z_i u_k^i = \\ = \min_{u_0^i} & x_1^{i\top} P_i x_1^i + x^{i\top}(t) W_i^\top Q W_i x^i(t) + \\ & u_0^{i\top} Z_i^\top R Z_i u_0^i && \text{(6a)} \\ \text{s.t. } & x_1^i = A_i x^i(t) + B_i u_0^i, && \text{(6b)} \\ & x_0^i = W_i^\top x(t) = x^i(t) && \text{(6c)} \\ & u_{\min} \leq u_0^i \leq u_{\max}, && \text{(6d)} \\ & u_k^i = 0, \; \forall k \geq 1 && \text{(6e)} \end{aligned}$$

where $P_i = P_i^\top \geq 0$ is the solution of the Lyapunov equation

$$A_i^\top P_i A_i - P_i = -W_i^\top Q W_i, \tag{7}$$

that exists by virtue of Assumption 2. Problem (6) corresponds to a finite horizon problem with control horizon $N_u = 1$.

At time t, each controller MPC #i measures (or estimates) the state $x^i(t)$ (usually corresponding to local and neighboring states), solves problem (6), and obtains the optimizer $u_0^{*i} = [u_0^{*i1}, \ldots, u_0^{*ii}, \ldots, u_0^{*im_i}]^\top \in \mathbb{R}^{m_i}$. In the simplified case $M = m$ and $I_{ui}^{\#} = i$, only the i-th sample of u_0^{*i}

$$u^i(t) = u_0^{*ii} \tag{8}$$

will determine the i-th component $u^i(t)$ of the input vector actually implemented to the process at time t.

The collection of the optimal inputs of all the M MPC controllers $u(t) = [u_0^{*11} \; \ldots \; u_0^{*ii} \; \ldots \; u_0^{*mm}]^\top$ is the actual input commanded to process (1). The optimizations (6) are repeated at time $t+1$, based on the new states $x^i(t+1) = W_i^\top x(t+1)$, according to the usual receding horizon control paradigm.

III. PACKET DROP

As mentioned in Section I, one of the issues raised in networked control systems is the unreliability of communication channels, which may result in data packet dropout. The non-triviality of this issue comes from the fact that if a set of measures for subsystem i is lost, this would not only affect the trajectory of subsystem i because of the improper control action u^i, but due to the dynamical coupling, also the trajectories of subsystems $j \in J$, where $J = \{j \mid i \in I_{xj} \cup I_{uj}\}$, and thus the performance of the overall system. In the following subsection, we will derive a sufficient condition for ensuring closed-loop stability of the overall system with the DMPC controllers in the case where packets containing measurements are lost for at most one time step. Next, we generalize this result to arbitrary, yet finite durations of data packet dropouts.

A. Single Packet Dropout

We recall here briefly the main theorem contained in [1].

Theorem 2: If one of the following conditions is satisfied for all $x \in \mathbb{R}^n$

$$(i)\ x^\top \left(\sum_{i=1}^{M} W_i W_i^\top Q W_i W_i^\top\right) x - \sum_{i=1}^{M} \Delta S_i(x) \geq 0 \quad (9)$$

$$(ii)\ x^\top \left(\sum_{i=1}^{M} W_i W_i^\top Q W_i W_i^\top\right) x - \alpha x^\top x - \sum_{i=1}^{M} \Delta S_i(x) + \sum_{i=1}^{M} u_0^{*i\top}(x) Z_i^\top R Z_i u_0^{*i}(x) \geq 0 \quad (10)$$

for some scalar $\alpha > 0$, then the decentralized MPC scheme defined in (6)–(8) in closed loop with (1) is globally asymptotically stable.

Condition (9) or condition (10) ensure closed-loop stability in the case where no packet loss occurs. Consider now the case where a single packet of measurements for controller i is lost at a generic instant t. This means that given $x(t-1), u(t-1)$, the next measurement of the state of subsystem i is available at time $t+1$. At time t, the optimization problem (6a)-(6d) can not be solved since constraint (6c) can not be directly fulfilled. Constraints (6b)-(6c) need to be replaced by the following equation, using an estimate $\tilde{x}^i(t)$ of the state $x^i(t)$, so that

$$\begin{cases} \tilde{x}_1^i(t) = A_i^2 W_i^\top x(t-1) + A_i B_i u_0^{*i}(t-1) + B_i u_0^i(t) \\ \tilde{x}^i(t) = A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1) \end{cases} \quad (11)$$

where $\tilde{x}^i(t)$ is the state at time $t+1$ estimated at time t from the available information. The computation of the input vector $u_0^{*i}(t)$ made by the i-th controller is based on the estimate $\tilde{x}^i(t)$. The value function V_i at time t is obtained by using (11) into (6a), is $V_i(\tilde{x}^i(t)) = (A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1))^\top (W_i^\top Q W_i)(A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1)) + (A_i(A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1)) + B_i u_0^{*i}(t))^\top P_i (A_i(A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1) + B_i u_0^{*i}(t)) + u_0^{*i\top}(t) Z_i^\top R Z_i u_0^{*i}(t)$. As the input $u_0^i(t) = 0$ satisfies the constraints $u_{min} \leq u_t^i \leq u_{max}$, the quantity $V_i(\tilde{x}(t)) - V_i(x(t-1))$ satisfies $V_i(\tilde{x}^i(t)) - V_i(x(t-1)) \leq (A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1))^\top (W_i^\top Q W_i)(A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1)) + (A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1))^\top A_i^\top P_i A_i (A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1)) - (W_i^\top x(t-1))^\top (W_i^\top Q W_i)(W_i^\top x(t-1)) - (A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1))^\top Pi (A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1)) - u_0^{*i\top}(t-1) Z_i^\top R Z_i u_0^{*i}(t-1)$.

By recalling (7) we obtain that

$$V_i(\tilde{x}^i(t)) - V_i(x(t-1)) \leq -(W_i^\top x(t-1))^\top (W_i^\top Q W_i) (W_i^\top x(t-1)) - u_0^{*i\top}(t-1) Z_i^\top R Z_i u_0^{*i}(t-1). \quad (12)$$

Inequality (12) implies that even if a packet loss occurs at time t for a generic subsystem i, the value function V_i of problem (6a) does not increase from time $t-1$ to time t.

State information for the i-th controller resumes at time $t+1$ by hypothesis. By rewriting $x(t+1) = Ax(t) + Bu(t) = A^2 x(t-1) + ABu(t-1) + BZ_i u_0^{*i}(t) + B\Delta u^i(t) = A(W_i(A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1)) + \Delta Y_i(x(t-1))) + BZ_i u_0^{*i}(t) + B\Delta u^i(t))$, where

$$\Delta Y_i(x(t-1)) = W_i W_i^\top (A W_i W_i^\top \Delta x^i(t-1) + B Z_i Z_i^\top \Delta u^i(t-1)) + \Delta A_i x(t-1) + \Delta B_i u(t-1) \quad (13a)$$

$$\Delta u^i(t) \triangleq u(t) - Z_i u_0^{*i}(t) \quad (13b)$$

$$\Delta x^i(t) \triangleq (I - W_i W_i^\top) x(t) \quad (13c)$$

$$\Delta A_i \triangleq (A - W_i W_i^\top A W_i W_i^\top) \quad (13d)$$

$$\Delta B_i \triangleq (B - W_i W_i^\top B Z_i Z_i^\top), \quad (13e)$$

it is easy to show that $V_i(x(t+1)) \leq (x(t+1))^\top W_i P_i W_i^\top (x(t+1)) = (\tilde{x}^i(t))^\top A_i^\top P_i A_i (\tilde{x}^i(t)) + \Delta S_i(x(t-1)) = (\tilde{x}^i(t))^\top P_i (\tilde{x}^i(t)) - (\tilde{x}^i(t))^\top W_i^\top Q W_i (\tilde{x}^i(t)) + \Delta S_i(x(t-1))$ and that

$$V_i(x(t+1)) - V_i(\tilde{x}^i(t)) \leq -(\tilde{x}^i(t))^\top W_i^\top Q W_i (\tilde{x}^i(t)) - u_0^{*i}(t)^\top Z_i^\top R Z_i u_0^{*i}(t) + \Delta S_i(x(t-1)). \quad (14)$$

As the input $u_0^i(t) = 0$ satisfies the constraints $u_{min} \leq u_t^i \leq u_{max}$, using (12) and (14) we finally obtain $V_i(x(t+1)) \leq V_i(x(t-1)) - (W_i^\top x(t-1))^\top (W_i^\top Q W_i + A_i^\top W_i^\top Q W_i A_i)(W_i^\top x(t-1)) - u_0^{*i\top}(t-1)(Z_i^\top R Z_i + B_i^\top W_i^\top Q W_i B_i) u_0^{*i}(t-1) - ((A_i W_i^\top x(t-1))^\top W_i^\top Q W_i B_i) u_0^{*i}(t-1) + \Delta S_i(x(t-1))$, where

$$\Delta S_i(x(t-1)) = 2(A(W_i(A_i W_i^\top x(t-1) + B_i u_0^{*i}(t-1))))^\top W_i^\top P_i W_i A(\Delta Y_i(x(t-1))) + (A(\Delta Y_i(x(t-1))))^\top W_i^\top P_i W_i A(\Delta Y_i(x(t-1))). \quad (15a)$$

We are now ready to state the following theorem:

Theorem 3: If condition (9) or (10) holds, and condition

$$x^\top \left(\sum_{i=1}^{M} W_i W_i^\top Q W_i W_i^\top + \sum_{i=1}^{M} W_i A_i^\top W_i^\top Q W_i \cdot A_i W_i^\top\right) x + x^\top \left(\sum_{i=1}^{M} W_i A_i^\top W_i^\top Q W_i B_i u_0^{*i}(x)\right) - \sum_{i=1}^{M} \Delta S_i(x) \geq 0 \quad (16)$$

is satisfied $\forall x \in \mathbb{R}^n$, then the DMPC scheme with single packet dropout defined in (6a), (11) in closed loop with (1) is globally asymptotically stable.

Proof: Define the function

$$V(x(t_k)) \triangleq \sum_{i=1}^{M} V_i(x(t_k)), \tag{17}$$

where t_k belongs to the subset $T \subseteq \mathbb{N}$ of time instants where no packet loss occurs. $V(x(t_k))$ satisfies the following properties

$$(i)\ V(x(t_k)) \geq 0, \tag{18a}$$

$$(ii)\ V(x(t_{k+1})) \leq V(x(t_k)) - Z_{t_k}, \tag{18b}$$

where

$$Z_{t_k} = \begin{cases} V(x(t_k)) - V(x(t_k-1)) & \text{no packet loss at time } t_k - 1 \\ V(x(t_k)) - V(x(t_k-2)) & \text{single packet loss at time } t_k - 1 \end{cases} \tag{19}$$

It follows from (9) or (10) and (16) that $Z_k \geq 0$ and hence $V(x(t_k))$ is non-increasing. Since $V(x(t_k)) \geq 0$, $\forall t_k \geq 0$, it follows that there exists $\lim_{t_k\to\infty} V(x(t_{k+1})) = \lim_{t_k\to\infty} V(x(t_k))$. Hence, let $T_1 \subseteq T$ be the subset of time instants t_k such that $Z_{t_k} = V(x(t_k)) - V(x(t_k-1))$, and let $t_{k_1} \in T_1$. Let $T_2 \subseteq T$ be the subset of time instants such that $Z_{t_k} = V(x(t_k)) - V(x(t_k-2))$, and let $t_{k_2} \in T_2$. As we are interested in asymptotic convergence, we consider the most general case in which both T_1 and T_2 are unbounded. By (9), it follows that $\lim_{t_{k_1}\to\infty} u_0^{*i}(x(t_{k_1})) Z_i^\top R Z_i u_0^{*i^\top}(x(t_{k_1})) = 0$, and by positive definiteness of $Z_i^\top R Z_i$, that $\lim_{t_{k_1}\to\infty} u_0^{*i}(x(t_{k_1})) = 0$, and hence that $\lim_{t_{k_1}\to\infty} u_0^{*ii}(x(t_{k_1})) = 0$, $\forall i = 1,\ldots,M$, which in turn implies $\lim_{t_{k_1}\to\infty} u(t_{k_1}) = 0$. Moreover, it also follows that $\lim_{t_{k_2}\to\infty} x(t_{k_2})^\top (\sum_{i=1}^M W_i W_i^\top Q W_i W_i^\top + W_i A_i^\top W_i^\top Q W_i A_i W_i^\top) x(t_{k_2}) + \sum_{i=1}^M u_0^{*i^\top}(x(t_{k_2}))(Z_i^\top R Z_i + B_i^\top W_i^\top Q W_i B_i) u_0^{*i}(x(t_{k_2})) + x^\top(t_{k_2}) \sum_{i=1}^M (W_i A_i^\top W_i^\top Q W_i B_i) u_0^{*i}(x(t_{k_2})) - \sum_{i=1}^M \Delta S_i(x(t_{k_2})) = 0$. Because of (16), it follows that $\lim_{t_{k_2}\to\infty} u_0^{*i}(x(t_{k_2}))(Z_i^\top R Z_i + B_i^\top W_i^\top Q W_i B_i) u_0^{*i^\top}(x(t_{k_2})) = 0$, and by positive definiteness of $Z_i^\top R Z_i + B_i^\top W_i^\top Q W_i B_i$, that $\lim_{t_{k_2}\to\infty} u_0^{*i}(x(t_{k_2})) = 0$, and hence that $\lim_{t_{k_2}\to\infty} u_0^{*ii}(x(t_{k_2})) = 0$, $\forall i = 1,\ldots,M$, which in turn implies $\lim_{t_{k_2}\to\infty} u(t_{k_2}) = 0$. Since $T = T_1 \cup T_2$, and by Assumption 1 the open-loop process (1) is linear and asymptotically stable, and therefore input-to-state stable, it also follows that $\lim_{t_k\to\infty} x(t_k) = 0$. It remains to show that $x(t_{\bar{k}}) \to 0$, where $t_{\bar{k}} \in \mathbb{N}\backslash T$, that is the subset of time instants where a packet loss occurs. If $\mathbb{N}\backslash T$ is bounded, clearly $\lim_{t\to\infty} x(t) = 0$ follows. Otherwise, consider $x(t_{\bar{k}}) = A_i x_i(t_{\bar{k}}-1) + B_i u_0^{*i}(t_{\bar{k}}-1) + \Delta A_i \Delta x^i(t_{\bar{k}}-1) + \Delta B_i \Delta u_0^{*i}(t_{\bar{k}}-1) = A_i W_i^\top x(t_{\bar{k}}-1) + B_i u_0^{*i}(t_{\bar{k}}-1) + \Delta A_i (I - W_i W_i^\top) x(t_{\bar{k}}-1) + \Delta B_i (u(t_{\bar{k}}-1) - Z_i u_0^{*i}(t_{\bar{k}}-1))$. Since $t_{\bar{k}}-1$ is a time instant when no packet loss occurs, it follows that $\lim_{t_{\bar{k}}\to\infty} x(t_{\bar{k}}-1) = 0$, and hence $\lim_{t_{\bar{k}}\to\infty} x(t_{\bar{k}}) = 0$. □

B. Multiple Packet Dropout

The conclusions of the previous section can be extended to the case of multiple packet losses. In this scenario, the malfunctioning of the network persists for a certain number N of time instants, $t \in T_s \triangleq [t,\ldots,t+N-1]$, $N \geq 2$. During the time interval T_s each subsystem i affected by the malfunctioning of the communication channel works in open- loop, computing its optimal vectors of moves $u_0^{*i}(t),\ldots,u_0^{*i}(t+N-1)$ using an estimate $\tilde{x}^i(t),\ldots,\tilde{x}^i(t+N-1)$ of its own states and of the states of the neighbors, based on the following linear prediction model

$$\tilde{x}^i(t+1) = A_i \tilde{x}^i(t) + B_i u_0^{*i}(t), t \in T_s, \tag{20a}$$

where $u_0^{*i}(t)$ is a function of $\tilde{x}(t)$ computed as in (6a)-(6d). We want to derive a condition for ensuring global asymptotical stability of the DMPC scheme in closed-loop with (1) in the presence of multiple packet dropouts. The non-triviality of this issue comes from the fact that the set of moves $u^{*i}(t), t \in T_s$, applied by agent i to the overall system, are computed on the basis of the information retrieved by the neighbors' states, and in this case of the prediction of the neighbors' states and moves. A bad choice of matrices W_i, Z_i would lead to a bad set of moves, and hence this would affect the performance and possibly the stability of the entire system.

Since, by Assumption 2, subsystem i is Lyapunov stable, we obtain that $V_i(\tilde{x}^i(t+N-k)) \leq V_i(\tilde{x}^i(t+N-k-1)) - (\tilde{x}^i(t+N-k-1))^\top W_i^\top Q W_i(\tilde{x}^i(t+N-k-1)) - (u_0^{*i}(t+N-k-1))^\top Z_i^\top R Z_i(u_0^{*i}(t+N-k-1))$ $\forall k = 1,..,N-1$. As the inputs $u_0^{*i}(t) = 0,..,u_0^{*i}(t+N-1) = 0$ satisfy the constraints $u_{min} \leq u_t^i \leq u_{max}$, the quantity $V_i(\tilde{x}^i(t+N-1)) - V_i(\tilde{x}^i(t))$ satisfies the following inequality

$$V_i(\tilde{x}^i(t+N-1)) \leq V_i(\tilde{x}^i(t)) - \sum_{k=0}^{N-2} (\tilde{x}^i(t+k))^\top W_i^\top Q W_i (\tilde{x}^i(t+k)). \tag{21}$$

Using the arguments of the previous section we can derive the following two inequalities. The first inequality describes the relation between the cost function V_i at time t, when the first set of measures is lost, and time $t-1$ when the last set of measures is successfully achieved by agent i.

$$V_i(\tilde{x}^i(t)) \leq V_i(x_i(t-1)) - (W_i^\top x(t-1))^\top (W_i^\top Q W_i) (W_i^\top x(t-1)) - u_0^{*i\top}(t-1) Z_i^\top R Z_i u_0^{*i}(t-1). \tag{22}$$

The second inequality describes the relation between the cost function V_i at time $t+N$, when the first set of measures is finally achieved again, and time $t+N-1$ when the final set of measures is lost, that is

$$V_i(x(t+N)) \leq V_i(\tilde{x}^i(t+N-1)) - (\tilde{x}^i(t+N-1))^\top (W_i^\top Q W_i)(\tilde{x}^i(t+N-1)) - u_0^{*i\top}(t+N-1) Z_i^\top R Z_i u_0^{*i}(t+N-1) + \Delta S_i^N(x(t-1)) \tag{23}$$

Where, $x(t+N) = A^{N+1}x(t-1) + A^N B u(t-1) + \sum_{j=0}^{N-1} A^j (B Z_i u_0^{*i}(t+N-1-j) + B\Delta u^i(t+N-1-j)) =$

$A^N(W_i(A_iW_i^\top x(t-1)+B_iu_0^{*i}(t-1))+\Delta Y_i(x(t-1)))+\sum_{j=0}^{N-1} A^j(BZ_iu_0^{*i}(t+N-1-j)+B\Delta u^i(t+N-1-j))$. As the inputs $u_0^{*i}(t)=0,..,u_0^{*i}(t+N-1)=0$ satisfy the constraints $u_{min} \leq u_t^i \leq u_{max}$, using (23), (21) and (22) we finally obtain $V_i(x(t+N)) \leq V_i(x(t-1))-(W_i^\top x(t-1))^\top(W_i^\top QW_i+(A_i^{N\top}W_i^\top QW_iA_i^N)+\sum_{k=0}^{N-2}(A_i^{k\top}W_i^\top QW_iA_i^k))(W_i^\top x(t-1))-u_0^{*i}(t-1)^\top(Z_i^\top RZ_i+(A_i^{N-1}B_i)^\top W_i^\top QW_iA_i^{N-1}B_i)u_0^{*i}(t-1)-(A_i^NW_i^\top x(t-1))^\top(W_i^\top QW_iA_i^{N-1}B_i+\sum_{k=0}^{N-2} A_i^{\top k}W_i^\top QW_iA_i^kB_i)u_0^{*i}(t-1)+\Delta S_i^N(x(t-1))$, where

$$\Delta S_i^N(x(t-1))=2(A(W_i(A_iW_i^\top x(t-1)+B_iu_0^{*i}(t-1))))^\top W_i^\top P_iW_iA^N(\Delta Y_i(x(t-1)))+(A^N(\Delta Y_i(x(t-1))))^\top W_i^\top P_iW_iA^N(\Delta Y_i(x(t-1))). \tag{24a}$$

We are now ready to state the following theorem

Theorem 4: If condition (9) or (10) holds, and condition

$$x^\top(\sum_{i=1}^M W_iW_i^\top QW_iW_i^\top+\sum_{i=1}^M W_iA_i^{N\top}W_i^\top QW_i A_i^NW_i^\top+\sum_{i=1}^M(\sum_{k=0}^{N-2} W_iA_i^{k\top}W_i^\top QW_iA_i^kW_i^\top))x+ x^\top\sum_{i=1}^M(W_iA_i^{\top N}W_i^\top QW_iA_i^{N-1}B_i+ \sum_{k=0}^{N-2} W_iA_i^{\top k}W_i^\top QW_iA_i^kB_iu_0^{*i}(x))-\sum_{i=1}^M\Delta S^i(x)\geq 0 \tag{25}$$

is satisfied $\forall x \in \mathbb{R}^n$, then the decentralized MPC scheme with N packets dropout defined in (6a), (20a) in closed loop with (1) is globally asymptotically stable.

Proof: Define the function

$$V(x(t_k)) \triangleq \sum_{i=1}^M V_i(x(t_k)), \tag{26}$$

where t_k belongs to the subset $T \subseteq \mathbb{N}$ of time instants where no packet loss occurs. $V(x(t_k))$ has the following properties

$$(i)\ V(x(t_k)) \geq 0, \tag{27a}$$
$$(ii)\ V(x(t_{k+1})) \leq V(x(t_k))-Z_{t_k} \tag{27b}$$

where

$$Z_{t_k}=\begin{cases} V(x(t_k))-V(x(t_k-1)), & \text{no packet loss} \\ V(x(t_k))-V(x(t_k-N-1)), & N \text{ packet losses.}\end{cases} \tag{28}$$

It follows from (9) or (10) and (16) that $Z_k \geq 0$ and hence $V(x(t_k))$ is non-increasing. Since $V(x(t_k)) \geq 0$, $\forall t_k \geq 0$, it follows that there exists $\lim_{t_k\to\infty} V(x(t_{k+1}))=\lim_{t_k\to\infty} V(x(t_k))$. Hence, let $T_1 \subseteq T$ be the subset of time instants such that $Z_{t_k}=V(x(t_k))-V(x(t_k-1))$, and let $t_{k_1} \in T_1$. and Let $T_2 \subseteq T$ be the subset of time instants such that $Z_{t_k}=V(x(t_k))-V(x(t_k-N-1))$, and let $t_{k_2} \in T_2$. As we are interested in asymptotic convergence, we consider the most general case in which both T_1 and T_2 are unbounded. By (9), it follows that $\lim_{t_{k_1}\to\infty} u_0^{*i\top}(x(t_{k_1}))Z_i^\top RZ_iu_0^{*i}(x(t_{k_1}))=0$. and by positive definiteness of $Z_i^\top RZ_i$, that $\lim_{t_{k_1}\to\infty} u_0^{*i\top}(x(t_{k_1}))=0$, and hence that $\lim_{t_{k_1}\to\infty} u_0^{*ii\top}(x(t_{k_1}))=0$, $\forall i=1,..,M$, which in turn implies $\lim_{t_{k_1}\to\infty} u(t_{k_1})=0$. Hence, because of (25), it also follows that $\lim_{t_{k_2}\to\infty} x^\top(t_{k_2})(W_iW_i^\top QW_iW_i^\top+W_iA_i^{N\top}W_i^\top QW_iA_i^NW_i^\top+\sum_{k=0}^{N-2} W_iA_i^{k\top}W_i^\top QW_iA_i^kW_i^\top)x(t_{k_2})+u_0^{*i}(x(t_{k_2}))^\top(Z_i^\top RZ_i+(A_i^{N-1}B_i)^\top W_i^\top QW_iA_i^{N-1}B_i)u_0^{*i}(x(t_{k_2}))+x^\top(t_{k_2})(W_iA_i^{N\top}W_i^\top QW_iA_i^{N-1}B_i+\sum_{k=0}^{N-2}(W_iA_i^k)^\top W_i^\top QW_iA_i^kB_i)u_0^{*i}(x(t_{k_2}))-\Delta S_i^N(x(t_{k_2}))=0$. Because of (16), it follows that $\lim_{t_{k_2}\to\infty}\sum_{i=1}^M u_0^{*i\top}(x(t_{k_2}))(Z_i^\top RZ_i+(A_i^{N-1}B_i)^\top W_i^\top QW_iA_i^{N-1}B_i)u_0^{*i}(x(t_{k_2})=0$, and by positive definiteness of $Z_i^\top RZ_i+(A_i^{N-1}B_i)^\top W_i^\top QW_iA_i^{N-1}B_i$, that $\lim_{t_{k_2}\to\infty} u_0^{*i}(x(t_{k_2}))=0$, and hence that $\lim_{t_{k_2}\to\infty} u_0^{*ii}(x(t_{k_2}))=0$, $\forall i=1,\dots,M$, which in turn implies $\lim_{t_{k_2}\to\infty} u(t_{k_2})=0$. Since $T=T_1\cup T_2$, and by Assumption 1 the open-loop process (1) is linear and asymptotically stable, and therefore input-to-state stable, it also follows that $\lim_{t_k\to\infty} x(t_k)=0$. It remains to show that $x(t_{\bar{k}})\to 0$, where $t_{\bar{k}} \in \mathbb{N}\backslash T$, that is the subset of time instants where a packet loss occurs. If $\mathbb{N}\backslash T$ is bounded, clearly $\lim_{t\to\infty} x(t)=0$ follows. Otherwise, consider

$$\begin{aligned}
x(t_{\bar{k}}) &= A_i^{N+1}x_i(t_{\bar{k}}-N-1)+A_i^NB_iu_0^{*i}(t_{\bar{k}}-N-1)+ \\
&\Delta A_i^{N+1}\Delta x^i(t_{\bar{k}}-N-1)+\Delta A_i^N\Delta B_i\Delta u_0^{*i}(t_{\bar{k}}-N-1)+ \\
&\sum_{j=0}^{N-1} A_i^jB_iu_0^{*i}(t_{\bar{k}}-j-1)+\sum_{j=0}^{N-1}\Delta A_i^j\Delta B_i\Delta u_0^{*i}(t_{\bar{k}}-j-1) \\
&= A_i^{N+1}W_i^\top x(t_{\bar{k}}-N-1)+A_i^NB_iu_0^{*i}(x(t_{\bar{k}}-N-1))+, \\
&\Delta A_i^{N+1}(I-W_iW_i^\top)x(t_{\bar{k}}-N-1)+ \\
&\Delta A_i^N\Delta B_i\Delta u_0^{*i}(x(t_{\bar{k}}-N-1))+\sum_{j=0}^{N-1} A_i^jB_iu_0^{*i}(t_{\bar{k}}-j-1)+ \\
&\sum_{j=0}^{N-1}\Delta A_i^j\Delta B_i\Delta u_0^{*i}(t_{\bar{k}}-j-1)
\end{aligned} \tag{29a}$$

Since $t_{\bar{k}}-N-1$ is a time instant when no packet loss has occurred, it follows that $\lim_{t_{\bar{k}}\to\infty} x(t_{\bar{k}}-N-1)=0$ and so $\lim_{t_{\bar{k}}\to\infty} u(x(t_{\bar{k}}-N-1))=0$. Since all the inputs $u(t_{\bar{k}}-j-1)$, $j=0,..,N-1$, are function of $u(x(t_{\bar{k}}-N-1))$ it follows that $\lim_{t_{\bar{k}}\to\infty} u(x(t_{\bar{k}}-j-1))=0, \forall j=0,..,N-1$ and hence $\lim_{t_{\bar{k}}\to\infty} x(t_{\bar{k}})=0$ $\square$

IV. NUMERICAL RESULTS

We consider the system examined in [1] composed by the linear system

$$\begin{cases} x(t+1) = \begin{bmatrix} 0.9429 & -0.02798 & -0.2611 \\ 0.02224 & 0.9798 & -0.02135 \\ 0.2616 & 0.01452 & 0.943 \end{bmatrix} x(t)+ \\ \quad + \begin{bmatrix} .009384 & .005471 & -.00072 \\ -.001563 & .00931 & -.00055 \\ -.002088 & -.00147 & .005401 \end{bmatrix} u(t) \end{cases} \tag{30}$$

in closed-loop with an asymptotically stabilizing decentralized MPC controller (see [1, Section IV] for the details).

Suppose that subsystem 1 loses 5 consecutive packets containing the measures of $[x_1(t), x_3(t)]$, $t = 5, .., 9$. Condition (25) leads to the following condition

$$x^\top \begin{bmatrix} 19.0912 & -0.6564 & -22.2504 \\ -0.7286 & 4.2577 & 0 \\ 6.8592 & 0 & 21.5841 \end{bmatrix} x \geq 0 \tag{31}$$

that is verified since the eigenvalues of the matrix associated with the quadratic form (44) are $20.3478 + 12.2847i$, $20.3478 - 12.2847i$, 4.2375. Figure 1 shows the state trajectories, starting from the initial state $x_0 = [-10\ 1\ 9]^\top$. It shows that even if subsystem 1 is the only one subject to packet losses, the trajectories of the other two subsystems are affected too through the dynamical coupling. In Figure 1 it is also reported the case when the number of packet lost by subsystem 1 is 10. In this case, condition (25) leads to the following condition

$$x^\top \begin{bmatrix} 25.0519 & -0.6564 & -17.5529 \\ -0.7286 & 4.2577 & 0 \\ -9.8879 & 0 & 32.6992 \end{bmatrix} x \geq 0 \tag{32}$$

that is verified since the eigenvalues of the matrix associated with the quadratic form (32) are 42.5980, 15.1856, 4.2252.

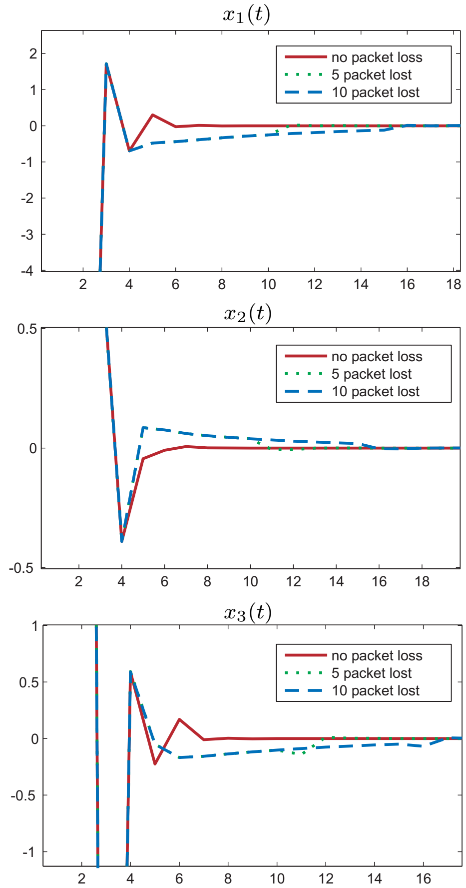

Fig. 1. Closed-loop trajectories if (i) no packet losses occur, (ii) if 5 packet losses starting from time $t = 5$ occurred, (iii) if 10 packet losses starting from time $t = 5$ occurred.

V. CONCLUSIONS

In this paper we have proposed a decentralized MPC scheme for large-scale systems where information is gathered through wireless (sensor) networks subject to possible dropouts of multiple packets. We proposed a sufficient stability criteria for testing asymptotic stability of the overall closed-loop system in the case where communication between neighbors is not guaranteed at every sampling time. In this paper we assumed the the open-loop process is asymptotically stable. Extensions of this approach to open-loop unstable systems by using Riccati terminal weights are currently under investigation.

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