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Observer Pole Placement Limitations for a Class of Offset-Free Model Predictive Controllers

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***Abstract*— Model predictive control (MPC) algorithms achieve offset-free control by augmenting the nominal system model with a disturbance model. The disturbance vector is used to predict the mismatch between the measured and predicted output vectors. In this paper, we consider an offset-free MPC framework that includes an output disturbance model and a Kalman filter to estimate the state and disturbance vectors. Using root locus techniques, we identify sufficient conditions for a class of nominal systems with at least one real positive pole for which the closed loop estimator poles cannot be arbitrarily selected. We present several examples illustrating the limitations of the closed loop estimator pole locations.**

I. INTRODUCTION

The main concept of Model Predictive Control (MPC) is to use a *model* of the plant to *predict* the future evolution of the system [2], [6]. At each time step t a certain performance index is optimized over a sequence of future input moves subject to operating constraints. The first of such optimal moves is the *control* action applied to the plant at time t. At time $t+1$, a new optimization is solved over a shifted prediction horizon.

Steady-state offset refers to asymptotically constant biases between the controlled output vector and the steady-state reference vector. MPC algorithms are designed to achieve offset-free steady-state tracking by augmenting the plant model with a disturbance model. This disturbance model is used to predict the bias error between the measured output vector and the output vector predicted using the nominal plant model. The general approach of these offset-free MPC algorithms is, first, to estimate the state and disturbance vectors using the measured output vector and, second, to use the estimated state and disturbance vectors to initialize the MPC optimization problem.

A number of disturbance models have been proposed and applied in offset-free MPC algorithms [4], [5], [7]–[12]. These disturbance models consist of integrating modes and are selected to capture the type of uncertainty affecting the nominal plant model. In this paper, we consider the following widely used offset-free MPC framework: (*i*) a discrete linear time-invariant nominal plant model, (*ii*) an output integrator disturbance model, and (*iii*) a linear time-invariant Kalman filter. Our experience with the application of this framework has shown a consistent limitation in the achievable closed loop system performance.

The objective of this paper is to present and study the source of the limitations for this class of offset-free MPC framework. In particular, we prove that if nominal single output plant models with at least one real positive pole satisfy certain conditions, then the resulting closed loop estimator has one real pole that cannot be arbitrarily selected. Furthermore, this limitation can be independent of the statistics of the nominal plant and disturbance models. The results of this paper have been extended to single output and multiple output augmented systems with postive and negative real poles in [1].

This paper is organized as follows. In Section II, we describe the nominal plant model and output disturbance model. In Section III, we review the steady-state Kalman filter and formulate the relationship between the closed loop estimator poles and the nominal plant model poles. In Section IV, we prove the limitations on the placement of the closed loop estimator poles for the defined offset-free MPC framework. In Section V, we present several examples illustrating the limitations described in Section IV.

II. PROBLEM FORMULATION

We consider the following nominal linear time-invariant (LTI) system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + G_x w_{x,k}, && \text{(1a)} \\ z_k &= Cx_k + v_k, && \text{(1b)} \end{aligned}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the system state and input vectors at time k, respectively. $z_k \in \mathbb{R}^p$ is both the measurement vector and the controlled output vector. $w_{x,k} \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^p$ are the state process noise and measurement noise vectors, respectively, and are modeled as zero-mean, Gaussian, uncorrelated white sequences:

$$\begin{aligned} &\mathrm{E}\left[w_{x,k}w_{x,k}^T\right] = Q_x = Q_x^T \geq 0 \\ &\mathrm{E}\left[w_{x,i}v_j^T\right] = 0 \\ &\mathrm{E}\left[v_k v_k^T\right] = R = R^T > 0 \\ &\forall i, j, k \end{aligned} \qquad (2)$$

where $Q_x \in \mathbb{R}^{n\times n}$ is the state process noise covariance matrix and $R \in \mathbb{R}^{p\times p}$ is the measurement noise covariance matrix. We assume that (A, B) is stabilizable, $(A, G_x\sqrt{Q_x})$ is stabilizable where $Q_x = \sqrt{Q_x}\sqrt{Q_x}^T$, and (C, A) is observable.

The objective of controller design is to formulate a control law that enables the controlled output vector to track an

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asymptotically constant reference vector z_{ref}, where $z_{ref} \in \mathbb{R}^p$. To achieve steady-state offset-free tracking of z_{ref}, we augment the nominal system (1) with an integrator disturbance model:

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + B_d d_k + G_x w_{x,k}, && (3a)\\
d_{k+1} &= d_k + G_d w_{d,k}, && (3b)\\
z_k &= Cx_k + C_d d_k + v_k, && (3c)
\end{aligned}
$$

where $d_k \in \mathbb{R}^{n_d}$ is the disturbance vector and $w_{d,k} \in \mathbb{R}^{n_d}$ is the disturbance process noise vector at time k. The vector $w_{d,k}$ is modeled as a zero-mean, Gaussian, white sequence uncorrelated with v_k:

$$
\begin{aligned}
&\mathrm{E}\left[w_{d,k} w_{d,k}^T\right] = Q_d = Q_d^T > 0,\ \forall k \quad G_d Q_d G_d^T > 0,\\
&\mathrm{E}\left[w_{d,i} v_j^T\right] = 0, \forall i, j,
\end{aligned} \tag{4}
$$

where $Q_d \in \mathbb{R}^{n_d \times n_d}$ is the disturbance process noise covariance matrix. We assume that the state and input constraints are inactive at steady-state throughout this paper.

Remark 1: We refer the reader to [11] for a description of the augmented system (3).

We denote I_n as the identity matrix belonging to $\mathbb{R}^{n \times n}$ and assume the following:

Assumption 1: We consider the output integrator disturbance model with $n_d = p$, $C_d = I_p$, and $B_d = 0$ in (3).

The augmented system (3) with Assumption 1 can be compactly written as:

$$
\begin{aligned}
X_{k+1} &= \Phi X_k + \bar{B} u_k + \Gamma w_k, && (5a)\\
z_k &= H X_k + v_k, && (5b)
\end{aligned}
$$

where $X_k \in \mathbb{R}^{n+n_d}$ is the augmented state vector at time k:

$$
X_k = \left[\begin{array}{cc} x_k^T & d_k^T \end{array}\right]^T, \tag{6}
$$

$w_k \in \mathbb{R}^{n+n_d}$ is the augmented process noise vector:

$$
\begin{aligned}
&w_k = \left[\begin{array}{cc} w_{x,k}^T & w_{d,k}^T \end{array}\right]^T,\\
&\mathrm{E}\left[w_k w_k^T\right] = Q \triangleq \left[\begin{array}{cc} Q_x & 0 \\ 0 & Q_d \end{array}\right],
\end{aligned} \tag{7}
$$

and

$$
\begin{aligned}
&\Phi = \left[\begin{array}{cc} A & 0 \\ 0 & I_{n_d} \end{array}\right], \quad \bar{B} = \left[\begin{array}{c} B \\ 0 \end{array}\right], \quad \Gamma = \left[\begin{array}{cc} G_x & 0 \\ 0 & G_d \end{array}\right],\\
&H = \left[\begin{array}{cc} C & I_p \end{array}\right].
\end{aligned} \tag{8}
$$

We assume that $(\Phi, \Gamma\sqrt{Q})$ is stabilizable where $Q = \sqrt{Q}\sqrt{Q}^T$ and (H, Φ) is observable. Necessary and sufficient conditions for the observability of the augmented system (5) are given in the following proposition.

Proposition 1: The augmented system (5) is observable if and only if (C, A) is observable and

$$
\left[\begin{array}{cc} A - I_n & 0 \\ C & I_p \end{array}\right] \text{ has full column rank.} \tag{9}
$$

Proof: See [8], [11].

Remark 2: In Assumption 1, we require $n_d = p$. In [11] it was proven that this condition guarantees offset-free steady-state tracking of z_{ref}. We refer the reader to [11] for a description of offset-free tracking for the case $n_d < p$.

Remark 3: To satisfy condition (9) of Proposition 1, the nominal system cannot have integrating modes.

III. Steady-State Kalman Filter

This section describes the steady-state Kalman filter used to estimate the augmented state vector governed by (5) and the frequency domain relationship between the steady-state closed loop estimator poles and the nominal system poles. The term steady-state will be dropped when describing the closed loop estimator poles because the following analysis applies at steady-state.

We use a Kalman filter [3] to estimate the augmented state vector. A steady-state Kalman filter exists for the augmented system (5) because the matrices Φ, Γ, H, Q, and R are time-invariant, $Q \geq 0$, $R > 0$, $(\Phi, \Gamma\sqrt{Q})$ is stabilizable, and (H, Φ) is observable. The closed loop estimator equations can be written as [3]:

$$
\begin{aligned}
\hat{X}_{k+1} &= \Phi(I_{n+n_d} - KH)\hat{X}_k + \bar{B}u_k + \Phi K z_k,\\
K &= P_\infty H^T (H P_\infty H^T + R)^{-1},\\
P_\infty &= \Phi P_\infty \Phi^T + \Gamma Q \Gamma^T -\\
&\quad - \Phi P_\infty H^T (H P_\infty H^T + R)^{-1} H P_\infty \Phi^T,
\end{aligned} \tag{10}
$$

where K is the Kalman gain, $P_\infty \geq 0$ is the solution to the discrete algebraic Riccati equation, $(HP_\infty H^T + R)$ is nonsingular, and $|\lambda_i(\Phi(I_{n+n_d} - KH))| < 1$ for all $i = 1, \ldots, n + n_d$ where $\lambda_i(M)$ denotes the i-th eigenvalue of the matrix M.

For a steady-state Kalman filter implemented for an LTI system, the frequency domain can be used to analyze the relationship between the closed loop estimator poles and the nominal system poles. We first define the following three transfer functions for analysis purposes. We denote $\Delta(z)$ as the characteristic polynomial of the augmented system (5):

$$
\begin{aligned}
\Delta(z) &= |zI_{n+n_d} - \Phi| = |zI_n - A||zI_{n_d} - I_{n_d}| =\\
&= |zI_n - A|(z-1)^{n_d},
\end{aligned} \tag{11}
$$

where $|\circ|$ refers to the determinant of $(\circ)$. We denote $\Delta_{cl}(z)$ as the characteristic polynomial of the closed loop estimator (10):

$$
\begin{aligned}
\Delta_{cl}(z) &= |zI_{n+n_d} - \Phi(I_{n+n_d} - KH)|\\
&= |I_{n+n_d} + \Phi K H (zI_{n+n_d} - \Phi)^{-1}||zI_{n+n_d} - \Phi|\\
&= |I_{n+n_d} + H(zI_{n+n_d} - \Phi)^{-1}\Phi K|\Delta(z)
\end{aligned}
$$

We denote $G(z)$ as the transfer function of the augmented system from $\bar{w}_k$ to z_k where $w_k = \sqrt{Q}\bar{w}_k$ and $\bar{w}_k$ is a white noise vector with unit covariance matrix:

$$
\begin{aligned}
G(z) &= H(zI_{n+n_d} - \Phi)^{-1}\Gamma\sqrt{Q} = \frac{H \mathrm{adj}(zI_{n+n_d} - \Phi)\Gamma\sqrt{Q}}{|zI_{n+n_d} - \Phi|}\\
&\triangleq \frac{N_G(z)}{\Delta(z)}.
\end{aligned} \tag{12}
$$

The Chang-Letov equation [3] can be used to establish the relationship between $\Delta(z)$, $\Delta_{cl}(z)$, and $G(z)$ and analyze the locations of the steady-state closed loop estimator poles:

$$
\begin{aligned}
\Delta_{cl}(z)\Delta_{cl}(z^{-1}) &= \left|G(z)G^T(z^{-1}) + R\right| \cdot \Delta(z)\Delta(z^{-1}) \cdot\\
&\quad \cdot \left|HP_\infty H^T + R\right|^{-1},
\end{aligned} \tag{13}
$$

$$\Delta_{cl}(z)\Delta_{cl}(z^{-1}) = \left|H(zI_{n+n_d}-\Phi)^{-1}\Gamma Q\Gamma^T(z^{-1}I_{n+n_d}-\Phi)^{-T}H^T+R\right|\cdot\Delta(z)\Delta(z^{-1}) = \Delta(z)\Delta(z^{-1})\left|\begin{bmatrix} C & I_{n_d}\end{bmatrix}\begin{bmatrix} zI_n-A & 0 \\ 0 & zI_{n_d}-I_{n_d}\end{bmatrix}^{-1}\begin{bmatrix} G_xQ_xG_x^T & 0 \\ 0 & G_dQ_dG_d^T\end{bmatrix}\begin{bmatrix} z^{-1}I_n-A & 0 \\ 0 & z^{-1}I_{n_d}-I_{n_d}\end{bmatrix}^{-T}\begin{bmatrix} C^T \\ I_{n_d}\end{bmatrix}+R\right| \quad (14)$$

$$\Delta_{cl}(z)\Delta_{cl}(z^{-1}) = \left|\frac{C\text{adj}(zI_n-A)G_xQ_xG_x^T\text{adj}(z^{-1}I_n-A^T)C^T}{|zI_n-A||z^{-1}I_n-A^T|}+\frac{G_dQ_dG_d^T}{(z-1)(z^{-1}-1)}+R\right|\Delta(z)\Delta(z^{-1}) \quad (15)$$

$$\Delta_{cl}(z)\Delta_{cl}(z^{-1}) = \left|-z^{-1}N(z)N^T(z^{-1})(z-1)^2+G_dQ_dG_d^T|zI_n-A||z^{-1}I_n-A^T|+ + R|zI_n-A||z^{-1}I_n-A^T|(z-1)(z^{-1}-1)\right|\frac{1}{|zI_n-A|^{p-1}|z^{-1}I_n-A^T|^{p-1}} \quad (16)$$

---

where $|HP_\infty H^T+R|$ is a scaling factor and will be ignored in the following analysis. We will rewrite (13) using these three transfer functions and then explain its importance. If (12) is substituted into (13), then $\Delta_{cl}(z)\Delta_{cl}(z^{-1})$ can be rewritten as:

$$\Delta_{cl}(z)\Delta_{cl}(z^{-1}) = \left|\frac{N_G(z)N_G^T(z^{-1})}{\Delta(z)\Delta(z^{-1})}+R\right|\Delta(z)\Delta(z^{-1}) = \frac{\left|N_G(z)N_G^T(z^{-1})+R\Delta(z)\Delta(z^{-1})\right|}{\Delta(z)^{p-1}\Delta(z^{-1})^{p-1}} \quad (17)$$

The right hand side of (17) is a polynomial whose stable roots are the closed loop estimator poles. Note that in (17) the term $\frac{1}{\Delta(z)^{p-1}\Delta(z^{-1})^{p-1}}$ simplifies with the polynomial resulting from the computation of $\left|N_G(z)N_G^T(z^{-1})+R\Delta(z)\Delta(z^{-1})\right|$.

The Chang-Letov equation (17) can be used to analyze the location of the closed loop estimator poles. As $R\to 0$, the $N_G(z)N_G^T(z^{-1})$ term predominates. Therefore, the closed loop estimator poles approach the stable roots of $\frac{|N_G(z)N_G^T(z^{-1})|}{\Delta(z)^{p-1}\Delta(z^{-1})^{p-1}}$ which are the stable zeros of $G(z)G^T(z^{-1})$ when $p = n_d = 1$. As $R\to\infty$, the $\Delta(z)\Delta(z^{-1})$ term predominates. Therefore, the closed loop estimator poles approach the stable zeros of $\Delta(z)\Delta(z^{-1})$ which are the stable poles of $G(z)G^T(z^{-1})$. In other words, the closed loop estimator poles approach either the augmented system's stable poles or the reflections of the augmented system's unstable poles inside the unit circle because the closed loop estimator is stable. Unstable poles are reflected inside the unit circle by taking their inverse.

The polynomial $\Delta_{cl}(z)\Delta_{cl}(z^{-1})$ can be written explicitly as a function of the augmented system matrices by substituting (8) into (13). The resulting equation, (14), is written at the top of the page. By explicating the inverse of the matrix in (14), we obtain (15) which can be compactly written as (16) where

$$N(z) = C\text{adj}(zI_n-A)G_x\sqrt{Q_x}. \quad (18)$$

Equation (16) is written as the top of the page.

IV. Main Result

We have often observed a limitation in the closed loop estimator performance when using the Kalman filter (10) to estimate the augmented state vector governed by (5). This section identifies and describes this performance characteristic by considering the root loci of the closed loop estimator poles. We consider nominal systems with at least one real pole.

Assumption 2: There exists an i such that $\lambda_i(A)\in\mathbb{R}$.

We define the following quantities for analysis purposes. We denote Λ_r as the set of real eigenvalues of A and their inverses, $\Lambda_r = \{\lambda_i,\ 1/\lambda_i \mid \lambda_i\in\mathbb{R},\ i=1,\ldots,n\}$. We denote $\lambda_{max,r}$ as the largest stable eigenvalue in Λ_r, $\lambda_{max,r} = \max_{|\lambda|<1}\lambda\in\Lambda_r$. We denote $Z(z)$ as $\Delta_{cl}(z)\Delta_{cl}(z^{-1})$ from (16) when $R=0$. $Z(z)$ is given in equation (20) at the top of the next page.

In the following, we consider a class of single output augmented systems with $\lambda_{max,r}>0$. For these systems, we provide general results regarding the root loci of the closed loop estimator poles, i.e., the stable roots of $\Delta_{cl}(z)\Delta_{cl}(z^{-1})$ as $0\to R\to\infty$. In particular we show that there are branches of the root loci that lie on the real axis and that these branches correspond to either slow or fast closed loop estimator poles which are independent of R. The existence of these branches is related to the assumption that the stable roots of $Z(z)$ lie within certain segments of the real axis. For this class of augmented systems, we provide sufficient conditions on $Z(z)$ that guarantee the existence of such roots.

Theorem 1: Consider the class of augmented systems (5) with $p=1$ that satisfy Assumption 2. If $\lambda_{max,r}>0$ and the following condition is satisfied:

$$\exists\bar{z}\in\mathbb{R}\mid\lambda_{max,r}<\bar{z}<1 \text{ and } Z(\bar{z})=0, \quad (19)$$

then, for all $R>0$, there exists $\lambda_{cl}\in\mathbb{R}$ such that $\Delta_{cl}(\lambda_{cl})=0$ and $\lambda_{max,r}\le\lambda_{cl}\le 1$.

Proof: As discussed in Section III, the branches of the root locus of $\Delta_{cl}(z)\Delta_{cl}(z^{-1})$ move from the zeros of $G(z)G^T(z^{-1})$ to the poles of $G(z)G^T(z^{-1})$ as $0\to R\to\infty$. These branches have the following properties: (1) they must begin at a zero (which can include ∞) and end at a

$$Z(z) \triangleq \frac{\left\lfloor -z^{-1}N(z)N^T(z^{-1})(z-1)^2 + G_dQ_dG_d^T|zI_n - A||z^{-1}I_n - A^T|\right\rfloor}{|zI_n - A|^{p-1}|z^{-1}I_n - A^T|^{p-1}}. \tag{20}$$

pole, (2) a real branch lies to the left of an odd number of real poles and zeros [13], (3) the branches are symmetric about the real axis, (4) the branches are inversely symmetric about the unit circle because if $g(z) = 0$ is a branch, then $g(z^{-1}) = 0$ is a branch, and (5) a branch can not cross the unit circle because the closed loop estimator is stable. $G(z)G^T(z^{-1})$ has exactly two poles at $z = 1$ (one pole from the disturbance model, one pole from its inverse, and no other poles at $z = 1$ from Remark 3). Property 1 indicates that two branches of the root locus must end at $z = 1$. Properties 4 and 5 indicate that one branch lies inside the unit circle and the second branch lies outside the unit circle. Property 3 indicates that these two branches must lie on the real axis. Property 2 indicates that the stable branch must begin at the largest zero of $G(z)G^T(z^{-1})$, $\bar{z}_{max}$, that satisfies condition (19). Therefore, there is a stable, real branch of the root locus of $\Delta_{cl}(z)\Delta_{cl}(z^{-1})$ that begins at $\bar{z}_{max}$ and ends at $z = 1$. □

Proposition 2: If $p = 1$, $\lambda_{max,r} > 0$, and

$$N(\lambda_{max,r})N^T(\lambda_{max,r}^{-1}) > 0, \tag{21}$$

then there exists a $\bar{z}$ satisfying condition (19) in Theorem 1.

Proof: Evaluate the right hand side of $Z(z)$ in (20) at $z = \lambda_{max,r}$ and $z = 1$:

$$Z(z)|_{z=\lambda_{max,r}} = -\lambda_{max,r}^{-1}N(\lambda_{max,r})N^T(\lambda_{max,r}^{-1})(\lambda_{max,r}-1)^2, \tag{22}$$

$$Z(z)|_{z=1} = G_dQ_dG_d^T|I_n - A||I_n - A^T|. \tag{23}$$

Equation (22) and hypothesis (21) imply that $Z(\lambda_{max,r}) < 0$. Equation (23) implies that $Z(1) > 0$ because $G_dQ_dG_d^T > 0$ from (4). The polynomial $Z(z)$ is continuous for all $z > 0$ and, thus, it must have a real root between $z = \lambda_{max,r}$ and $z = 1$. □

The results of Theorem 1 have been extended to single output and multiple output augmented systems with both positive and negative real poles in [1].

In summary, Theorem 1 applies to single output augmented systems that have at least one positive eigenvalue, $\lambda_{max,r}$. By definition, $\lambda_{max,r}$ corresponds to the slowest non-oscillating mode of either the nominal system's stable modes or the reflection of the nominal system's unstable modes inside the unit circle. Theorem 1 shows that if condition (19) holds, then the closed loop estimator always has a real pole between $\lambda_{max,r}$ and $z = 1$ for all R. Therefore, the poles of the closed loop estimator cannot be arbitrarily changed by tuning R and the closed loop estimator will always have a non-oscillating mode slower than $\lambda_{max,r}$.

Finally, we remark that, from a practical viewpoint, it might be simpler to test the existence of the slow and fast modes in Theorem 1 by a direct numerical verification process where the closed loop estimator poles are computed for a large selection of weights R, Q_x and Q_d within given bounds.

V. EXAMPLES

Example 1: Single State, Single Output Nominal Model. Consider the following augmented system (5):

$$X_{k+1} = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} X_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w_k, \tag{24}$$

$$z_k = \begin{bmatrix} c & 1 \end{bmatrix} X_k + v_k \tag{25}$$

where $\lambda > 0, c, Q_x, Q_d, R \in \mathbb{R}$, $\Delta(z) = (z-\lambda)(z-1)$, and $N(z) = c\ \mathrm{adj}(z-\lambda)\sqrt{Q_x} = c\sqrt{Q_x}$.

We denote the stable eigenvalue in Λ_r as $a = \min\{\lambda, 1/\lambda\}$. Therefore, $|N(a)N^T(a^{-1})| = c^2Q_x > 0$ for all $Q_x > 0$. From Proposition 2 and Theorem 1 we can conclude that the closed loop estimator always has a real pole between $z = a$ and $z = 1$ for all $R > 0$. In particular, the closed loop estimator will have a stable, real branch of the root locus of $\Delta_{cl}(z)\Delta_{cl}(z^{-1})$ that begins from $a < \bar{z} < 1$ and ends at $z = 1$ where $\bar{z} = \min\{z_1, z_1^{-1}\}$ and:

$$z_1 = \frac{2c^2Q_x + (1+a^2)Q_d}{2c^2Q_x + 2aQ_d} + \frac{\left[(a-1)^2Q_d\left[4c^2Q_x + (a+1)^2Q_d\right]\right]^{1/2}}{2c^2Q_x + 2aQ_d}. \tag{26}$$

Therefore, the closed loop estimator always has a non-oscillating mode slower than $z = a$. The root locus plot is shown in Figure 1. The zeros of $G(z)G^T(z^{-1})$ are represented by circles and the poles of $G(z)G^T(z^{-1})$ are represented by crosses.

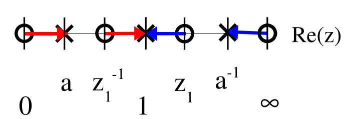

Fig. 1. Root locus plot for Example 1

Example 2: Multiple State, Single Output Nominal Model. Consider the transfer function:

$$F(z) = \frac{z + 0.3}{z^3 - 1.804z^2 + 1.626z - 0.670} \tag{27}$$

The state-space matrices of (1) are extracted from $F(z)$ in a controller canonical form. The nominal system's eigenvalues are $\lambda(A) = 0.4930 \pm 0.7592j, 0.8180$ so that the largest stable eigenvalue in Λ_r is $\lambda_{max,r} = 0.8180$. The remaining matrices and statistics of the augmented system (5) were selected as $G_x = I_3$, $G_d = 1$, $Q_x = \alpha I_3$ where $\alpha > 0$,

$Q_d = 1$, and $R \in \mathbb{R}$. For $R = 0$, the closed loop estimator poles are $\lambda(\Phi(I_{n+n_d} - KH)) = 0.1812 \pm 0.3767j, 0, 0.9203$.

By direct computation, $N(z)N^T(z^{-1}) = \alpha(0.3165z^2 - 0.9871z + 5.524 - 0.9871z^{-1} + 0.3165z^{-2})$, and, thus, $N(\lambda_{max,r})N^T(\lambda_{max,r}^{-1}) = 4.195 > 0$ for all $\alpha > 0$. From Proposition 2 and Theorem 1, we can conclude that the closed loop estimator always has a real pole between $z = \lambda_{max,r}$ and $z = 1$ for all $\alpha, R > 0$. Therefore, the closed loop estimator always has a non-oscillating mode slower than $z = \lambda_{max,r}$. The root locus plot is shown in Figure 2. One stable real branch of $\Delta_{cl}(z)\Delta_{cl}(z^{-1})$ begins at $z = 0.9203$ and ends at $z = 1$

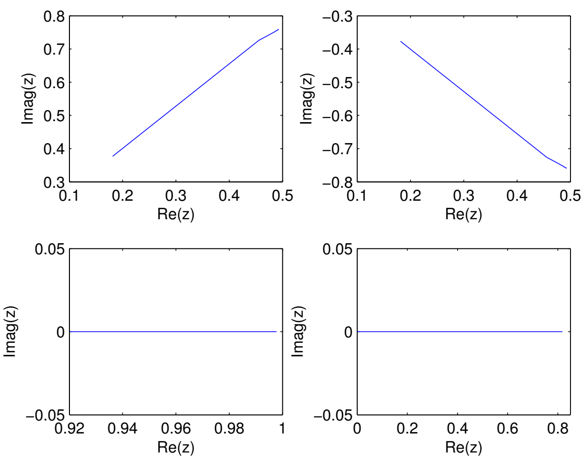

Fig. 2. Root locus plots for Example 2

Example 3: Multiple State, Single Output Nominal Model. Consider the transfer function:

$$F(z) = \frac{z - 0.9}{z^5 + 0.496z^4 - 1.923z^3 + 1.987z^2 - 0.566z - 0.402} \quad (28)$$

The state-space matrices of (1) are extracted from $F(z)$ in a controller canonical form. The nominal system's eigenvalues are $\lambda(A) = -2.000, 0.4930 \pm 0.7592j, 0.8180, -0.3000$ so that the largest stable eigenvalue in Λ_r is $\lambda_{max,r} = 0.8180$. The remaining matrices and statistics of the augmented system (5) were selected as $G_x = I_5$, $G_d = 1$, $Q_x = \text{diag}[0.001, 0.1, 1, 1, 0.1]$, $Q_d = 1$, and $R \in \mathbb{R}$. For $R = 0$, the closed loop estimator poles are $\lambda(\Phi(I_{n+n_d} - KH)) = 0.5002 \pm 0.6299j, -0.5193, 0, 0.0209, 0.8178$.

By direct computation, $N(z)N^T(z^{-1}) = 1.076z^3 - 1.935z^2 - 2.222z + 6.165 - 2.222z^{-1} - 1.935z^{-2} + 1.076z^{-3}$, and, thus, $N(\lambda_{max,r})N^T(\lambda_{max,r}^{-1}) = -0.001 < 0$. This system violates the hypothesis of Theorem 1 because $N(\lambda_{max,r})N^T(\lambda_{max,r}^{-1}) < 0$ and $Z(z)$ has no roots between $z = \lambda_{max,r}$ and $z = 1$ for $R = 0$. It should be noted that Proposition 2 and Theorem 1 are sufficient conditions. A real branch of the closed loop estimator can approach and cross $\lambda_{max,r}$, and then move toward $z = 1$ for $R > 0$ regardless of the location of the closed loop estimator poles when $R = 0$. The root locus plot is shown in Figure 3 and illustrates this very characteristic.

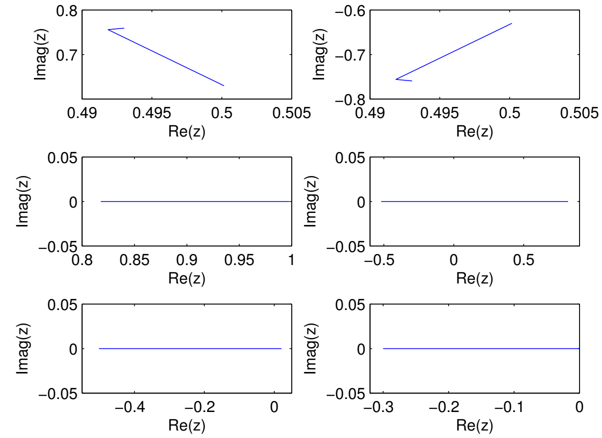

Fig. 3. Root locus plots for Example 3

VI. Conclusion

In this paper, we considered an offset-free MPC framework with a discrete LTI nominal system model, an output disturbance model, and a steady-state Kalman filter. We identified sufficient conditions for a class of systems with at least one postive real pole for which the closed loop estimator pole locations could not be arbitrarily selected. In particular, using root locus techniques, we showed that the closed loop estimator always has a non-oscillating mode slower than the slowest, stable, positive nominal system real mode. These limitations on the closed loop estimator poles restrict the locations of the closed loop MPC poles and, thus, restrict the closed loop controller performance.

The limitations on the closed loop estimator pole locations are a result of using the Kalman filter for the defined offset-free MPC framework. A different observer design method might not lead to these limitations. For example, in [10], the observer gain matrix was designed by H_∞ techniques and limitations on closed loop estimator pole locations were not reported. We also remark that the limitations on the closed loop estimator pole locations are a characteristic of the proposed disturbance model. These limitations may not exist if other types of disturbance models are used to augment the nominal system model.

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