2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

A Necessary and Sufficient Condition for Semi-stability of the Recursive Kalman Filter

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Abstract— **This paper studies semi-stability for Kalman filters in the context of linear time-varying systems with incorrect noise information. Semi-stability is a key property, as it ensures that the actual estimation error does not diverge exponentially. As the main result of the paper we present a necessary and sufficient condition for the recursive Kalman filter to be semi-stable, relying on the relevant data of the system and noise. The condition does not involve limiting gains nor the solution of Riccati equations, as they can be difficult to obtain numerically and may not exist.**

I. Introduction

It is a well-known fact that Kalman filters (KFs) may present the phenomena of divergence under incorrect model and noise information, in such a manner that the state estimate diverges from the actual value with a high, sometimes exponential, rate. There may be no indication of divergence when calculating the KF, making the phenomena more problematic.

Divergence was studied for KFs in the context of time-invariant, periodic and time-varying systems, see e.g. [7], [5], [8], [9]. The available conditions for avoiding divergence present some degree of conservativeness, as they rely on detectability/uniform observability assumptions or on the existence of a stationary solution to the filtering problem (equivalently, a solution to an algebraic Riccati equation (ARE)). However, there may exist neither a solution to the ARE nor a limiting gain, even in time-invariant context, see Example 3. Moreover, the available results concern only the stability of the KF, meaning (from the standpoint of divergence analysis) that the actual covariance matrix of the estimation error is *bounded* for *any* incorrect noise covariances. However, incorrect initial error covariance Σ can be handled without requiring stability and we mention, as illustration, that $\Sigma > 0$ ensures bounded error but does not ensure stability, see Example 1; furthermore, in many situations it may be acceptable that the actual error does not diverge exponentially, or diverges with a "specified" rate.

In this paper we fill some of the aforementioned gaps. We start showing that exponential divergence makes no distinction among the incorrect data being related to persistent or non-persistent noise, see Lemma 3. We derive some links between the *actual* and *calculated* error covariances, provided the latter structurally describes the former, in the sense of Lemma 1 and Lemma 4, which allows to present the exact role of the condition $\Sigma > 0$. We also present, in Corollary 3, an orthogonality result involving the Kalman gain and the covariance matrix of the plant. These results are combined to obtain our main result: a necessary and sufficient condition for semi-stability of the KF (equivalently, for avoiding exponential divergence) relying on the nominal data employed in the KF calculation, and involving neither detectability assumptions, nor existence of limiting gains, nor calculations of the Kalman gain and of the solution of the associated Riccati difference equation (RDE).

Since we are concerned with actual error divergence in absence of nominal error divergence, we assume that the nominal error covariance matrices are bounded, which is much weaker than detectability or existence of solution to the ARE. We also require "structural invariance" of unstable subspaces, in the sense of Assumption 2, which holds trivially for time-invariant or periodic systems.

The paper is organised as follows. Section II presents some preliminary results, and Section III presents stability notions and some related results. In Section IV we address a link between the Kalman gain and the plant. The main result is presented in Section V. Finally, Section VI provides some conclusions, and the Appendix contains the proof of a technical fact.

II. Preliminary results

Let $\mathbb{R}^n$ denote the $n-$th dimensional Euclidean space. Let $\mathcal{R}^{r,s}$ (respectively, $\mathcal{R}^r$) represent the normed linear space formed by all $r \times s$ real matrices (respectively, $r \times r$) and $\mathcal{R}^{r*}$ ($\mathcal{R}^{r0}$) the cone $\{U \in \mathcal{R}^r : U = U'\}$ (the closed convex cone $\{U \in \mathcal{R}^r : U = U' \geq 0\}$) where U' denotes the transpose of U. For $U \in \mathcal{R}^n$, $\lambda(U)$ stands for the eigenvalues of U and, for $U \in \mathcal{R}^{n0}$, $\lambda_{-}(U)$ is the smallest positive eigenvalue of U. Let $\mathbb{D}$ (respectively $\bar{\mathbb{D}}$) be the open (closed) unit disk in the complex plane. Following the terminology of [1] we say that $A \in \mathcal{R}^n$ is semi-stable (stable) if $\lambda(A) \subset \bar{\mathbb{D}}$ ($\lambda(A) \subset \mathbb{D}$), and we say that $v \in \mathbb{R}^n$

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Research supported in part by FAPESP Grants 06/02004-0 and 06/04210-6 and the EPSRC Research Grant EP/E057438, Nonlinear observation theory with applications to Markov jump systems.

is an unstable[1] eigenvector when it is associated with $\lambda_i(A) \notin \bar{\mathbb{D}}$. Let $\mathcal{H}^{r,n}$ denote the linear space formed by sequences of matrices $H = \{H_i \in \mathcal{R}^{r,s}; i \in \mathcal{Z}\}$ such that $\sup_{i\in\mathcal{Z}} ||H_i|| < \infty$; also, $\mathcal{H}^n \equiv \mathcal{H}^{n,n}$ and $\|H\|_\infty = \sup_{i\in\mathcal{Z}} \|H_i\|$.

Consider the linear, time-varying, stochastic system defined in a fundamental probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\Phi : \begin{cases} x(k+1) = A_k x(k) + B_k w(k), \quad x(0) = x_0, \\ y(k) = C_k x(k) + D_k v(k), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^r$ is the observed variable, $w \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$ form stationary zero-mean independent white noise processes satisfying $\mathcal{E}\{w(k)w(k)'\} = I$ and $\mathcal{E}\{v(k)v(k)'\} = I$, and the (independent) random variable x_0 has Gaussian distribution satisfying $\mathcal{E}\{x_0\} = \bar{x}_0$ and $\mathcal{E}\{x_0 x_0'\} = \Psi$. It is assumed that $A \in \mathcal{H}^n$, $B \in \mathcal{H}^{n,p}$, $C \in \mathcal{H}^{r,n}$ and $D \in \mathcal{H}^{r,q}$, with $D_i D_i' > 0$, $i \geq 0$ (nonsingular measurement noise). Much attention has been devoted to the problem of estimating x and the associated RDE

$$P_{k+1} = A_k[P_k - P_k C_k'(C_k P_k C_k' + D_k D_k')^{-1} C_k P_k]A_k' + E_k E_k' \tag{2}$$

with initial condition $P_0 = \Sigma \in \mathcal{R}^{n0}$, where $E \in \mathcal{H}^{n,p}$. Σ and E are the *available* information on the *actual* Ψ and B, respectively. The state estimate is given by $\widehat{x}(0) = \bar{x}_0$ and

$$\widehat{x}(k+1) = A_k \widehat{x}(k) + L_k[y(k) - C_k \widehat{x}(k)] \tag{3}$$

where

$$L_k = A_k P_k C_k' [C_k P_k C_k' + D_k D_k']^{-1}. \tag{4}$$

Results concerning existence of solutions, uniqueness, convergence and other aspects of RDEs and AREs can be traced back to the sixties, and were extended at some degree to periodic, time-varying and other classes of systems, as well as to generalised RDEs and AREs. We mention for illustration [1], [10], [11].

Assumption 1. *For each $\Sigma \in \mathcal{R}^{n0}$ there is $\bar{X} \in \mathcal{R}^{n0}$ such that $P_k \leq \bar{X}$, $k \geq 0$.*

Remark 1. Assumption 1 holds trivially provided that P_k in (2) converges as $k \to \infty$ or that (A, C) is detectable [1], thus generalising assumptions in literature. It is connected to the fact that we address divergence of the actual error in *absence* of divergence of the nominal error.

Consider the state transition matrix $\mathcal{A}(k+t, t) = A_{k+t} \cdots A_t$, $k, t \geq 0$, and let $M(k+t, t)$ stand for the orthogonal projection onto the space spanned by the unstable eigenvectors of $\mathcal{A}(k+t, t)$.

Assumption 2. *There exists T such that $M((k+1)T, kT) = M(T, 0)$, $k \geq 0$.*

Remark 2. Assumption 2 is related to the structure of semi-stable and unstable eigenvectors. The assumption holds trivially for time-invariant or periodic systems, T being the period. Systems that do not satisfy the assumption are difficult to deal with; even conditions like $\Sigma > 0$ are not relevant for avoiding divergence, e.g. $\Sigma = 1$, $A_0 = 0$, $A_k = 2$, $k \geq 1$, $B = 0$ and $C = 1$ lead to an unstable KF with $A_k + L_k C_k = 2$, $k \geq 1$.

The actual estimation error is directly obtained from (1) and (3): $\tilde{x}(0) = x(0) - \bar{x}_0$ and

$$\tilde{x}(k+1) = (A_k - L_k C)\tilde{x}(k) + B_k w(k) - L_k D_k v_k, \tag{5}$$

where L_k is the Kalman gain (4). Next we present a well-known result, see e.g. [8].

Proposition 1. *Consider the Kalman gains L_k associated with the available data E and Σ, and the sequence defined by $\tilde{X}(0,0) = \Psi$, and*

$$\begin{aligned}\tilde{X}(k+1, t, B, \Psi) = (A_k - L_k C_k)\tilde{X}(k, t, \Psi)(A_k - L_k C_k)' \\ + L_k D_k D_k' L_k' + B_k B_k', \quad k \geq t \geq 0.\end{aligned} \tag{6}$$

Then $\mathcal{E}\{\tilde{x}(k)\tilde{x}(k)'\} = \tilde{X}(k, 0, B, \Psi)$.

It is simple to check by inspection that $\tilde{X}(k, 0, E, \Sigma) = P_k$ and $\tilde{X}(k, k_0, E, P_{k_0}) = P_k$, $k \geq k_0$, thus connecting (2) and (6). The homogeneous solution of (6) is given by

$$\begin{aligned}\tilde{X}_{\mathrm{h}}(k, t, V) = (A_k - L_k C_k) \cdots (A_t - L_t C_t) V \cdot \\ \cdot (A_t - L_t C_t)' \cdots (A_k - L_k C_k)', \quad k \geq t \geq 0.\end{aligned} \tag{7}$$

For convenience we define $\tilde{X}_{\mathrm{h}}(k-1, k, V) = V$ and $\tilde{X}_{\mathrm{h}}(k-1, k+\ell, V) = 0$, $\ell \geq 1$. Similarly to Proposition 1, if we define $X(0,0,B,\Psi) = \Psi$ and

$$X(k+1, t, B, \Psi) = A_k X(k, t) A_k' + B_k B_k', \quad k \geq t \geq 0, \tag{8}$$

then $X(k, 0, B, \Psi) = E\{x(k)x(k)'\}$. We also define

$$X_{\mathrm{h}}(k, t, V) = A_k \cdots A_t V A_t' \cdots A_k', \quad k \geq t \geq 0.$$

We omit the variables t, V and B when $t = 0$, $\Psi = \Sigma$ and $B_k = E_k$, $k \geq 0$, respectively. For instance, we denote $\tilde{X}(k, 0, E, \Sigma)$ simply by $\tilde{X}(k)$. We now gather, without giving the proofs, some inequalities relating the above quantities, which are either basic ones or are adapted from the literature of RDEs, see e.g. [1].

Proposition 2. *The following statements hold.*
(i) For each $\Sigma \in \mathcal{R}^{n0}$ there exists $\rho \geq 0$ such that $\|L_k\| \leq \rho$ for $k \geq 0$.
(ii) For each $\Sigma \in \mathcal{R}^{n0}$ there exists $\delta, \varepsilon \geq 0$ such that $\tilde{X}(k+1, k, \Sigma) \leq \delta\|\Sigma\| + \varepsilon$.
(iii) Let $0 \leq \alpha \leq 1$ and $V_0, V_1 \in \mathcal{R}^{n0}$ and assume $V_1 \geq V_0$. Then $\tilde{X}(k, \alpha V_1) \geq \alpha \tilde{X}(k, V_0)$, $k \geq 0$. Similarly, $\tilde{X}_h(k, \alpha V_1) \geq \alpha \tilde{X}_h(k, V_0)$, $k \geq 0$.
(iv) Let $M = M(T, 0)$ be as in Assumption 2 and $V \in \mathcal{R}^{n0}$. There exist $\nu, \mu > 0$ such that $\lambda_-(M P_{tT} M') \geq \nu$, $\tilde{X}_h((k+t)T, tT, MM') \geq \mu MM'$, $k, t \geq 0$. Also, $\tilde{X}_h((k+t)T, tT, (I-M)V(I-M)') \leq (I-M)V(I-M)'$, $k, t \geq 0$.
(v) for $V, H \in \mathcal{R}^{n0}$ and for each scalar ε we have that $(1-\varepsilon^2)HVH' + (1-\varepsilon^{-2})(I-H)V(I-H)' \leq V \leq (1+\varepsilon^2)HVH' + (1+\varepsilon^{-2})(I-H)V(I-H)'$.

[1] We prefer the terminology "unstable" to the terminology "anti-stable" used in [1].

III. Divergence and stability for the KF

The following stability notion is standard for the KF and parallels mean square stability for linear systems with additive noise, see e.g. [6].

Definition 1 (KF stability)**.** *We say that the KF is stable if, for each* $B \in \mathcal{H}^{n,p}$ *and* $\Psi \in \mathcal{R}^{n0}$*, there exists* $\bar{X}$ *such that* $\tilde{X}(k,B,\Psi) \le \bar{X}$*,* $k \ge 0$*.*

We extend stability to semi-stability, in a parallel with the fact that $A \in \mathcal{R}^n$ is semi-stable if and only if ξA is stable for all $0 \le \xi < 1$. Semi-stability only requires that $E\{\tilde{x}(k)'\tilde{x}(k)\} = \tilde{X}(k,B,\Psi)$ does not diverge exponentially; polynomial divergence is allowed, as in Example 1.

Definition 2 (KF semi-stability)**.** *We say that the KF is semi-stable if, for each* $B \in \mathcal{H}^{n,p}$*,* $\Psi \in \mathcal{R}^{n0}$ *and* $0 \le \xi < 1$*, there exists* $\bar{X}$ *such that* $\tilde{X}(k,B,\Psi) \le \xi^{-2k}\bar{X}$*,* $k \ge 0$*.*

For the study of semi-stability we can assume that $\Psi = I$ and $B = E$. That is, under Assumptions 1 and 2, the occurrence of divergence (in opposition to the rate and the degree of the divergence) makes no distinction between the incorrect noise being related to Σ or E, i.e., between non-persistent and persistent noise incorrect data. We make this notion precise, as follows. Consider $\tilde{X}_\xi(k)$, $k \ge 0$, defined recursively by

$$\begin{aligned}\tilde{X}_\xi(k+1) = (\xi(A-L_kC))\tilde{X}_\xi(k+1)(\xi(A-L_kC)) \\ + L_kDD'L_k' + BB', \quad \tilde{X}_\xi(0) = I.\end{aligned} \tag{9}$$

Lemma 1. *The KF is semi-stable if and only if, for each* $0 \le \xi < 1$*, there exists* $\bar{X}$ *such that* $\tilde{X}_\xi(k) \le \bar{X}$*,* $k \ge 0$*.*

Proof. (Necessity). For ρ as in Proposition 2 (i) let $\kappa = \rho\|D\|$, in such a manner that $\kappa^2 I = \rho^2\|D\|^2 I \ge L_kDD'L_k'$. One can check from (6) and (9) that $\tilde{X}_\xi(k) \le \xi^{2k}\tilde{X}(k, B+\kappa I, I)$, and from the semi-stability of the KF we obtain $\tilde{X}_\xi(k) \le \xi^{2k}\tilde{X}(k,B+\kappa I,I) \le \bar{X}$, hence the claim. (Sufficiency). Assume that $B = E$, that is, there is no error in the persistent noise data. For each $\Psi \in \mathcal{M}^{n0}$ there exists $\kappa \ge 0$ for which $\Psi \le \kappa I$, hence from Proposition 2 (iii) and (9) we obtain

$$\tilde{X}(k,\Psi) \le \kappa\xi^{-2k}X_\xi(k) \le \xi^{-2k}(\kappa\bar{X}). \tag{10}$$

In order to incorporate $B \ne E$, we make use of the fact that there is always a "margin of semi-stability" (in the sense that for each $0 \le \xi < 1$ there exists $\xi < \bar{\xi} < 1$ and, by hypothesis, $\tilde{X}_{\bar{\xi}}(k) \le \bar{X}$) that allows for accommodation of persistent noise. Assume now that the inequality

$$\tilde{X}_{\mathrm{h}}(k+t,t,BB') \le \delta^{2T}\bar{\xi}^{-2\bar{k}T}\bar{\bar{X}}\|B\|^2, \quad k,t \ge 0. \tag{11}$$

holds, where δ is as in Proposition 2 (ii), T is as in Assumption 2 and $\bar{k}$ is defined in such a manner that $k + t - \bar{k}T \ge 0$. This assumption is shown to hold in the Appendix. Setting χ such that $\bar{\xi} = (1+\chi)\xi$, yields

$$\begin{aligned}\xi^{2k}\tilde{X}(k,B,I) = \xi^{2k}\tilde{X}(k,B,I) + \\ + (1+\chi)^{-2k}\big(\bar{\xi}^{2k}\tilde{X}_{\mathrm{h}}(k-1,k-2,BB') + \\ + \cdots + \bar{\xi}^{2k}\tilde{X}_{\mathrm{h}}(k-1,1,BB')\big),\end{aligned}$$

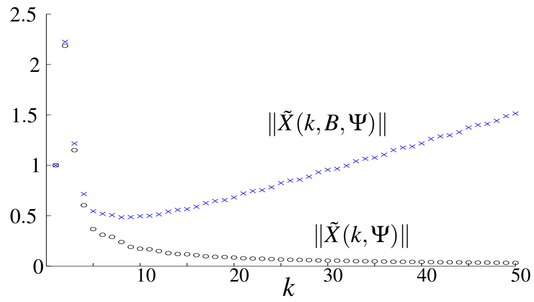

Fig. 1. Error covariances for the KF in Example 1.

hence (11) with $t = 0$ leads to

$$\begin{aligned}\xi^{2k}\tilde{X}(k,B,I) &\le \xi^{2k}\tilde{X}(k,B=0,I) + \\ &\quad + (1+\chi)^{-2k}\big(\bar{\xi}^{2k}\bar{\xi}^{-2\bar{k}T}\delta^{2T}\bar{\bar{X}}\|B\|^2 + \\ &\quad + \cdots + \bar{\xi}^{2k}\bar{\xi}^{-2\bar{k}T}\delta^{2T}\bar{\bar{X}}\|B\|^2\big) \\ &= \xi^{2k}\tilde{X}(k,B=0,I) \\ &\quad + (1+\chi)^{-2k}(k-1)\bar{\xi}^{2(k-\bar{k}T)}\delta^{2T}\bar{\bar{X}}\|B\|^2 \\ &\le \bar{X} + (1+\chi)^{-2k}(k-1)\delta^{2T}\bar{\bar{X}}\|B\|^2 \\ &\le \bar{X} + \left(\frac{\delta^{2T}\bar{\bar{X}}\|B\|^2(1+\chi)^{-2}}{1-2(1+\chi)^{-2}+(1+\chi)^4}\right).\end{aligned}$$

The proof for (11) is in the Appendix. □

Dissimilarly to Lemma 1, there is a distinction between accommodation of non-persistent and persistent noise incorrect models, as stability is concerned, and the next stability notion is specific to the latter case.

Definition 3 (KF stability w.r.t. Ψ)**.** *We say that the KF is stable with respect to (w.r.t.)* Ψ *when, for each* $\Psi \in \mathcal{R}^{n0}$ *there exists* $\bar{X}$ *such that* $\tilde{X}(k,\Psi) \le \bar{X}$*,* $k \ge 0$*.*

Lemma 2. *Stability w.r.t.* Ψ *is weaker than stability, stronger than semi-stability and equivalent to stability w.r.t.* I*.*

Proof. The first statement follows from Definitions 2 and 3. The second statement follows straightforwardly from Lemma 1. The proof for the third statement is analogous to the proof of Lemma 1, in particular the part where $B = E$. □

Example 1. *Consider system* Φ*,*

$$A_i = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, E_i = 0 \text{ and } \Sigma = C_i = D_i = I,\ i \ge 0.$$

The KF leads to bounded $\tilde{X}(k,\Psi = I)$ *(it is enough to consider* $\Psi = I$*, according to Lemma 2) and linearly divergent* $\tilde{X}(k,B = 0.2I,\Psi = I)$*, see Figure 1, yielding that the KF is stable w.r.t.* Ψ *but, clearly, not stable.*

One interesting property of the KF related to stability w.r.t. Ψ is that, provided that at time instant k the actual error covariance is structurally described by the nominal

P, then it remains described by P at successive instants, as discussed hereafter.

Lemma 3. *Assume that* $V \in \mathcal{R}^{n0}$ *and* $t \geq 0$ *are such that* $\ker\{P_t\} \subset \ker\{V\}$. *Then there exists* $\bar{X}$ *such that* $\tilde{X}(k+t,t,V) \leq \bar{X}$, $k \geq 0$. *Moreover,* $\ker\{P_{k+t}\} \subset \ker\{\tilde{X}(k+t,t,V)\}$, $k \geq 0$.

Proof. For V such that $\ker\{V\} \supset \ker\{P_t\}$ we can pick a scalar $0 < \kappa \leq 1$ such that $\kappa V \leq P_t$. Therefore, Proposition 2 (iii) (with $\alpha = 0$) and Assumption 1 yield

$$\tilde{X}(k+t,t,\kappa V) \leq \tilde{X}(k+t,t,P_t) = P_{k+t} \leq \bar{X}, \quad k \geq 0, \tag{12}$$

We set $\alpha = \kappa$ and $V_0 = V_1 = V$ in Proposition 2 (iii) to obtain $\kappa\tilde{X}(k+t,t,V) \leq \tilde{X}(k+t,t,\kappa V)$; this and (12) provide

$$\tilde{X}(k+t,t,V) \leq \kappa^{-1}\bar{X}, \quad k \geq 0.$$

Regarding the second statement, it follows immediately from (12) that

$$\ker\{P_{k+t}\} = \ker\{\tilde{X}(k+t,k,P_t)\} \subset \ker\{\tilde{X}(k+t,t,\kappa V)\}. \tag{13}$$

Now, for $\eta \in \ker\{\tilde{X}(k+t,t,\kappa V)\}$ we can write

$$\begin{aligned} 0 &= \eta'\tilde{X}(k+t,t,\kappa V)\eta \\ &= \eta'\big(\tilde{X}_{\rm h}(k+t,t,\kappa V) + \tilde{X}(k+t,t,\Psi=0)\big)\eta \\ &= \eta'\big(\kappa\tilde{X}_{\rm h}(k+t,t,V) + \tilde{X}(k+t,t,\Psi=0)\big)\eta \end{aligned}$$

yielding that $\tilde{X}_{\rm h}(k+t,t,V)\eta = 0$ and $\tilde{X}(k+t,t,\Psi=0)\eta = 0$, which lead to $\tilde{X}(k+t,t,V)\eta = \big(\tilde{X}_{\rm h}(k+t,t,V) + \tilde{X}(k+t,t,\Psi=0)\big)\eta = 0$, i.e., $\eta \in \ker\{\tilde{X}(k+t,t,V)\}$. We have shown that $\ker\{\tilde{X}(k+t,t,\kappa V)\} \subset \ker\{\tilde{X}(k+t,t,V)\}$, and substituting this relation in (13) yields the result. □

Lemma 3 leads immediately to the next result, retrieving the well-known fact that KFs for periodic or time-invariant systems are semi-stable when $\Sigma > 0$, see e.g. [4] and [8], and making clear the role of $\Sigma > 0$.

Corollary 1. *If* $\Sigma > 0$, *then the KF is stable w.r.t.* Ψ.

The upper bound on $\tilde{X}$ provided by Lemma 3 is dependent on V and t. This lack of uniformity can not be removed in general. However, for the non-forced solution $\tilde{X}_{\rm h}$ of (6) we can employ the second statement of Lemma 3 to derive the following result, the proof of which is omitted.

Lemma 4. *Let* $t \geq 0$ *and* P_t *be given by* (2). *There exists* X^+ *such that* $\tilde{X}_h(k+t,t,V) \leq \|V\|X^+$, $k \geq 0$, *for all* V *for which* $\ker\{V\} \supset \ker\{P_t\}$.

IV. A structural link between the Kalman gain and the plant

This section addresses the connection among the Kalman gain L and the covariance matrices X and P. We start showing an orthogonality result involving L_kC_k and the null space of $X(k+1)$, with the interpretation that the KF behaves exactly as the plant in subspaces with no associated noise. Then we show that the null spaces of P and X coincide, thus extending the orthogonality result to P; in the time-invariant case, this is to some extent analogous to [5, Theorem 1].

Lemma 5. *The following statements hold.*

(i) $\ker\{P_{k+1}\} = \ker\{X(1,0,P_k\}$, $k \geq 0$.
(ii) *For each* $v \in \mathbb{R}^n$, L_kC_kv *is orthogonal to* $\ker\{P_{k+1}\}$.

Proof. We start showing that $\ker\{P_{k+1}\} \supset \ker\{X(1,0,P_k)\}$. By optimality of the KF, we have that

$$P_{k+1} = \tilde{X}(k+1) \leq X(k+1,k,\tilde{X}(k)) = X(1,0,P_k),$$

hence for $\eta \in \ker\{X(1,0,P_k)\}$,

$$\eta'P_{k+1}\eta \leq \eta'X(1,0,P_k)\eta = 0. \tag{14}$$

Conversely, if we pick an arbitrary $\eta \in \ker\{P_{k+1}\}$, we can employ Proposition 1 to write

$$\begin{aligned} 0 &= \eta'P_{k+1}\eta = \eta'\tilde{X}(k+1)\eta \\ &= \eta'[(A_k+L_kC_k)\tilde{X}(k)(A_k+L_kC_k)' + L_kD_kD_k'L_k' + E_kE_k']\eta \end{aligned} \tag{15}$$

which, in particular (and recalling that $D_iD_i' > 0$, $i \geq 0$), means that

$$L_k'\eta = 0, \quad \eta \in \ker\{P_{k+1}\}, \tag{16}$$

allowing to re-evaluate (15) as

$$\begin{aligned} 0 &= \eta'[(A+L_kC)\tilde{X}(k)(A_k+L_kC_k)' + L_kD_kD_k'L_k' + E_kE_k']\eta \\ &= \eta'[A_k\tilde{X}(k)A_k' + E_kE_k']\eta = \eta'X(1,0,P_k)\eta, \end{aligned}$$

completing the proof of (i). For (ii), note that (16) immediately leads to $\eta'(L_kC_kv) = 0$ for each $\eta \in \ker\{P_{k+1}\}$, thus L_kC_kv is orthogonal to $\ker\{P_{k+1}\}$. □

Corollary 2. $\ker\{P_k\} = \ker\{X(k)\}$, $k \geq 0$.

Proof. We proceed inductively. For $k = 0$, $P_0 = X(0,0,\Sigma) = \Sigma$ by definition. Now assume that $\ker\{P_k\} = \ker\{X(k)\}$ holds for k, and note that

$$\ker\{A_kP_kA_k' + E_kE_k'\} = \ker\{A_kX(k)A_k' + E_kE_k'\}. \tag{17}$$

Lemma 5 and (8) yield

$$\ker\{P_{k+1}\} = \ker\{X(k+1,k,P_k)\} = \ker\{A_kP_kA_k' + E_kE_k'\}. \tag{18}$$

Hence (17) and (18) lead to $\ker\{P_{k+1}\} = \ker\{A_kX(k)A_k' + E_kE_k'\} = \ker\{X(k+1)\}$, completing the induction. □

We conclude this section with one more result.

Corollary 3. *The following statements hold:*

(i) *for each* $v \in \mathbb{R}^n$, $L_kC_kv \perp \ker\{P_{k+1}\}$;
(ii) $H_{k+1}L_kC_k = 0$;

where $H_k \in \mathbb{R}^n$, $k \geq 0$, *represents the orthogonal projection onto* $\ker\{P_k\}$.

Proof. Statement (i) follows immediately from Lemma 5 (ii) and Corollary 2. Regarding (ii), since H_{k+1} stands for the orthogonal projection onto $\ker\{P_{k+1}\}$, we have that $\ker\{H_{k+1}\} \perp \ker\{P_{k+1}\}$. On the other hand, the statement (i) leads to $L_kC_kv \perp \ker\{P_{k+1}\}$. Then, $L_kC_kv \in \ker\{H_{k+1}\}$, and $H_{k+1}L_kC_kv = 0$, $v \in \mathbb{R}^n$. □

V. Semi-stability for the KF

In this section we present a necessary and sufficient condition for the KF to be semi-stable relying on semi-stability of an auxiliary system, defined as follows. Recall that H_k stands for the projection onto $\ker\{P_k\}$ (equivalently onto $\ker\{X(k)\}$ as stated in Corollary 2, which is easier to calculate). Let $Z_k \in \mathcal{R}^{n0}$, $k \geq 0$, be defined by $Z_0 = H_0 H_0'$ and

$$Z_k = (H_k A_k) Z_{k-1} (H_k A_k)', \quad k \geq 1. \tag{19}$$

Z_k is strongly linked with the KF dynamics, as follows.

Lemma 6. $Z_k = [H_k(A_k + L_{k-1}C_k)] Z_{k-1} [H_k(A_k + L_{k-1}C_k)]'$, $k \geq 0$.

Proof. The result is immediate from Corollary 3. □

The next lemma provide some useful evaluations involving $\tilde{X}_\xi$, $\tilde{X}_\mathrm{h}$ (defined in (7) and (9)) and Z, the proofs of which are omitted.

Lemma 7. *The following statements hold for any $\varepsilon \neq 0$:*

(i)
$$\begin{aligned}\tilde{X}_\xi(k,0,I) &\leq \tilde{X}_\xi(k,0,0) + \xi^{2k}(1+\varepsilon^{-2})^{k+1} Z_k \\ &+ \xi^{2k}(1+\varepsilon^2)\tilde{X}_h(k-1,0,(I-H_0)I(I-H_0)') \\ &+ \xi^{2k}(1+\varepsilon^2)\sum_{\ell=1}^{k}[\times(1+\varepsilon^{-2})^\ell \\ &\times \tilde{X}_h(k-1,\ell,(I-H_\ell)\tilde{X}_h(\ell-1,\ell-1,Z_{\ell-1})(I-H_\ell)')],\end{aligned}$$

(ii)
$$\begin{aligned}\tilde{X}_\xi(k,0,I) &\geq \tilde{X}_\xi(k,0,0) + \xi^{2k}(1/2)(1-\varepsilon^{-2})^k Z_k \\ &- \xi^{2k}\tilde{X}_h(k-1,0,(I-H_0)I(I-H_0)') \\ &- \xi^{2k}(\varepsilon^2-1)\sum_{\ell=1}^{k}[(1-\varepsilon^{-2})^\ell \times \\ &\times \tilde{X}_h(k-1,\ell,(I-H_\ell)\tilde{X}_h(\ell-1,\ell-1,Z_{\ell-1})(I-H_\ell)')].\end{aligned}$$

Theorem 1. *The KF is semi-stable if and only if there exists $\bar{Z} \in \mathcal{R}^{n0}$ such that $\xi_Z^{2k} Z_k \leq \bar{Z}$, $k \geq 0$, $0 \leq \xi_Z < 1$.*

Proof. (Sufficiency). For each $0 \leq \xi < 1$ we set $\varepsilon > 0$ and $0 \leq \xi_Z < 1$ in such a manner that

$$\xi^2 \xi_Z^{-2}(1+\varepsilon^{-2}) < 1.$$

We start taking into account each term on the right hand side of statement (i) of Lemma 7, separately. For the first term Proposition 2 (iii) yields

$$\tilde{X}_\xi(k,0,0) \leq \tilde{X}_\xi(k,0,\Sigma) \leq P_k \leq \bar{X}. \tag{20}$$

For the second term it is simple to check that

$$\begin{aligned}&(\xi^2(1+\varepsilon^{-2}))^k(1+\varepsilon^{-2})Z_k \\ &\quad \leq (\xi_Z^{-2}\xi^2(1+\varepsilon^{-2}))^k(1+\varepsilon^{-2})\bar{Z} \leq (1+\varepsilon^{-2})\bar{Z}.\end{aligned} \tag{21}$$

For the third term, since H_0 is the projection onto $\ker\{P_0\}$, we have that $\ker\{(I-H_0)I(I-H_0)'\} \supset \ker\{P_0\}$ yielding, from Lemma 4,

$$\begin{aligned}&\xi^{2k}(1+\varepsilon^2)\tilde{X}_\mathrm{h}(k-1,0,(I-H_0)I(I-H_0)') \\ &\quad \leq \xi^{2k}(1+\varepsilon^2)\|(I-H_0)I(I-H_0)'\|X^+ \\ &\quad \leq \xi^{2k}(1+\varepsilon^2)X^+.\end{aligned} \tag{22}$$

For the last term Proposition 2 (ii) and (iii) lead to $\|\tilde{X}_\mathrm{h}(\ell-1,\ell-1,Z_{\ell-1})\| \leq \|\tilde{X}_\mathrm{h}(\ell-1,\ell-1,\xi_Z^{-2\ell}\bar{Z})\| \leq \xi_Z^{-2\ell}\delta\|\bar{Z}\| + \gamma$, $\ell = 1,\ldots,k$, and

$$\|(I-H_\ell)\tilde{X}_\mathrm{h}(\ell-1,\ell-1,Z_{\ell-1})(I-H_\ell)'\| \leq (\xi_Z^{-2\ell}\delta\|\bar{Z}\| + \gamma)I.$$

Then, we employ Lemma 4 (note that, similarly to the case k=0, $\ker\{(I-H_\ell)\tilde{X}_\mathrm{h}(\cdot)(I-H_\ell)'\} \supset \ker\{P_k\}$) and the above inequality, respectively, to evaluate

$$\begin{aligned}&(1+\varepsilon^2)\xi^{2k}\sum_{\ell=1}^{k}[(1+\varepsilon^{-2})^\ell \times \\ &\quad \times \tilde{X}_\mathrm{h}(k-1,\ell,(I-H_\ell)\tilde{X}_\mathrm{h}(\ell-1,\ell-1,Z_{\ell-1})(I-H_\ell)')] \\ &\leq (1+\varepsilon^2)\xi^{2k}\sum_{\ell=1}^{k}[(1+\varepsilon^{-2})^\ell \times \\ &\quad \times \|(I-H_\ell)\tilde{X}_\mathrm{h}(\ell-1,\ell-1,Z_{\ell-1})(I-H_\ell)'\|X^+] \\ &\leq (1+\varepsilon^2)\xi^{2k}\sum_{\ell=1}^{k}[(1+\varepsilon^{-2})^\ell(\xi_Z^{-2\ell}\delta\|\bar{Z}\|+\gamma)X^+] \\ &\leq (1+\varepsilon^2)X^+\sum_{\ell=1}^{k}(\xi^2\xi_Z^{-2}(1+\varepsilon^{-2}))^\ell(\delta\|\bar{Z}\|+\gamma) \leq \zeta X^+,\end{aligned} \tag{23}$$

where we have defined $\zeta = (1+\varepsilon^2)(\delta\|\bar{Z}\|+\gamma)q(1-q)^{-1}$ with $q = \xi^2\xi_Z^{-2}(1+\varepsilon^{-2}) < 1$. Substituting (20)–(23) in statement (i) of Lemma 7 we obtain, for $k \geq 0$:

$$\tilde{X}_\xi(k,0,I) \leq \bar{X}_\Sigma + (1+\varepsilon^{-2})\bar{Z} + \xi^{2k}(1+\varepsilon^2)X^+ + \zeta X^+ \tag{24}$$

and Lemma 1 leads to the result.

(Necessity.) Assuming that the KF is semi-stable, we have in particular that for each $0 \leq \xi < 1/2$ there exists $\bar{X}$ such that $X_\xi(k,0,I) \leq \bar{X}$, $k \geq 0$, and from Lemma 7 (ii), with $\varepsilon > 1$ such that $\xi^2(1-\varepsilon^{-2}) < 1$, we obtain

$$\begin{aligned}\bar{X} &\geq \tilde{X}_\xi(k,0,I) \geq \tilde{X}_\xi(k,0,0) + \xi^{2k}(1/2)(1-\varepsilon^{-2})^k Z_k \\ &- \xi^{2k}\tilde{X}_\mathrm{h}(k-1,0,(I-H_0)I(I-H_0)') \\ &- \xi^{2k}(\varepsilon^2-1)\sum_{\ell=1}^{k}(1-\varepsilon^{-2})^\ell \times \\ &\quad \times \tilde{X}_\mathrm{h}(k-1,\ell,(I-H_\ell)\tilde{X}_\mathrm{h}(\ell-1,\ell-1,Z_{\ell-1})(I-H_\ell)').\end{aligned} \tag{25}$$

Similarly to the proof of the sufficiency, one can check that the third and fourth terms on the right hand side of (25) are bounded from bellow by $-X^+$ and $-\eta X^+$ where η is set similarly to ζ in the proof of sufficiency. Note that $\tilde{X}_\xi(k,0,0) \geq 0$. These elements can be combined to obtain $\bar{X} \geq (1/2)\xi^{2k}(1-\varepsilon^{-2})^k Z_k - X^+ - \eta X^+$, and if we set $\varepsilon = 2$ and $\xi_Z = 2\xi$, we get that for each $0 \leq \xi_Z < 1$,

$$Z_k \leq 2\xi_Z^{-2k}[\bar{X} + X^+ + \eta X^+]. \quad \square$$

Example 2 (Example 1 continued)**.** *Example 1 has already established that the KF is not stable. Since (A,C) is detectable, one can easily check that Assumption 1 holds, e.g. by employing the results in [1], [2]. Assumption 2 holds trivially, see Remark 2. Stability w.r.t. Ψ follows from Corollary 1, since $\Sigma = I$. Moreover, $\Sigma > 0$ yields $H_0 = 0$ and from (19) it follows that $Z_k = 0$, and semi-stability is confirmed by Theorem 1.*

Example 3. *Consider system* Φ,

$$A_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B_i = C_i = 0,\ D_i = I,\ i \geq 0.$$

Implementing the KF one can check that P_k *is periodic (and conditions available in the literature can not be employed). Thus* P_k *is bounded and satisfies Assumption 1. Assumption 2 holds trivially, see Remark 2. It is simple to check that* Z_k *is bounded (in fact,* $Z_k = P_{k+1}$*), in such a manner the KF is semi-stable, as stated in Theorem 1.*

Example 4 (A simple non-observable, non-periodic system)**.** *Consider* $T \geq 1$ *and a sequence of discrete random variables* K_m, $m \geq 0$, *satisfying* $0 \leq K_m(\omega) \leq T$, $\omega \in \Omega$. *Given a realization* $\{K_0(\omega), K_1(\omega), \ldots\}$, *let* $A_{K_m} = 0$, $m \geq 0$; *for each* $i \neq K_m$, $m \geq 0$, *let* A_i *be defined in such a manner that* $A_i \leq Q$, $Q \in \mathcal{R}^n$. *Let* $C_i = E_i = \Sigma = 0$ *and* $D_i = I$, $i \geq 0$. *Assumptions 1 and 2 are satisfied with* $\bar{X} = 0$ *and* $M(T,0) = 0$. *One can check from* (19) *that* $Z_k = 0$, $k \geq T$, *hence Theorem 1 shows that the KF is semi-stable.*

VI. Concluding remarks

In this paper we explored the structure of the KF from the perspective of divergence of the actual error covariance $\tilde{X}_k$ under incorrect noise measurements, assuming that the calculated error covariance P_k is bounded. We have shown that stability w.r.t. Ψ (accommodation of incorrect Σ, exclusively) is weaker than stability, whereas semi-stability makes no distinction between imprecise Σ, E or both, see Lemma 1. These results and some structural properties of KF (as the one in Lemma 3 and the orthogonality property of the Kalman gain $H_{k+1}L_kC_k = 0$) together allow us to derive a necessary and sufficient condition for semi-stability of the KF or, equivalently, for avoiding exponential divergence of $\tilde{X}_k$. The condition relies on the existence of an upper bound for the auxiliary system in (19), the dynamics of which only involves the relevant data A, E and Σ, and involves neither conditions on C (hence are valid for non-detectable systems) nor calculations of RDEs. The results in this paper provide the basic elements to investigate, as detailed in [3], *algebraic* necessary and sufficient conditions for stability, stability w.r.t. to Ψ and semi-stability of KFs in the time-invariant context.

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Appendix

Proof of Lemma 1 (continued). We now prove inequality (11). From Proposition 2 (v) we have that $BB' \leq (1+\varepsilon^2)MBB'M' + \left(1+\varepsilon^{-2}\right)(I-M)BB'(I-M)'$ and employing Proposition 2 (iii) we evaluate

$$\begin{aligned} \tilde{X}_{\rm h}((k+t)T, tT, BB') &\leq \tilde{X}_{\rm h}\big((k+t)T, tT, (1+\varepsilon^2)MBM' \\ &\qquad + \left(1+\varepsilon^{-2}\right)(I-M)BB'(I-M)'\big) \\ &= (1+\varepsilon^2)\tilde{X}_{\rm h}((k+t)T, tT, MBB'M') \\ &\qquad + \left(1+\varepsilon^{-2}\right)\tilde{X}_{\rm h}((k+t)T, tT, (I-M)BB'(I-M)') \end{aligned} \tag{26}$$

For the first term on the right-hand side of (26), setting μ as in Proposition 2 (iv) yields $\mu\|B\|^{-2}MBB'M' \leq \mu MM' \leq \tilde{X}_{\rm h}(tT-1, 0, MM')$, while Proposition 2 (iii) leads to

$$\begin{aligned} &\tilde{X}_{\rm h}((k+t)T, tT, \mu\|B\|^{-2}MBB'M') \\ &\quad \leq \tilde{X}_{\rm h}((k+t)T, tT, \tilde{X}_{\rm h}(tT-1, 0, MM')) \\ &\quad = \tilde{X}_{\rm h}((k+t)T, 0, MM) \leq \tilde{X}((k+t)T, 0, I). \end{aligned}$$

As a result, from (10) with $\kappa = 1$ and ξ replaced by $\bar{\xi} = (1+\chi)\xi$, recalling that $\chi > 0$ is defined in such a manner that $\xi < \bar{\xi} < 1$, we obtain

$$\begin{aligned} &\tilde{X}_{\rm h}((k+t)T, tT, \mu\|B\|^{-2}MBB'M') \\ &\qquad \leq \tilde{X}((k+t)T, 0, I) \leq \bar{\xi}^{-2(k+t)T}\bar{X}. \end{aligned} \tag{27}$$

For the second term on the right-hand side of (26) we have from Proposition 2 (iv) that

$$\begin{aligned} &X_{\rm h}((k+t)T, kT, (I-M)BB'(I-M)') \leq \\ &\qquad \leq (I-M)BB'(I-M)' \leq \|B\|^2 I. \end{aligned} \tag{28}$$

By substituting (27) and (28) in (26), we obtain

$$\tilde{X}_{\rm h}((k+t)T, tT, BB') \leq \bar{\xi}^{-2(k+t)T}\|B\|^2\bar{\bar{X}} \tag{29}$$

where $\bar{\bar{X}} = (1+\varepsilon^2)\mu^{-1}\bar{X} + \left(1+\varepsilon^{-2}\right)I$. Note that (29) represents an evaluation for the maximal expansion of $\tilde{X}_{\rm h}$ along time intervals of the form $[tT, (k+t)T]$. The inequality (11) follows by extending the above evaluation to general intervals $[t, k+t]$, defining $\bar{k}$ as the largest integer for which $\bar{k}T \leq k+t$ and replacing $k+t$ in (29) by $\bar{k}$, combined with the evaluation for the expansion of $\tilde{X}$ provided in Proposition 2 (ii). □