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# Extension of Harmonic Balance Principle and its Application to Analysis of Convergence Rate of Second-Order Sliding Mode Control Algorithms

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***Abstract*—The harmonic balance condition is revisited. The existing methodology of analysis of periodic motions is extended to analysis of vanishing oscillations of variable frequency. Transient processes in the systems controlled by second-order sliding mode (SOSM) algorithms are analyzed. A simple criterion of the existence of finite-time convergence is proposed. It is shown that the convergence rate depends on the angle between the high-frequency asymptote of the Nyquist plot of the plant and the low-amplitude asymptote of the negative reciprocal of the describing function of the controller, which is named the *phase deficit*.**

## I. INTRODUCTION

HARMONIC balance principle is widely used in many areas of science and engineering to solve the problems of finding parameters of self-excited periodic motions. For a system with one nonlinearity and linear dynamics, it can be illustrated by drawing the Nyquist plot of the linear dynamics and the plot of the negative reciprocal of the describing function (DF) [1] of the nonlinearity in the complex plane and finding the point of intersection of the two plots, which would correspond to the self-excited periodic motion in the system. Therefore, the harmonic balance principle treats the system as a loop connection of the linear dynamics and of the nonlinearity. It is also possible to reformulate the harmonic balance, so that the format of the system analyzed is not a loop connection but the denominator of the closed-loop system. This would imply a different interpretation of the harmonic balance. It is shown in the present paper that this representation would allow one to extend the harmonic balance principle to analysis of not only self-excited periodic motions but also other types of oscillatory motions.

One of the types of the systems that exhibit vanishing oscillatory motion is the conventional and second-order sliding mode (SM) control system. There are a number of second-order SM (SOSM) algorithms available now, the most popular of which are “twisting”, “super-twisting”, “twisting-as-a-filter” [2], [3], “sub-optimal” [4], [5], and a number of other algorithms [6]. The problem of convergence rate is a valid problem in the conventional SM control and “terminal SM” [7], [8] control too. Therefore, some common approach to the problem of the convergence rate assessment, including qualitative (finite-time or asymptotic) and quantitative assessment, is of high importance.

The frequency-domain approach to assessment of convergence rate would provide a number of advantages over the direct solution/estimates of the system differential equations. The most important one would be the unification of the treatment of all the algorithms based on some frequency-domain characteristics. This in turn may lead to formulation of some criteria that should be satisfied for a SOSM algorithm to provide a finite-time convergence, which can also lead to relatively simple rules that would allow one to develop new SOSM algorithms.

The paper is organized as follows. At first the harmonic balance principle is considered and its different representation is proposed. Then a system comprising a second-order plant and an asymptotic SOSM (relay) controller is analyzed with the use of the approach proposed. Such characteristics as frequency and amplitude of oscillations as functions of time are derived. After that a system comprising the twisting SOSM controller and a second-order plant is analyzed with the use of the proposed approach. Finally, an approach to analysis of the type of convergence based on the frequency-domain characteristics is considered.

## II. HARMONIC BALANCE REVISITED

Consider the system that includes linear dynamics given by the transfer function $W_l(s) = P(s)/Q(s)$, which is a ratio of two polynomials, and a symmetric nonlinearity for which the describing function is $N(a)$, where $a$ is the amplitude of the oscillations at the input to this nonlinearity. Assume also an autonomous mode, so that the input to the nonlinearity is the output of the linear dynamics, and the output of the nonlinearity is the input to the linear dynamics. The harmonic balance condition is formulated as

$$W_l(j\Omega)N(a) = -1, \quad (1)$$

where $\Omega$ is the frequency and $a$ is the amplitude of the self-excited periodic motion. Find the closed-loop transfer function $W_{cl}(s)$ of this system.

$$W_{cl}(s) = \frac{W_l(s)N(a)}{1+W_l(s)N(a)} = \frac{P(s)N(a)}{Q(s)+P(s)N(a)} \quad (2)$$

Let us note that (1) is equivalent to

$$R(a, j\Omega) = Q(j\Omega) + P(j\Omega)N(a) = 0, \quad (3)$$

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which means that the denominator of the closed-loop transfer function turns into zero when the frequency and the amplitude become equal to the frequency and the amplitude of the periodic motion. Equation (3) is also sometimes used for finding a periodic solution, especially via algebraic methods. However, equation (3) usually is not attributed to the denominator of the closed-loop transfer function but considered a direct result of (1). Assuming that $R(a,s)$ can be represented in the form $R(a,s)=(s-s_1)(s-s_2)\cdot\ldots\cdot(s-s_n)$, where $s_i$ are roots of the characteristic polynomial, we must conclude that there must be at least one pair of complex conjugate root with zero real parts. That would correspond to the existence of the conservative component in the transfer function $W_{cl}(s)$. Indeed, we can consider the existence of non-vanishing oscillations as a result of the existence of the component $\left(s^2+\rho^2\right)$ in the denominator of $W_{cl}(s)$, where $\rho$ is a parameter that depends on the amplitude *a*. Let us refer to formula (3), implying the denominator of the closed-loop transfer function, as to the *reformulated harmonic balance principle*. Conditions (1) and (3) are fully equivalent, and there may be only technical advantages of one over another. However, the consideration of the denominator of the closed-loop transfer function offers an extension of the harmonic balance principle.

Assume now that the characteristic polynomial of the closed-loop system (with parametric dependence on the amplitude of the oscillations) has a pair of complex conjugate roots with negative real parts. Then a vanishing oscillation of certain frequency and amplitude occurs. The idea of considering equations of vanishing oscillations is similar to the one of the Krylov-Bogoliubov method [9]. However, the latter can only deal with small "deviations" from the harmonic oscillator and is limited to second-order systems. In the present approach, the "equivalent damping" is not limited to small values. Let us consider instantaneous values of the oscillatory process and formulate the idea of the extension of the harmonic balance principle as follows. *A. At every time, the characteristic equation of the closed-loop system provides an equation of an oscillator of a certain instantaneous frequency, amplitude and amplitude decay (decay can be positive, negative or zero). B. The coefficients of the characteristic polynomial depend on the instantaneous amplitude of the oscillation (and possibly on the frequency and the decay in more complex structures of the control system) that can be found via the DF method or assessed using other techniques. C. For every given amplitude of the oscillation, corresponding values of the instantaneous frequency and decay can be obtained by finding the roots of the characteristic equation and considering the solutions of the oscillator(s) that occur if there is at least one pair of complex conjugate roots. D. The process with variable amplitude (and frequency and decay) can be obtained by solving the differential equation relating the instantaneous amplitude and the instantaneous decay of the amplitude.* Consider the following example that illustrates it.

## III. Analysis of Convergence Rate – Frequency-Domain Approach

Carry out frequency-domain analysis of the transient process in the asymptotic SOSM controlled system, which is the relay feedback system with the plant of relative degree two (in particular, it is a second-order plant). The time-domain analysis of such system was done by [10]. Let the system be given as follows:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}+\mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}, \qquad (4)$$

$$u=-c\cdot\operatorname{sgn} y, \qquad (5)$$

where the linear part is given by (4), and the relay nonlinearity is given by (5). We shall consider only the case of the second-order system with $\mathbf{A}=\begin{bmatrix}0 & 1\\ -a_1 & -a_2\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}0 & b_2\end{bmatrix}^T$, $\mathbf{C}=\begin{bmatrix}1 & 0\end{bmatrix}$, $a_1\ge 0, a_2>0, b_2>0$. We shall also use the transfer function of the linear part: $W_l(s)=b_2/(s^2+a_2 s+a_1)$. Apply the DF method in the following modified form to analysis of system (4), (5). Assume that the linear part is a low-pass filter, so that $y(t)$ is a damped harmonic oscillation of variable frequency, $a(t)$ is the instantaneous amplitude, and $\Omega(t)$ is the instantaneous frequency of oscillations of $y(t)$. Replace the nonlinearity in equation (5) with its DF:

$$u=-N(a)\cdot y, \qquad (6)$$

where $N(a)=4c/\pi a$ (7)

is the DF [1]. Obtain the transfer function of the closed-loop system (4), (6) using the DF (7):

$$W_{cl}(s)=\frac{N(a)b_2}{s^2+a_2 s+a_1+N(a)b_2}.$$

The characteristic equation of the closed-loop system is, therefore,

$$s^2+2\xi\omega_n s+\omega_n^2=0, \qquad (8)$$

where $\omega_n=\sqrt{a_1+N(a)b_2}$, $\xi=0.5a_2/\sqrt{a_1+N(a)b_2}$. Assuming that $\xi<1$ (which always holds at least starting from a certain instant – due to the growth of $N(a)$) we can write an analytical solution. Introduce *instantaneous amplitude*, *instantaneous frequency* and *instantaneous phase angle*. We shall consider that for every time *t* the solution of (8) provides instantaneous values of the parameters of the oscillations, so that the solution can be written as follows:

$$y(t)=a_0\, e^{\sigma t}\sin\Psi(t), \qquad (9)$$

where $\sigma=-\xi\omega_n=-0.5a_2$ is the decay constant, $a_0$ is the initial amplitude, $\Psi(t)$ is the instantaneous phase,

$\Psi(t)=\int_0^t \Omega(\tau)d\tau+\phi$, $\phi$ is selected to satisfy initial conditions, $\Omega=0.5\sqrt{4(a_1+N(a)b_2)-a_2{}^2}$ is the instantaneous frequency, and

$$a(t)=a_0\,e^{\sigma t}=a_0\,e^{-0.5a_2 t} \qquad (10)$$

is the instantaneous amplitude. It follows from formula (10) that the amplitude decreases exponentially, with constant decay $\sigma$. Therefore, *an asymptotic convergence of $y(t)$ to zero takes place*, because the oscillations have non-zero amplitude at any finite time. The frequency of the oscillations grows according to the following formula:

$$\Omega(t)=0.5\sqrt{4\left(a_1+\frac{4cb_2}{\pi a(t)}\right)-a_2{}^2}\,. \qquad (11)$$

Simulations of the process in the system with $W_l(s)=1/(s^2+s+1)$ show a good match to the analytical results, providing the actual value of the decay a little higher than the theoretical one, due to the approximate DF method.

Now let us carry out similar analysis for a system with SOSM. Assume as before that the linear part is given by (4). Let the controller be the "twisting" SOSM controller [2] given as follows:

$$u=-c_1\cdot\operatorname{sgn} y-c_2\cdot\operatorname{sgn}\dot{y}\,, \qquad (12)$$

Apply the DF analysis to this system. Since the twisting algorithm includes two relay nonlinearities, apply to them two describing functions – like in [11]. For the first relay:

$$N_1(a)=4c_1/(\pi a)\,, \qquad (13)$$

and for the second relay

$$N_2(a^*)=4c_2/(\pi a^*)\,, \qquad (14)$$

where $a^*$ is the instantaneous amplitude of $\dot{y}(t)$, which still needs to be obtained. In the case of the twisting SOSM controller, the decay will not be constant any longer (it is shown below). For that reason, a different representation of the system output is used. We shall consider the following output formula:

$$y(t)=a(t)\sin\Psi(t)\,, \qquad (15)$$

where the instantaneous phase $\Psi(t)$ is given by the same formula as above, $a(t)$ is the instantaneous amplitude. Find this amplitude by differentiating (15).

$$\dot{y}(t)=a(t)\left[\sigma(t)\sin\Psi(t)+\Omega(t)\cos\Psi(t)\right], \qquad (16)$$

where $\sigma(t)$ and $\Omega(t)$ are instantaneous decay and frequency respectively. Therefore, $a^*=a\sqrt{\sigma^2(t)+\Omega^2(t)}$, and the DF for the second relay can be rewritten as

$$N_2(a)=\frac{4c_2}{\pi a\sqrt{\sigma^2+\Omega^2}}\,. \qquad (17)$$

In the same way as for the asymptotic SOSM controlled system, write the formula for the closed-loop system for the case of the twisting controller

$$W_{cl}(s)=\frac{(N_1(a)+sN_2(a))b_2}{s^2+(a_2+N_2(a)b_2)s+a_1+N_1(a)b_2}\,.$$

The characteristic equation of the closed-loop system is given by (8) where now $\omega_n=\sqrt{a_1+N_1(a)b_2}$ $\xi=0.5(a_2+N_2(a)b_2)/\sqrt{a_1+N_1(a)b_2}$. The decay is now time-varying with the instantaneous value being:

$$\sigma(t)=-\xi\omega_n=-0.5(a_2+N_2(a)b_2) \qquad (18)$$

Formula (18) provides the instantaneous rate of the amplitude change, and not related to the initial amplitude (unlike formula (10)): $\sigma(t)=\dot{a}(t)/a(t)$. The instantaneous amplitude can be found via solving the following differential equation:

$$\dot{a}(t)=a(t)\sigma(t),\quad a(0)=a_0\,. \qquad (19)$$

Therefore, the formulas for the instantaneous decay and instantaneous frequency are as follows.

$$\sigma(t)=-0.5\left(a_2+\frac{4c_2b_2}{\pi a(t)\sqrt{\sigma^2(t)+\Omega^2(t)}}\right), \qquad (20)$$

$$\Omega(t)=0.5\sqrt{4\left(a_1+\frac{4c_1b_2}{\pi a(t)}\right)-\left(a_2+\frac{4c_2b_2}{\pi a(t)\sqrt{\sigma^2(t)+\Omega^2(t)}}\right)^2} \qquad (21)$$

The formulas for the instantaneous amplitude (19), instantaneous decay (20) and instantaneous frequency (21) make a set of one differential and two algebraic equations. The proposed solution algorithm is as follows. Express $\Omega$ from (20) as follows:

$$\Omega=\sqrt{\frac{16c_2{}^2b_2{}^2}{\pi^2a^2(2\sigma+a_2)^2}-\sigma^2} \qquad (22)$$

and substitute the expression in formula (21). Solve the resulting equation for $\sigma$.

$$\sigma=-\frac{2c_2b_2}{\sqrt{\pi^2a^2a_1+4\pi ac_1b_2}}-\frac{a_2}{2} \qquad (23)$$

Substitution of (23) in (19) yields the following differential equation for $a(t)$.

$$\dot{a}=-\frac{2c_2b_2a}{\sqrt{\pi^2a^2a_1+4\pi ac_1b_2}}-\frac{a_2}{2}a,\quad a(0)=a_0\,. \qquad (24)$$

Formula (24) is a first-order nonlinear differential equation of the type:

$$\dot{z}=-\lambda z-g(z),\quad z(0)=z_0>0\,, \qquad (25)$$

where $g(z)=\dfrac{\alpha}{\sqrt{1+\beta/z}}$, $\alpha=\dfrac{2c_2b_2}{\pi\sqrt{a_1}}$, $\beta=\dfrac{4c_1b_2}{\pi a_1}$, $\lambda=\dfrac{a_2}{2}$, $z$=$a$. The nonlinear function $g(z)$ has infinite derivative at $z=0$, which makes the finite-time convergence of the process given by (24) possible. Prove it and assess the convergence time. At first consider the following lemma.

*Lemma 1* (given without proof, which can be based upon consideration of time being function of *z*). For the first-order nonlinear differential equation

$$\dot{z} = -g(z), \tag{26}$$

where $g(z) > 0$ for all $z > 0$, and $g(0) = 0$, and the initial condition $z(0) = z_0 > 0$ the following holds. If the solution of equation (26) is $z(t)$, such that a finite-time convergence to zero takes place $z(T_g) = 0$, $z(t) \in [0; z_0]$, and there exists function $h(z)$, such that $h(z) \le g(z)$ for all $z \in [0; z_0]$, $h(z) > 0$ for all $z > 0$, and $h(0) = 0$, then the finite-time convergence to zero in the equation

$$\dot{z} = -h(z) \tag{27}$$

takes place too, with the convergence time $T_h$ ($z(T_h) = 0$), so that $T_h \ge T_g$.

*Theorem 1*. The process of conversion of the amplitude described by (24) from the initial value $a_0$ provides finite-time conversion with the conversion time not exceeding

$$T^* = \frac{2}{\lambda}\left(\ln\left(\lambda\sqrt{z_0} + \frac{\alpha}{\sqrt{z_0+\beta}}\right) - \ln\frac{\alpha}{\sqrt{z_0+\beta}}\right).$$

*Proof*. Consider equation (25), which is a reformulated (24). Replace nonlinearity $g(z)$ in it with another nonlinearity $h(z)$ such that $h(z) \le g(z)$, $z \in [0; z_0]$, for which the finite-time conversion property can be (has been) proved. Select $h(z)$ to be $h(z) = \rho\sqrt{z}$, $\rho > 0$. Select parameter $\rho = \frac{\alpha}{\sqrt{z_0+\beta}}$ to satisfy the requirement $h(z) \le g(z)$, $z \in [0; z_0]$. Also, note that $h(z_0) = g(z_0)$. Therefore, since $g^2(z) = \frac{\alpha^2 z}{z+\beta}$ and $h^2(z) = \frac{\alpha^2 z}{z_0+\beta}$, $g^2(z) \ge h^2(z)$ for all $z \in [0; z_0]$. The nonlinear functions $g(z)$ and $h(z)$ for $c_1 = 50$, $c_2 = 5$ and other parameters of the above example are presented in Fig. 1. Via the substitute $z_1 = \sqrt{z}$, and respectively $\dot{z} = 2z_1\dot{z}_1$, equation containing the square root function is transformed into a linear equation: $\dot{z}_1 = -0.5\rho z_1 - 0.5\lambda$, which has a solution $z_1(t) = -\frac{\rho}{\lambda}(1 - e^{-0.5\lambda t}) + z_1(0)e^{-0.5\lambda t}$. By solving the equation $z_1(T^*) = 0$ find $T^*$:

$$T^* = \frac{2}{\lambda}\left(\ln\left(\lambda\sqrt{z_0} + \rho\right) - \ln\rho\right). \tag{28}$$

The considered first-order system with the square root nonlinearity (assuming also symmetric properties of the square root function for negative *z*) is known as having a *terminal sliding mode* (or *power-fractional sliding mode*) [7], [8], which has finite-time convergence. Since $h(z) \le g(z)$, $z \in [0; z_0]$, according to Lemma 1, the system (26) provides a faster convergence than the system with the square root nonlinearity. Time $T^*$ serves as a higher estimate of the convergence time in system (24).;

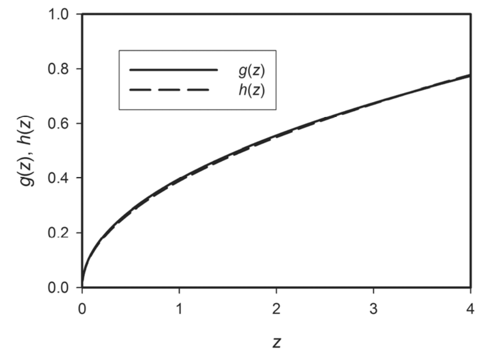


Fig. 1. Functions $g(z)$ and $h(z)$ of differential equation for amplitude

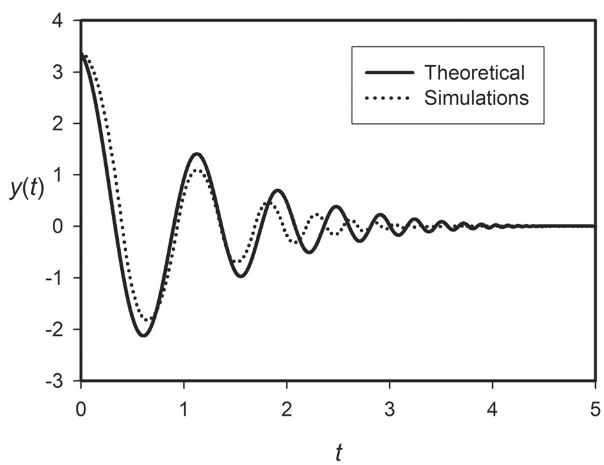


Fig. 2. Example of analysis of twisting SOSM controlled system

An example of analysis of the system with the linear plant $W_l(s) = 1/(s^2 + s + 1)$ and the twisting controller with $c_1 = 50$, $c_2 = 5$ is given in Fig. 2. The theoretical value of the higher estimate of the convergence time assessed per (28) is $T^* = 4.85$, which is close to the actual convergence time due to closeness of functions $g(z)$ and $h(z)$ (Fig. 1). The simulations show that the theoretical value of the decay is again a little smaller than the actual one. Yet, the proposed approach provides a good estimate of the SOSM transient dynamics.

## IV. FREQUENCY-DOMAIN CHARACTERISTICS AND CONVERGENCE RATE

Let us now consider the problems of the existence of periodic motions, asymptotic and finite-time convergence.

Periodic motions can exist in the system if the Nyquist plot of the linear part $W(j\omega)$ intersects the negative reciprocal of the DF $-N^{-1}(a)$ (Fig. 3). In Fig. 3, two Nyquist plots corresponding to the second- $W_1(j\omega)$ and third-order $W_2(j\omega)$ linear parts and two negative reciprocal DFs corresponding to the relay control $-N_1^{-1}(a)$ and to the twisting algorithm $-N_2^{-1}(a)$ [11] are depicted. Intersection of $W_2(j\omega)$ and either of the DFs provides a periodic solution (points A or B) of finite frequency. Plot $W_1(j\omega)$ does not have any points of intersection with either $-N_1^{-1}(a)$ or $-N_2^{-1}(a)$ except the origin. However, the character of the process in the system is different – depending on whether the control is a conventional ideal relay (plot $-N_1^{-1}(a)$) or the SOSM control (plot $-N_2^{-1}(a)$). In the former case the convergence is asymptotic, in the second one – it is finite-time.

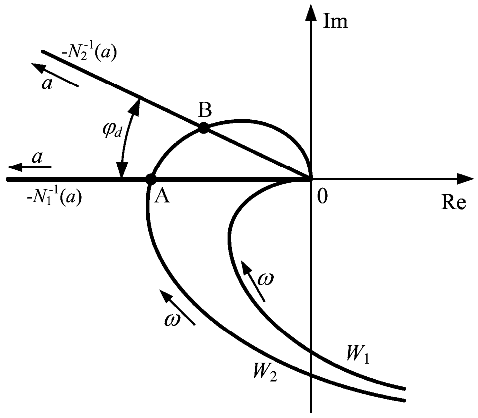


Fig. 3. Determination of periodic motions and decaying oscillations

Let us consider the condition of the phase balance that is a part of the harmonic balance condition. For a periodic motion to occur in the system the following must hold:

$$\varphi_l(\Omega) + \arg N(a) = -\pi\,, \qquad (29)$$

where $\Omega$ is the frequency and $a$ is the amplitude of the self-excited periodic motion, $\varphi_l(\omega) = \arg W_l(j\omega)$ is the phase characteristic of the linear part. Considering linear part $W_1(j\omega)$ we should note that there is a significant difference between the controls with $-N_1^{-1}(a)$ and $-N_2^{-1}(a)$. In the first case, formally speaking, there is a frequency at which the phase balance condition (29) holds. This frequency is $\Omega = \infty$. Therefore, we might say that in the system with $W_1(j\omega)$ and $-N_1^{-1}(a)$, a periodic motion of infinite frequency occurs. As for the second option, a periodic motion cannot occur at any frequency (including $\Omega = \infty$). There is a condition that we shall further refer to as the *phase deficit*. Quantitatively, let us call the *phase deficit* the minimum value of phase that needs to be added (with the negative sign) to the phase characteristic of the linear part to make the phase balance condition hold at some frequency (including the case of $\Omega = \infty$). Note: we do not consider now the case of non-monotone frequency characteristics. The *phase deficit* is depicted in Fig. 3 as $\varphi_d$. Therefore,

$$\varphi_l(\Omega) - \varphi_d + \arg N(a) = -\pi\,, \qquad (30)$$

assuming that $\varphi_d \ge 0$ and $\arg N(a) \ge 0$ for SOSM.

Now consider controllers that have a nonlinearity with infinite derivative in zero. For this type of nonlinearity, the DF $N(a) \to \infty$ if $a \to 0$ and, therefore, $-N^{-1}(a) \to 0$ if $a \to 0$. Also, assume that $-N^{-1}(a)$ is a straight line on the complex plane (other types of $-N^{-1}(a)$ will be considered below). Formulate the following theorem.

*Theorem 2*. For the second-order linear part given by (4) and the controller containing at least one ideal relay function, and having the controller describing function $N(a)$, so that the ratio $\operatorname{Im} N(a)/\operatorname{Re} N(a) = \text{const}$ (the negative reciprocal DF of the controller is a straight line on the complex plane), the following three modes of oscillations can occur (*necessary conditions*). *A. A periodic motion occurs if the phase deficit value is negative. B. An oscillation having asymptotic convergence of amplitude to zero (periodic process of infinite frequency and zero amplitude) occurs if the phase deficit value is zero. C. An oscillation having finite-time convergence of amplitude to zero occurs if the phase deficit value is positive.*

*Proof.* A. If the phase deficit is negative there always exists a point of intersection of the Nyquist plot of the linear part and of the negative reciprocal of the DF of the controller (follows from the definition of the phase deficit). Therefore, there is a solution of the harmonic balance equation [1], and a self-excited periodic motion occurs.

C. It follows from the definition of the nonlinearity of the controller that

$$N(a) = \frac{k_1}{r(a)} + j\frac{k_2}{r(a)}\,, \qquad (31)$$

where $k_1 > 0$, $k_2 > 0$ are constant coefficients, $r(a)$ is an increasing function of the amplitude $a$: $\frac{dr(a)}{da} > 0$ for all $a \in [0;\infty)$, such that $r(0) = 0$ (examples of this function can be $r(a) = a$, $r(a) = \sqrt{a}$, etc.). The negative reciprocal of (31) becomes

$$-N^{-1}(a) = -\frac{r(a)}{k_1^2 + k_2^2}(k_1 - jk_2)$$

Therefore, the *phase deficit* for this system is $\varphi_d = \arctan(k_2/k_1)$. Considering that $y(t) = a(t)\sin\Psi(t)$, and $\dot{y}(t) = a(t)[\sigma(t)\sin\Psi(t) + \Omega(t)\cos\Psi(t)]$ represents the response of the nonlinear controller to signal $y(t)$ as an expansion in the basis of functions $y(t)$, $\dot{y}(t)$ (weighted sum) the following holds:

$$u(t) \approx -(p_1 y(t) + \frac{p_2}{\Omega}\dot{y}(t)) = -a\left((p_1 + p_2\frac{\sigma}{\Omega})\sin\Psi + p_2\cos\Psi\right),$$

where the sign “-“ is attributed to the negative feedback, the "approximate equality" is due to the use of the approximate DF method. Weight $p_2$ is $p_2 = k_2 / r(a)$; weight $p_1$ can be determined for a particular controller. It becomes $p_1 = k_1 / r(a)$ when $\sigma = 0$. Therefore, the controller output can be represented as follows:

$$u(t) \approx -\left( p_1 + \frac{p_2}{\Omega} s \right) y(t),$$

where $s = \frac{d}{dt}$. Having found $\dot{y}(t)$ similarly to (16), we can write the following formula for the instantaneous decay:

$$\sigma = -0.5\left( a_2 + \frac{p_2(a)}{\Omega} b_2 \right) = -0.5\left( a_2 + \frac{k_2}{r(a)\Omega} b_2 \right) \quad (32)$$

and instantaneous frequency (similar to (20), (21)):

$$\Omega = 0.5\sqrt{4(a_1 + p_1(a)b_2) - \left( a_2 + \frac{k_2}{r(a)\Omega} b_2 \right)^2}.$$

As an auxiliary result, find the following limit from the last formula:

$$\lim_{a \to 0} r(a)\Omega = \lim_{a \to 0} 0.5 r(a)\sqrt{4(a_1 + p_1(a)b_2) - \left( a_2 + \frac{k_2}{r(a)\Omega} b_2 \right)^2} = 0,$$

considering that $\Omega \to \infty$ when $a \to 0$ and $r(0) = 0$. Therefore, considering the equation for the amplitude

$$\dot{a} = a\sigma = -\frac{k_2}{2r(a)\Omega} b_2 a - \frac{a_2}{2} a, \quad (33)$$

one can see that the nonlinearity that it has is the one with $g(0) = 0$ and infinite derivative at $a = 0$:

$g(a) = \frac{k_2 b_2 a}{2r(a)\Omega}$, $\lim_{a \to 0} g(a) = 0$ (follows from $\frac{dr(a)}{da} > 0$),

$$g'(a) = \frac{k_2 b_2}{2}\left( \frac{1}{r(a)\Omega} - \frac{ar'(a)}{r^2(a)\Omega} - \frac{a\, d\Omega/da}{r(a)\Omega^2} \right),$$

$\lim_{a \to 0} g'(a) = \infty$ (due to the first term in the brackets; considering also boundedness of the second term, and $d\Omega/da < 0$). These dynamics have a terminal sliding mode and finite convergence time as shown above and in [8].

B. For this option, coefficient $k_2$ in (31) is zero. As follows from (33) $\dot{a} = -0.5 a_2 a$, thus, providing exponential (asymptotic) convergence. ;

The relationship between the DF of the controller and the possibility of a particular mode of the transient process to occur was established above for the controllers that satisfy the condition $\operatorname{Im} N(a) / \operatorname{Re} N(a) = \text{const}$. Among known controllers, this applies to the twisting controller [2], and the sub-optimal algorithm [4], [5]. Yet, the fact of finite-time convergence depends on the configuration of $-N^{-1}(a)$ only in the vicinity of the origin (in the complex plane), so that if the convergence process starts from a certain amplitude, only the amplitudes in the range from the initial one to zero will be realized. Therefore, what is important is the location of the low amplitude asymptote of the plot $-N^{-1}(a)$. For that reason, let us reformulate the definition of the *phase deficit* as the angle between the high-frequency asymptote of the Nyquist plot of the linear part and the low-amplitude asymptote of the negative reciprocal DF of the controller (considering also the sign of this value). Simulations prove that the finite-time convergence occurs only if the *phase deficit* is positive.

## V. Conclusion

An extended harmonic balance and a frequency-domain method of analysis of transient oscillatory processes have been developed. The proposed method is applied to analysis of the convergence rate of SOSM controlled systems. It leads to a simple criterion of the existence of a finite-time or asymptotic conversion, which involves just one characteristic – the *phase deficit, which must be positive for the finite-time convergence to occur*.

## References


[1] D.P. Atherton, *Nonlinear Control Engineering - Describing Function Analysis and Design*. Van Nostrand Company Limited: Workingham, Berks, UK, 1975.

[2] A. Levant (Levantovsky, L.V.). "Sliding order and sliding accuracy in sliding mode control", *International Journal of Control*, **58**, No. 6, pp.1247-1263, 1993.

[3] A. Levan, "Universal SISO sliding-mode controllers with finite-time convergence", *IEEE transactions on Automatic Control*, **46**, No. 9, pp. 1447-1451, 2001.

[4] G. Bartolini, A. Ferrara, E. Usai, "Chattering Avoidance by Second-Order Sliding Mode Control", *IEEE Transactions on Automatic Control,* Vol.43, No.2, pp. 241-246, 1998.

[5] G. Bartolini, A. Ferrara, A. Levant, E. Usai, "On Second Order Sliding Mode Controllers", In *Variable Structure Systems, Sliding Mode and Nonlinear* (*Lecture Notes in Control and Information Science, 247*) (K.D. Young and U. Ozguner (eds.)), pp. 329-350. Springer-Verlag, London, 1999.

[6] Y.B. Shtessel, I.A. Shkolnikov, M.J. Brown, "An asymptotic second-order smooth sliding mode control", *Asian Journal of Control*, **5**, No. 4, pp. 498-504, 2003.

[7] Z. Man, A.P. Poplinsky, H.R. Wu, "A robust MIMO terminal sliding mode control for rigid robotic manipulators", *IEEE Trans. Automat. Contr.*, **39**, No. 12, pp. 2464-2468, 1994.

[8] X. Yu, Y. Wu, Z. Man, "On global stabilization of nonlinear dynamical systems", In: *Variable Structure Systems, Sliding Mode and Nonlinear Control* (K.D. Toung, U. Ozguner (Eds.)), p. 109-122. Springer, London, 1999.

[9] N.M. Krylov, N.N. Bogoliubov, *Introduction to Nonlinear Mechanics*, Princeton University Press, Princeton, 1947.

[10] D.V. Anosov, "On stability of equilibrium points of relay systems", *Automatica i telemechanica (Automation and Remote Control)*, **20**, no. 2, pp. 135-149, 1959.

[11] I. Boiko, M., L. Fridman, M.I. Castellanos, "Analysis of second-order sliding mode algorithms in the frequency domain", *IEEE Trans. Automat. Contr.*, **49**, No. 6, pp. 946-950, 2004.