

# On Constrained Steady-state Optimal Control: Dynamic KKT Controllers

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**Abstract**—This paper presents a solution to the problem of controlling a general linear time-invariant dynamical system (plant) to a time-varying economically optimal operating point. The plant is characterized by a set of exogenous inputs as an abstraction of time-varying loads and disturbances. The economically optimal operating point is implicitly defined as a solution to a given constrained convex optimization problem, which is related to steady-state operation of the plant. A subset of the plant's states and the exogenous inputs represent respectively the decision variables and the parameters in the optimization problem. The proposed control structure, which is proven to solve the considered control problem, is explicitly defined and is based on the dynamic extension of the Karush-Kuhn-Tucker (KKT) optimality conditions for the steady-state related optimization problem.

**Index Terms**—Real-time optimization, complementarity systems, constrained control.

## I. INTRODUCTION

In many production facilities, the optimization problem reflecting economical benefits of production is associated with a *steady-state operation* of the system. The control action is then required to maintain the production in an optimal regime in spite of various disturbances, and to efficiently and rapidly respond to changes in demand. Furthermore, it is desirable that the system settles in a steady-state that is optimal for novel operating conditions. The vast majority of control literature is focused on regulation and tracking with respect to known setpoints or trajectories, while coping with different types of uncertainties and disturbances in both plant and its environment. Typically, setpoints are determined off-line by solving an appropriate optimization problem, and they are updated in an open-loop manner. The optimization problem usually reflects variable costs of production and economical benefits under the current market conditions, e.g. fuel or electricity prices, and accounts for physical and security limits of the plant.

If a production system is required to follow a time-varying demand in real-time, e.g. if produced commodities cannot be efficiently stored in large amounts, it becomes crucial to perform economic optimization on-line. A typical example of such a system are electrical power systems. Increase of the frequency with which the economically optimal setpoints are updated can result in a significant increase of economic benefits accumulated in time. If the time-scale on which economic optimization is performed approaches the time-scale

of the underlying physical system, i.e. of the plant dynamics, dynamic interaction in between the two has to be considered. Economic optimization then becomes a challenging control problem, even more since it has to cope with inequality constraints that reflect the physical and security limits of the plant.

In this paper we present a novel feedback control design procedure as a solution to the problem of regulating a general linear time-invariant dynamical system to a time-varying economically optimal operating point. The considered dynamical system is characterized with a set of exogenous inputs as an abstraction of time-varying loads and disturbances acting on the system. Economic optimality is defined through a convex constrained optimization problem with a set of system states as decision variables, and with the values of exogenous inputs as parameters in the optimization problem.

**Nomenclature.** For a matrix  $A \in \mathbb{R}^{m \times n}$ ,  $[A]_{ij}$  denotes the element in the  $i$ -th row and  $j$ -th column of  $A$ . For a vector  $x \in \mathbb{R}^n$ ,  $[x]_i$  denotes the  $i$ -th element of  $x$ . A vector  $x \in \mathbb{R}^n$  is said to be nonnegative (nonpositive) if  $[x]_i \geq 0$  ( $[x]_i \leq 0$ ) for all  $i \in \{1, \dots, n\}$ , and in that case we write  $x \geq 0$  ( $x \leq 0$ ). The nonnegative orthant of  $\mathbb{R}^n$  is defined by  $\mathbb{R}_+^n := \{x \in \mathbb{R}^n \mid x \geq 0\}$ . The operator  $\text{col}(\cdot, \dots, \cdot)$  stacks its operands into a column vector. For  $u, v \in \mathbb{R}^k$  we write  $u \perp v$  if  $u^\top v = 0$ . We use the compact notational form  $0 \leq u \perp v \geq 0$  to denote the complementarity conditions  $u \geq 0, v \geq 0, u \perp v$ . For a scalar-valued differentiable function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\nabla f(x)$  denotes its gradient at  $x = \text{col}(x_1, \dots, x_n)$  and is defined as a *column vector*, i.e.  $\nabla f(x) \in \mathbb{R}^n$ ,  $[\nabla f(x)]_i = \frac{\partial f}{\partial x_i}$ . For a vector-valued differentiable function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $f(x) = \text{col}(f_1(x), \dots, f_m(x))$ , the Jacobian at  $x = \text{col}(x_1, \dots, x_n)$  is the matrix  $Df(x) \in \mathbb{R}^{m \times n}$  and is defined by  $[Df(x)]_{ij} = \frac{\partial f_i(x)}{\partial x_j}$ . For a vector valued function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , we will use  $\nabla f(x)$  to denote the transpose of the Jacobian, i.e.  $\nabla f(x) \in \mathbb{R}^{n \times m}$ ,  $\nabla f(x) \triangleq Df(x)^\top$ , what is consistent with the gradient notation  $\nabla f$  when  $f$  is a scalar-valued function. *With a slight abuse of notation we will often use the same symbol to denote a signal, i.e. a function of time, as well as possible values that the signal may take at any time instant.*

## II. PROBLEM FORMULATION

In this section we formally present the constrained steady-state optimal control problem considered in this paper. Furthermore, we list several standing assumptions, which will be instrumental in the subsequent sections.

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Consider an LTI system  $\Sigma$  described by a state-space realization

$$\begin{aligned} \dot{x} = \begin{pmatrix} \dot{x}_p \\ \dot{x}_q \end{pmatrix} &= \begin{pmatrix} A_{pp} & A_{pq} \\ A_{qp} & A_{qq} \end{pmatrix} \begin{pmatrix} x_p \\ x_q \end{pmatrix} + \begin{pmatrix} F_p \\ F_q \end{pmatrix} w + \begin{pmatrix} B_p \\ B_q \end{pmatrix} u, \\ &\triangleq Ax + Fw + Bu, \end{aligned} \quad (1a)$$

$$y = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} x_p \\ x_q \end{pmatrix} \triangleq Cx, \quad (1b)$$

where  $x(t) \in \mathbb{R}^n$  is the state variable,  $u(t) \in \mathbb{R}^m$  is the control input,  $w(t) \in \mathbb{R}^{n_w}$  is an exogenous input and  $y(t) \in \mathbb{R}^m$  is the measured output. The state  $x$  is partitioned into  $x_p \in \mathbb{R}^m$  and  $x_q \in \mathbb{R}^{n-m}$ , inducing the corresponding partitioning of the matrices  $A \in \mathbb{R}^{n \times n}$ ,  $F \in \mathbb{R}^{n \times n_w}$ ,  $B \in \mathbb{R}^{n \times m}$  as indicated in (1a). With  $W \subset \mathbb{R}^{n_w}$  denoting a known bounded set and for a constant  $w \in W$ , consider the following convex optimization problem associated with the partial state vector  $x_p$  of the dynamical system (1):

$$\min_{x_p} J(x_p) \quad (2a)$$

$$\text{subject to} \quad Lx_p = h(w), \quad (2b)$$

$$q_i(x_p) \leq r_i(w), \quad i = 1, \dots, k, \quad (2c)$$

where  $J: \mathbb{R}^m \rightarrow \mathbb{R}$  is a strictly convex and continuously differentiable function,  $L \in \mathbb{R}^{l \times m}$  is a constant matrix,  $h: \mathbb{R}^{n_w} \rightarrow \mathbb{R}^l$  and  $r_i: \mathbb{R}^{n_w} \rightarrow \mathbb{R}$ ,  $i = 1, \dots, k$  are continuous functions, while  $q_i: \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $i = 1, \dots, k$  are convex, continuously differentiable functions. For the matrix  $L$  we require  $\text{rank} L = l < m$ . For a constant exogenous signal  $w(t) = w \in W$ , the optimization problem (2) reflects the corresponding optimal steady-state operating point for the state  $x_p$  in (1). The inequality constraints (2c) represent the security-type “soft” constraints for which some degree of transient violation may be accepted, but whose feasibility is required for steady-state operation. The state vector  $x_p$  collects only the states which appear explicitly in (2). Note that in general not all of the elements of  $x_p$  that appear in the constraints (2b), (2c) need to appear in the objective function  $J$ , and *vice versa*. The objective of the control input  $u$  is to drive the state  $x_p$  to the optimal steady-state operating point given by (2). We continue by listing several assumptions concerning the dynamics (1) and the optimization problem (2).

Let  $\mathcal{I}_l$  denote the set of indices  $i$  for which the function  $q_i$  in (2c) is a linear function, and let  $\mathcal{I}_n$  denote the set of indices for which  $q_i$  is a nonlinear function. We make the following assumption:

**Assumption II.1** For each  $w \in W$  the set

$$\begin{aligned} \{x_p \mid Lx_p = h(w), \quad q_i(x_p) < r_i(w) \text{ for } i \in \mathcal{I}_n, \\ q_i(x_p) \leq r_i(w) \text{ for } i \in \mathcal{I}_l\} \end{aligned}$$

is nonempty.  $\square$

Assumption II.1 states that the convex optimization problem (2) satisfies Slater’s constraint qualification [1] for each  $w \in W$ , implying that strong duality holds for the considered problem. Note also that due to strict convexity of the

objective function in (2), the optimization problem has a unique minimizer  $\tilde{x}_p(w)$  for each  $w \in W$ .

**Assumption II.2** For each  $w \in W$ , in the optimization problem (2) the minimum is attained.  $\square$

**Assumption II.3** The matrix  $A$  is Hurwitz. The sub-matrix<sup>1</sup> of  $A^{-1}B$  formed by taking the first  $m$  rows of  $A^{-1}B$  has full rank.  $\square$

Assumption II.3 guarantees that for all constant  $w(t) = w \in W$ , the partial state vector  $x_p$  can be driven to an arbitrary steady-state point, which is then characterized by a unique, constant value of the input signal  $u$ . In other words, Assumption II.3 implies that the steady-state relations from (1) do not pose any additional constraints to the optimization problem (2). Although seemingly restrictive, this assumption is in practice almost always fulfilled since  $x_p$  represents the states which directly appear in the economical objective of the plant, e.g. the optimization problem (2). Well designed systems allow complete steady-state control (in the sense of the above stated assumption) of these “economically relevant” states. However, it is also possible to relax this assumption. Then, any constraint on the steady-state imposed by (1) should be included in (2b). The assumption that  $A$  is Hurwitz is also reasonable. Thinking of  $u$  as a setpoint signal for steady-state operation, (1) represents the plant which already includes a stabilizing controller.

**Assumption II.4** In (1) the output matrix  $C$  in (1b) is given by  $C = \begin{pmatrix} I & 0 \end{pmatrix}$ , i.e. the state vector  $x_p$  can be measured. Furthermore, all components of  $w$  that appear in (2) are known at all time instants.  $\square$

Assumption II.4 implies that violations of the constraints (2b) and (2c) are available for control. In practice, and in contrast to the above assumption, violations of the constraints are often directly measurable, and not only indirectly through  $x_p$  and  $w$ . To illustrate this, consider an example of an electrical power system. Demand for electrical power, which corresponds to the exogenous signal  $w$ , is never explicitly known. However, the network frequency serves as a direct measure of production-demand imbalance, i.e. as a direct measure of the violation of an equality constraint in (2b). Assumption II.4 is made only for the purpose of simplifying the presentation in the next section and in Remark III.2 this assumption is relaxed.

With the definitions and assumptions made so far, we are now ready to formally state the control problem considered in this paper.

**Problem II.5** *Constrained steady-state optimal control.*

For a dynamical system  $\Sigma$  given by (1), design a feedback controller that has  $y$  as input signal and  $u$  as output signal, such that the following objective is met for any constant-valued exogenous signal  $w(t) = w \in W$ : the state of the

<sup>1</sup>Note that this is a square matrix.

closed-loop system globally converges to an equilibrium point with  $x_p = \tilde{x}_p(w)$ , where  $\tilde{x}_p(w)$  denotes the minimizer of the optimization problem (2) for some  $w \in W$ .  $\square$

### III. DYNAMIC KKT CONTROLLERS

In this section we present two controllers that guarantee the existence of an equilibrium point with  $x_p = \tilde{x}_p(w)$  as described in Problem II.5.

Assumption II.1 implies that for each  $w \in W$ , the first order Karush-Kuhn-Tucker (KKT) conditions are *necessary and sufficient* conditions for optimality. For the optimization problem (2) these conditions are given by the following set of equalities and inequalities:

$$\nabla J(x_p) + L^\top \lambda + \nabla q(x_p) \mu = 0, \quad (3a)$$

$$Lx_p - h(w) = 0, \quad (3b)$$

$$0 \leq -q(x_p) + r(w) \perp \mu \geq 0, \quad (3c)$$

with the abbreviations  $q(x_p) = \text{col}(q_1(x_p), \dots, q_k(x_p))$ ,  $r(w) = \text{col}(r_1(w), \dots, r_k(w))$  and  $\lambda \in \mathbb{R}^l$ ,  $\mu \in \mathbb{R}^k$  denoting Lagrange multipliers. Since the above conditions are necessary and sufficient conditions for optimality, it is apparent that the existence of an equilibrium point with  $x_p = \tilde{x}_p(w)$  is implied if for each  $w \in W$  the controller guarantees the existence of the vectors  $\lambda$  and  $\mu$ , such that the conditions (3) are fulfilled in a steady-state of the closed-loop system.

**Max-based KKT controller.** Let  $K_\lambda \in \mathbb{R}^{l \times l}$ ,  $K_\mu \in \mathbb{R}^{k \times k}$ ,  $K_c \in \mathbb{R}^{m \times m}$  and  $K_o \in \mathbb{R}^{k \times k}$  be diagonal matrices with non-zero elements on the main diagonal and  $K_o \succ 0$ . Consider a dynamic controller with the following structure:

$$\dot{x}_\lambda = K_\lambda (Lx_p - h(w)), \quad (4a)$$

$$\dot{x}_\mu = K_\mu (q(x_p) - r(w) + v), \quad (4b)$$

$$\dot{x}_c = K_c (L^\top x_\lambda + \nabla q(x_p) x_\mu + \nabla J(x_p)), \quad (4c)$$

$$0 \leq v \perp K_o x_\mu + q(x_p) - r(w) + v \geq 0, \quad (4d)$$

$$u = x_c, \quad (4e)$$

where  $x_\lambda$ ,  $x_\mu$  and  $x_c$  denote the controller states and the matrices  $K_\lambda$ ,  $K_\mu$ ,  $K_c$  and  $K_o$  represent the controller gains. Note that the input vector  $v(t) \in \mathbb{R}^k$  in (4b) is at any time instant required to be a solution to a finite-dimensional linear complementarity problem (4d).  $\square$

**Saturation-based KKT controller.** Let  $K_\lambda \in \mathbb{R}^{l \times l}$ ,  $K_\mu \in \mathbb{R}^{k \times k}$  and  $K_c \in \mathbb{R}^{m \times m}$  be diagonal matrices with non-zero elements on the main diagonal and  $K_\mu \succ 0$ . Consider a dynamic controller with the following structure:

$$\dot{x}_\lambda = K_\lambda (Lx_p - h(w)), \quad (5a)$$

$$\dot{x}_\mu = K_\mu (q(x_p) - r(w) + v), \quad (5b)$$

$$\dot{x}_c = K_c (L^\top x_\lambda + \nabla q(x_p) x_\mu + \nabla J(x_p)), \quad (5c)$$

$$0 \leq v \perp x_\mu \geq 0, \quad (5d)$$

$$u = x_c, \quad (5e)$$

$$x_\mu(0) \geq 0, \quad (5f)$$

where  $x_\lambda$ ,  $x_\mu$  and  $x_c$  denote the controller states and the matrices  $K_\lambda$ ,  $K_\mu$  and  $K_c$  represent the controller gains. Note that the input vector  $v(t) \in \mathbb{R}^k$  in (5b) is at any time

instant required to be a solution to a finite-dimensional linear complementarity problem (5d).  $\square$

The choice of names *max-based KKT controller* and *saturation-based KKT controller* will become clear later in this section. Notice that both controllers belong to the class of complementarity systems [2], [3].

**Theorem III.1** *Let  $w(t) = w \in W$  be a constant-valued signal, and suppose that Assumption II.1 and Assumption II.3 hold. Then the closed-loop system, i.e. the system obtained from the system (1) connected with controller (4) or (5) in a feedback loop, has an equilibrium point with  $x_p = \tilde{x}_p(w)$ , where  $\tilde{x}_p(w)$  denotes the minimizer of the optimization problem (2) for some  $w \in W$ .  $\square$*

*Proof.* We first consider the closed-loop system with the max-based KKT controller, i.e. controller (4). By setting the time derivatives of the closed-loop system states to zero and by exploiting the non-singularity of the matrices  $K_\lambda$ ,  $K_\mu$  and  $K_c$ , we obtain the following complementarity problem:

$$0 = A \begin{pmatrix} x_p \\ x_q \end{pmatrix} + Bx_c + Fw, \quad (6a)$$

$$0 = Lx_p - h(w), \quad (6b)$$

$$0 = q(x_p) - r(w) + v, \quad (6c)$$

$$0 = L^\top x_\lambda + \nabla q(x_p) x_\mu + \nabla J(x_p), \quad (6d)$$

$$0 \leq v \perp K_o x_\mu + q(x_p) - r(w) + v \geq 0, \quad (6e)$$

with the closed-loop system state vector  $x_{cl} := \text{col}(x_p, x_q, x_\lambda, x_\mu, x_c)$  and the vector  $v$  as variables. Any solution  $x_{cl}$  to (6) is an equilibrium point of the closed-loop system. By substituting  $v = -q(x_p) + r(w)$  from (6c) and utilizing  $K_o \succ 0$ , the complementarity condition (6e) reads as  $0 \leq -q(x_p) + r(w) \perp x_\mu \geq 0$ . With  $\lambda := x_\lambda$  and  $\mu := x_\mu$ , the conditions (6b),(6c),(6d),(6e) therefore correspond to the KKT conditions (3) and, under Assumption II.1, they necessarily have a solution. Furthermore, for any solution  $(x_p, x_\lambda, x_\mu, v)$  to (6b),(6c),(6d),(6e), it necessarily holds that  $x_p = \tilde{x}_p(w)$ . It remains to show that (6a) admits a solution in  $(x_q, x_c)$  for  $x_p = \tilde{x}_p(w)$ . This, however, readily follows from Assumption II.3. Moreover, Assumption II.3 implies uniqueness of  $x_q$  and  $x_c$  in an equilibrium. Now, consider the closed-loop system with the saturation-based KKT controller, i.e. controller (5). The difference in this case comes only through (5b) and (5d). It is therefore sufficient to show that (5b) and (5d) imply  $0 \leq -q(x_p) + r(w) \perp x_\mu \geq 0$  in a steady-state. This implication is obvious since  $K_\mu \succ 0$ .  $\square$

**Remark III.2** This remark concerns Assumption II.4. Suppose that the output  $y$  of the system (1), instead of being given by (1b), has the form  $y = \text{col}(\alpha(x, w), \beta(x, w), x_p)$ , where  $\alpha : \mathbb{R}^n \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}^l$  and  $\beta : \mathbb{R}^n \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}^k$  are continuous functions, characterized by the following *steady-state-related* properties:

$$[\alpha(x, w)]_i = 0 \Leftrightarrow [Lx_p - h(w)]_i = 0, \quad i = 1, \dots, l, \quad (7a)$$

$$[\beta(x, w)]_j \leq 0 \Leftrightarrow [q(x_p) - r(w)]_j \leq 0, \quad j = 1, \dots, k. \quad (7b)$$

Here, by *steady-state-related* we mean that the above properties hold when  $w \in W$  is a given constant signal and the system is in a steady-state. The values of  $\alpha$  and  $\beta$  therefore carry the information of the violation of the constraints, and as such they can be directly used for control purposes. From (7) it follows that by replacing  $Lx_p - h(w)$  by  $\alpha(x, w)$  and  $q(x_p) - r(w)$  by  $\beta(x, w)$  in (4a),(4b),(4d),(5a), and (5b), the statement of Theorem III.1 still holds.  $\square$

### A. Complementarity integrators

The main distinguishing feature between the max-based KKT controller (4) and the saturation-based KKT controller (5) is in the way the steady-state complementarity slackness condition (3c) is enforced. Although characterized by the same steady-state relations, the two controllers, and therefore the corresponding closed-loop systems, have some significantly different *dynamical* features which will be discussed further in this section. In the following two paragraphs our attention is on the equations (4b),(4d) and (5b),(5d), and the goal is to show the following:

- The *max-based KKT controller*, i.e. controller (4), can be represented as a dynamical system in which certain variables are coupled by means of static, *continuous*, piecewise linear characteristics;
- The *saturation-based KKT controller*, i.e. controller (5), can be represented as a dynamical system with state saturations.

**Max-based complementarity integrator.** Let  $a, b$  and  $c$  be real scalars related through the complementarity condition  $0 \leq c \perp a + b + c \geq 0$ . It is easily verified, e.g. by checking all possible combinations, that this complementarity condition can equivalently be written as  $b + c = \max(a + b, 0) - a$ . With  $\max(\cdot, \cdot)$  defined for vectors as an elementwise maximum, i.e. for  $v, w \in \mathbb{R}^n$ ,  $(z = \max(v, w)) \Leftrightarrow ([z]_i = \max([v]_i, [w]_i), i = 1, \dots, n)$ , the above equivalence holds as well for the case when  $a, b, c$  are vectors of the same dimension. Now, by taking  $c = v$ ,  $a = K_o x_\mu$  and  $b = q(x_p) - r(w)$ , it follows that (4b) and (4d) can be equivalently described by

$$\dot{x}_\mu = K_\mu (\max(K_o x_\mu + q(x_p) - r(w), 0) - K_o x_\mu). \quad (8)$$

With  $\beta := q(x_p) - r(w)$ , Figure 1 presents a block diagram representation of the  $i$ -th row in (8). The block labeled “Max” in the figure, represents a scalar max relation as a static piecewise linear characteristics.

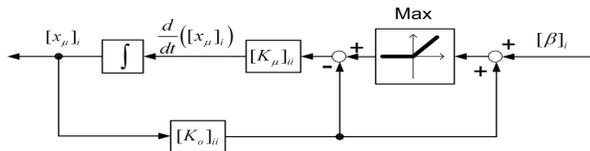


Fig. 1. Max-based complementarity integrator.

With  $[K_o]_{ii} > 0$ , it is easy to verify that if the system in Figure 1 is in steady-state, then the value of its input signal  $[\beta]_i$  and the value of its output signal  $[x_\mu]_i$  necessarily satisfy the complementarity condition  $[x_\mu]_i \geq 0$ ,  $[\beta]_i \leq 0$ ,  $([x_\mu]_i [\beta]_i) = 0$ .  $\square$

**Saturation-based complementarity integrator.** The differential algebraic equations (5b), (5d) restrict the state vector  $x_\mu$  to the nonnegative orthant  $\mathbb{R}_+^k$ . For  $x_\mu > 0$ , i.e. when

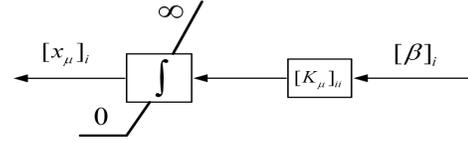


Fig. 2. Saturation-based complementarity integrator.

$x_\mu$  is in the interior of  $\mathbb{R}_+^k$ , its dynamics is described by the equation  $\dot{x}_\mu = K_\mu \beta(x, w)$ , where  $\beta(x, w) := q(x_p) - r(w)$ . However, on the boundary this dynamics is modified to prevent the solution from leaving  $\mathbb{R}_+^k$ . Precisely, the dynamics of the  $i$ -th element in  $x_\mu$  is given by

$$[\dot{x}_\mu]_i = \begin{cases} 0 & \text{if } [x_\mu]_i = 0 \text{ and } [K_\mu]_{ii}[\beta]_i < 0, \\ [K_\mu]_{ii}[\beta]_i & \text{otherwise.} \end{cases} \quad (9)$$

Figure 2 presents a block diagram representation of (9), which is a saturated integrator with the lower saturation point equal to zero. The equivalence of the dynamics (5b), (5d) and the saturated integrators defined by (9) directly follows from the equivalence of *gradient-type complementarity systems* (GTCS) ((5b), (5d) belong to the GTCS class) and *projected dynamical systems* (PDS) ((9) belongs to the PDS class). For the precise definitions of GTCS and PDS system classes and for the equivalence results see [4] and [5].

With  $[K_\mu]_{ii} > 0$ , it is easy to verify that in steady-state the value of the input signal  $[\beta]_i$  and the value of the output signal  $[x_\mu]_i$  necessarily satisfy the complementarity condition  $[x_\mu]_i \geq 0$ ,  $[\beta]_i \leq 0$ ,  $([x_\mu]_i [\beta]_i) = 0$ .  $\square$

The above presented complementarity integrators provide the basic building blocks for imposing steady-state complementarity conditions. We will use the term *max-based complementarity integrator* to refer to a system with the structure as depicted in Figure 1, and the term *saturation-based complementarity integrator* for the system in Figure 2. Together with a pure integrator, complementarity integrators form the basic building block of a KKT controller.

### B. Well-posedness and stability of the closed-loop system

In this subsection we shortly address some results concerning the well-posedness and stability analysis problems of the closed-loop system, i.e. of the system (1) interconnected with a dynamic KKT controller in a feedback loop. We refer to [6] for a more detailed treatment of these topics.

**Well-posedness.** Since the function  $\max(\cdot, 0)$  is globally Lipschitz continuous, for checking well-posedness of the closed-loop system with max-based KKT controller one can resort on standard Lipschitz continuity conditions. The closed-loop system with saturation-based KKT controller belongs to a specific class of gradient-type complementarity systems for which sufficient conditions for well-posedness have been presented in [4] and [5].

**Stability analysis.** Theorem III.1 states that for any constant-valued exogenous signal  $w(t) \in W$ , the closed-loop

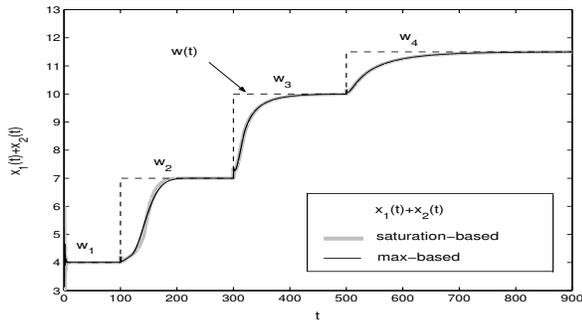


Fig. 3. The values of  $w$  and  $x_1 + x_2$ , i.e. the right hand and the left hand side of the equality constraint (11b), as a function of time.

system necessarily has an equilibrium. Furthermore, from the proof of this theorem it follows that for *all* corresponding equilibrium points the values of the state vectors  $(x_p, x_q, x_c)$  are *unique*. For a given  $w(t) = w \in W$ , the necessary and sufficient condition for uniqueness of the remaining closed-loop system state vectors  $(x_\lambda, x_\mu)$ , and therefore necessary and sufficient condition for uniqueness of the closed-loop system equilibrium, corresponds to the condition for uniqueness of the Lagrange multipliers in (3). This condition is known as *strict Mangasarian-Fromovitz constraint qualification* (SMFCQ) and is presented in [7]. Since both types of complementarity integrators can be presented in a piecewise affine framework [8], for a given  $w(t) = w \in W$  characterized by unique equilibrium, one can perform global asymptotic stability analysis based on: *i*) the analysis procedures from [9]–[11] in case when (2) is a quadratic program; *ii*) the analysis procedure from [12] in case when (2) is given with a (higher order) polynomial objective function and (higher order) polynomial inequality constraints. In the case when  $w(t) = w \in W$  is such that the SMFCQ does not hold, the closed-loop system is characterized by a set of equilibria (not a singleton), which is then an invariant set for the closed-loop system. Each equilibrium in this set is characterized by different values of the state vectors  $(x_\lambda, x_\mu)$ , but unique values of the remaining states. Under additional generalized Slater constraint qualification, see [13] for details, the set of equilibria is guaranteed to be bounded. For stability analysis with respect to this set, one has to invoke LaSalle's invariance theorem, see [14] for a general introduction and [15]–[17], including the references therein, for generalizations of the invariance theorem to hybrid systems. Finally, to perform stability analysis for all possible constant values of the exogenous signal  $w(t)$ , i.e. for all  $w(t) = w$  where  $w$  is any constant in  $W$ , it is possible to reformulate this problem into suitably defined robust stability analysis problem of a LTI system affected by structured uncertainties, see [6] for details. In [6] several remedies for dealing with non-unique equilibria were proposed.

#### IV. EXAMPLE

As already reported in [18] and [19], the dynamic KKT controllers (with certain application oriented modifications)

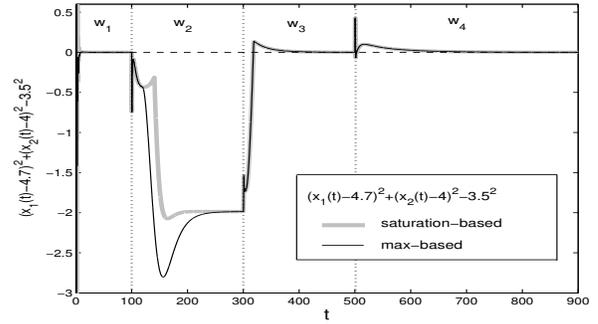


Fig. 4. Violation of the inequality constraint (11c) as a function of time. When the curves are above zero (horizontal dashed line), the constraint is violated.

have a large potential for application in real-time, *price-based* power balance and network congestion control of electrical energy transmission systems, what is considered to be one of the toughest problems in operation and control of restructured, market-based power systems [20]. Specifically, the KKT control structure is suitable for this particular application since it explicitly manipulates with the Lagrange multipliers, which, in electrical power systems, have an interpretation of nodal prices for electricity. Due to space limitations, in this section we will illustrate the effectiveness of the KKT controllers on an academic example.

Consider a third-order system of the form (1) with (1a) given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -2.5 & 0 & -5 \\ 0 & -5 & -15 \\ 0.1 & 0.1 & -0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -0.1 \end{pmatrix} w + \begin{pmatrix} 2.5 & 0 \\ 0 & 5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad (10)$$

where  $x_p = \text{col}(x_1, x_2)$ ,  $x_q = x_3$  and  $u = \text{col}(u_1, u_2)$ . The associated steady-state related optimization problem is defined as follows:

$$\min_{x_p} \frac{1}{2} x_p^\top H x_p + a^\top x_p \quad (11a)$$

$$\text{subject to } x_1 + x_2 = w, \quad (11b)$$

$$(x_1 - 4.7)^2 + (x_2 - 4)^2 \leq 3.5^2, \quad (11c)$$

where  $H = \text{diag}(6, 2)$ ,  $a = \text{col}(-4, -4)$ , and the value of the exogenous signal  $w$  is limited to be in the interval  $W = [4, 11.5]$ . It can be verified that for this  $W$  and the constraints (11b) and (11c), the Assumption II.1 holds true. Furthermore, it can easily be verified that the condition in the Assumption II.3 is fulfilled. We assume that the complete state vector is available for control, i.e. that the output equation (1b) is given by  $y = x$ . From the dynamics of the state  $x_3$ , it follows that in steady-state the equality  $x_1 + x_2 - 2x_3 = w$  holds. Therefore, in steady-state,  $x_3 = 0$  implies fulfilment of the constraint (11b). This implies that for control we can directly use the value of the state  $x_3$  as a measure for violation of this constraint (see Remark III.2). Simulations of the closed-loop system response to the stepwise exogenous input  $w(t)$ , which is presented in Figure 3, have been performed.

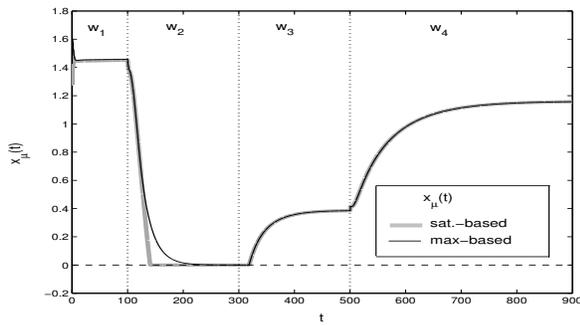


Fig. 5. Simulated trajectory of the controller state  $x_\mu$ .

Figures 3-6 present both the results of the simulation when the system is controlled with a saturation-based and when it is controlled with a max-based KKT controller. Both controllers were implemented with the gains  $K_\lambda = 0.15$ ,  $K_\mu = 0.1$ ,  $K_c = -0.7I_2$ , and the gain  $K_o$  in the max-based controller was set to 0.5. Figure 3 and Figure 4 clearly illustrate that the controllers continuously drive the closed-loop system towards the steady-state where the constraints (11b), (11c) are satisfied. Figures 4 and 5 illustrate fulfilment of the complementarity slackness condition (3c) in steady-state. Finally, Figure 6 illustrates that the controllers drive the system towards correspondent optimal operating point as defined by (1). In this figure the straight dashed lines labeled  $w_i$ ,  $i = 1, \dots, 4$ , represent the equality constraint  $x_1 + x_2 = w_i$  where the values of  $w_i$  are the ones given in Figure 3. The dashed circle represents the inequality constraint (11c), i.e. the steady-state feasible region for  $x_p$  is within this circle. Thin dotted lines represent the contour lines of the objective function (11a), while the dash-dot line represents the locus of the optimal point  $\tilde{x}_p(w)$  for the whole range of values  $w$  in the case when the inequality constraint (11c) would be left out from the optimization problem.

## V. CONCLUSIONS

We have presented a control design procedure as a solution to the problem of regulating a general linear time-invariant dynamical system to a time-varying economically optimal operating point. The system was characterized with a set of exogenous inputs as an abstraction of time-varying loads and disturbances acting on the system. Economic optimality was defined through a constrained convex optimization problem with a set of system states as decision variables, and with the values of exogenous inputs as parameters. A distinguishing, advantageous feature of the presented approach is that it offers an explicitly defined dynamic controller as a solution, i.e. the resulting controller is not based on solving on-line the corresponding optimization problem.

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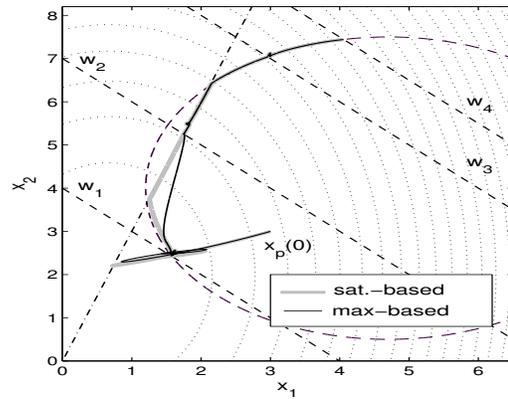


Fig. 6. Simulated trajectory of the state vector  $x_p$  for the close-loop system.

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