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$\mathcal{L}_1$ Adaptive Controller for Multi–Input Multi–Output Systems in the Presence of Unmatched Disturbances

Chengyu Cao and Naira Hovakimyan

***Abstract*— $\mathcal{L}_1$ adaptive control architecture introduced in [1]–[4] ensures that the input and the output of an uncertain linear system track the input and output of a desired linear system during the transient phase, in addition to asymptotic tracking. In this paper, we extend the $\mathcal{L}_1$ adaptive controller to multi–input multi–output systems in the presence of unmatched time-varying disturbances. Simulation results illustrate the theoretical findings.**

I. Introduction

The paper extends the results of [1]–[4] to a class of multi–input multi–output (MIMO) uncertain systems in the presence of time-varying parameters, unknown high-frequency gain, unmatched time-varying unknown disturbances that cannot be attenuated by backstepping type controllers. The control signal is defined as the output of a low-pass filter that appropriately attenuates the high-frequencies in the control signal typical for large adaptation rates. The $\mathcal{L}_\infty$ norm bounds for the error signals between the closed-loop adaptive system and the closed-loop reference system can be systematically reduced by fast adaptation.

The paper is organized as follows. Section II states some preliminary definitions, and Section III gives the problem formulation. Section IV introduces the $\mathcal{L}_1$ adaptive control architecture. Stability and uniform transient tracking bounds of the $\mathcal{L}_1$ adaptive controller are presented in Section V. Section VI discusses design details. In Section VII simulation results are presented, while Section VIII concludes the paper. All proofs are in the Appendix.

II. Preliminaries

Throughout this paper, $\mathbb{I}$ indicates the identity matrix of appropriate dimension, $||H(s)||_{\mathcal{L}_1}$ denotes the $\mathcal{L}_1$ gain of $H(s)$, $||x||_{\mathcal{L}_\infty}$ denotes the $\mathcal{L}_\infty$ norm of $x(t)$, $||x_t||_{\mathcal{L}_\infty}$ denotes the truncated $\mathcal{L}_\infty$ norm of $x(t)$ at the time instant t, and $\|x\|_2$ and $\|x\|_\infty$ indicate the 2- and ∞- norms of the vector x respectively. The next lemma extends the results of Example 5.2 ([6], page 199) to general MIMO systems.

Lemma 1: For a stable proper MIMO system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have $\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r_t\|_{\mathcal{L}_\infty}$, $\forall\, t \geq 0$.

III. Problem Formulation

Consider the following multi-input multi-output system:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + B(\theta(t)x(t) + \omega u(t)) + \sigma(t), \quad (1) \\ y(t) &= C^\top x(t)\,, \; x(0) = x_0\,, \end{aligned}$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}^m$ is the control signal, $y \in \mathbb{R}^m$ is the regulated output, $B, C \in \mathbb{R}^{n\times m}$ are known constant matrices, A_m is a known $n\times n$ Hurwitz matrix, (A_m, B) is controllable, $\theta(t) \in \mathbb{R}^{m\times n}$ is a matrix of time-varying unknown parameters, $\omega \in \mathbb{R}^{m\times m}$ is a constant non-singular unknown matrix, and $\sigma(t) \in \mathbb{R}^n$ is a vector of time-varying unmatched disturbances. We assume conservative knowledge of unknown parameters and uncertainties, i.e. there exist compact sets Ω, Θ, and Δ such that $\omega \in \Omega$, $\theta(t) \in \Theta$, $\sigma(t) \in \Delta$, for all $t \geq 0$. We further assume $\theta(t)$ and $\sigma(t)$ are differentiable with bounded derivatives, i.e. there exist finite d_θ and d_σ such that

$$\sqrt{\mathrm{tr}(\dot{\theta}^\top(t)\dot{\theta}(t))} \leq d_\theta\,, \;\; \|\dot{\sigma}(t)\|_2 \leq d_\sigma\,, \;\forall\, t \geq 0\,. \quad (2)$$

Let $H_o(s) = C^\top H(s)\,, \;\; H(s) = (s\mathbb{I} - A_m)^{-1}B\,.$

Assumption 1: The poles of $H_o^{-1}(s)$ are located in the left half plane.

The control objective is to design an adaptive controller to ensure that $y(t)$ tracks a given bounded continuous reference signal $r(t) \in \mathbb{R}^m$ *both in transient and steady state*, following a given reference model

$$y(s) \approx H_o(s) K_g r(s)\,, \quad (3)$$

where $K_g \in \mathbb{R}^{m\times m}$ is a constant matrix.

IV. $\mathcal{L}_1$ Adaptive controller

In this section, we develop a novel adaptive control architecture that permits complete transient characterization for both system's both input and output signals. We consider the following state predictor (or passive identifier) for generation of the adaptive laws:

$$\dot{\hat{x}}(t) = A_m\hat{x}(t) + B(\hat{\theta}(t)x(t) + \hat{\omega}(t)u(t)) + \hat{\sigma}(t), \hat{x}(0) = x_0. \quad (4)$$

The adaptive estimates $\hat{\omega}(t), \hat{\theta}(t), \hat{\sigma}(t)$ are defined as:

$$\begin{aligned} \dot{\hat{\omega}}(t) &= \Gamma\mathrm{Proj}(\hat{\omega}(t), -(\tilde{x}^\top(t)PB)^\top u^\top(t))\,, \\ \dot{\hat{\theta}}(t) &= \Gamma\mathrm{Proj}(\hat{\theta}(t), -(\tilde{x}^\top(t)PB)^\top x^\top(t))\,, \\ \dot{\hat{\sigma}}(t) &= \Gamma\mathrm{Proj}(\hat{\sigma}(t), -(\tilde{x}^\top(t)P)^\top)\,, \quad (5) \end{aligned}$$

in which $\tilde{x}(t) = \hat{x}(t) - x(t)$ is the tracking error between the system dynamics in (1) and the state predictor in (4), $\Gamma > 0$ is the adaptation gain, and $P = P^\top > 0$ is the solution of the algebraic Lyapunov equation $A_m^\top P + PA_m = -Q$, $Q > 0$, while $\mathrm{Proj}(\cdot,\cdot)$ denotes the projection operator [8], defined over the sets Θ, Ω and Δ. The control signal is generated through gain feedback of the following system:

$$\chi(s) = D(s)\bar{r}(s)\,, \quad u(s) = -K\chi(s)\,, \quad (6)$$

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The authors are with Aerospace & Ocean Engineering, Virginia Polytechnic Institute & State University, Blacksburg, VA 24061-0203, e-mail: chengyu, nhovakim@vt.edu

where $K \in \mathbb{R}^{m\times m}$,

$$\begin{aligned}\bar{r}(t) &= \hat{\theta}(t)x(t) + \hat{\omega}(t)u(t) + \bar{\sigma}(t) - K_g r(t)\,, \\ \bar{\sigma}(s) &= H_o^{-1}(s)C^\top(s\mathbb{I}-A_m)^{-1}\hat{\sigma}(s)\,, \end{aligned} \quad (7)$$

while $D(s)$ is a matrix of m by m transfer functions. Letting

$$F(s) = \omega K(\mathbb{I} + D(s)\omega K)^{-1}D(s)\,, \quad (8)$$

the choice of $D(s)$ and K needs to ensure that

(i) $F(s)$ is strictly proper and stable with $F(0)=\mathbb{I}$; (9)

(ii) $F(s)H_o^{-1}(s)$ is proper and stable; (10)

(iii) $\|\bar{G}(s)\|_{\mathcal{L}_1} L < 1\,,$ (11)

where

$$\begin{aligned} L &= \max_{\theta\in\Theta}\|\theta\|_{\mathcal{L}_1} = \max_i(\sum_j |\theta_{ij}|)\,, \quad (12) \\ \bar{G}(s) &= (s\mathbb{I}-A_m)^{-1}B(\mathbb{I}-F(s))\,. \quad (13) \end{aligned}$$

The complete $\mathcal{L}_1$ adaptive controller consists of (4), (5), (6) subject to (9)-(11).

A. Design of Control Law

In this section, we address how to choose $D(s)$ and K to satisfy (9)-(11). For simplicity, we consider the following most common choice of $D(s)$ as $D(s) = (1/s)\mathbb{I}$, which leads to

$$F(s) = \omega K(s\mathbb{I} + \omega K)^{-1}\,. \quad (14)$$

Hence, the requirement in (9) is equivalent to determining a K that renders $-\omega K$ Hurwitz. Using (14), it can be derived straightforwardly that the impulse response for $\mathbb{I} - F(s)$ is $\mathbb{I}\delta(t) - \omega K \mathrm{e}^{-\omega K t}$, where $\delta(t)$ denotes the impulse function. Let $\lambda_{\max}(-\omega K)$ be the maximum eigenvalue of the Hurwitz matrix $-\omega K$. If K is chosen to have arbitrarily small $\lambda_{\max}(-\omega K)$, then $\omega K \mathrm{e}^{-\omega K t}$ can approximate the delta function $\mathbb{I}\delta(t)$ arbitrarily closely. Hence, we have

$$\lim_{\lambda_{\max}(-\omega K)\to-\infty} \|\mathbb{I} - F(s)\|_{\mathcal{L}_1} = 0\,. \quad (15)$$

It follows from Lemma 1 that

$$\|\bar{G}(s)\|_{\mathcal{L}_1} \le \|(s\mathbb{I}-A_m)^{-1}B\|_{\mathcal{L}_1}\|\mathbb{I}-F(s)\|_{\mathcal{L}_1}\,. \quad (16)$$

Since $\|(s\mathbb{I}-A_m)^{-1}B\|_{\mathcal{L}_1}$ and L are constants, it follows from (15) and (16) that (11) can always be satisfied by making $\lambda_{\max}(-\omega K)$ small via appropriate choice of K.

If ω is a diagonal matrix, the design of K is straightforward. Assuming that the signs of the diagonal elements ω_i, $i = 1, .., m$ are known, K is selected to be a diagonal matrix with its i^{th} element K_i having the same sign as ω_i. Hence, it follows from (14) that $F(s)$ is a diagonal matrix with its i^{th} diagonal element being

$$F_i(s) = K_i\omega_i/(s + K_i\omega_i)\,. \quad (17)$$

We notice that (9) is obviously satisfied, while (11) can be verified if we increase $|K_i|$. Instead of the first order low-pass filter in (17), we can increase the relative degree of $F(s)$ by choosing $D(s)$ with larger relative degree as $D(s) = \frac{1}{s^2}\mathbb{I}$ or $D(s) = \frac{3a^2s+a^3}{s^3+3as^2}\mathbb{I}$, $a>0$. Hence, $F(s)H_o^{-1}(s)$ can always be made proper via the choice of $D(s)$. Once $F(s)H_o^{-1}(s)$ is proper, its stability follows from Assumption 1 straightforwardly.

If $m = 1$, then $F(s) = K\omega/(s+\omega K)$, where K is scalar. We notice that in this case the condition in (11) degenerates into the one for SISO systems stated in [1]–[4]. Also, Assumption 1 degenerates into conventional minimum phase requirement for $H_o(s)$.

V. Analysis of $\mathcal{L}_1$ Adaptive Controller

The analysis of the above stated control architecture is an extension of the results from [1]–[4]. The main difference as compared to [1]–[4] is that the system in this paper has multiple inputs and multiple outputs and unmatched time-varying disturbances. Let

$$\begin{aligned} F_1(s) &= H(s)F(s)H_o^{-1}(s)C^\top\,, \quad (18) \\ F_2(s) &= \omega^{-1}F(s)H_o^{-1}(s)C^\top\,. \quad (19) \end{aligned}$$

It follows from (10) that both $F_1(s)$ and $F_2(s)$ are proper and stable and hence their $\mathcal{L}_1$ gains are finite.

A. Closed-loop Reference System

We consider the following closed-loop LTI reference system with its control signal and system response being defined as follows:

$$\begin{aligned} &\dot{x}_{ref}(t) = A_m x_{ref}(t) + B(\theta(t)x_{ref}(t) + \omega u_{ref}(t)) \\ &+\sigma(t)\,, \quad x_{ref}(0) = x_0 \quad (20) \\ &y_{ref}(t) = C^\top x_{ref}(t) \quad (21) \\ &u_{ref}(s) = \omega^{-1}F(s)\eta_{ref}(s)\,, \quad (22) \end{aligned}$$

where $\eta_{ref}(s)$ is the Laplace transformation of the signal

$$\begin{aligned} \eta_{ref}(t) &= -\theta(t)x_{ref}(t) - \bar{\sigma}_{ref}(t) + K_g r(t)\,, \\ \bar{\sigma}_{ref}(s) &= H_o^{-1}(s)C^\top(s\mathbb{I}-A_m)^{-1}\sigma(s)\,. \quad (23) \end{aligned}$$

We note that the condition in (10) implies that the transfer function between $\sigma(t)$ and $u_{ref}(t)$ is proper and stable, and there is no singularity or differentiation involved in the generation of $u_{ref}(t)$. Since $u_{ref}(t)$ uses unknown signals and parameters ω, $\theta(t)$ and $\sigma(t)$, this closed-loop reference system is not implementable and is only used for analysis purposes. The next Lemma establishes stability of the closed-loop system in (20)-(22).

Lemma 2: If K and $D(s)$ verify (9)-(11), the reference system in (20)-(22) is stable.

B. Guaranteed Transient Performance of $\mathcal{L}_1$ Adaptive Controller

Let

$$\gamma_0 = \sqrt{\theta_m/(\lambda_{\min}(P)\Gamma)}\,, \quad (24)$$

where

$$\begin{aligned} \theta_m \triangleq\, & 4\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\Big(d_\theta \max_{\theta\in\Theta}\mathrm{tr}\sqrt{\theta^\top\theta} + d_\sigma \max_{\sigma\in\Delta}\|\sigma\|_2\Big) \\ &+4\Big(\max_{\theta\in\Theta}\mathrm{tr}(\theta^\top\theta) + \max_{\omega\in\Omega}\mathrm{tr}(\omega^\top\omega) + \max_{\sigma\in\Delta}(\sigma^\top\sigma)\Big)\,. \quad (25) \end{aligned}$$

The performance of $\mathcal{L}_1$ adaptive controller is stated in the following Theorem.

Theorem 1: Given the system in (1), the reference system in (20)-(22), and the $\mathcal{L}_1$ adaptive controller defined via (4), (5) and (6), subject to (9)-(11), we have:

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \gamma_0 \,, \quad (26)$$
$$\|x - x_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_1 \,, \quad (27)$$
$$\|y - y_{ref}\|_{\mathcal{L}_\infty} \leq \|C^\top\|_{\mathcal{L}_1}\gamma_1 \,, \quad (28)$$
$$\|u - u_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_2 \,, \quad (29)$$

where γ_0 is defined in (24) and

$$\gamma_1 = \|F_1(s)\|_{\mathcal{L}_1}\gamma_0/(1 - \|\bar{G}(s)\|_{\mathcal{L}_1}L) \,, \quad (30)$$
$$\gamma_2 = \|\omega^{-1}F(s)\|_{\mathcal{L}_1}L\gamma_1 + \|F_2(s)\|_{\mathcal{L}_1}\gamma_0 \,. \quad (31)$$

Theorem 1 states that $x(t)$, $y(t)$ and $u(t)$ follow $x_{ref}(t)$, $y_{ref}(t)$ and $u_{ref}(t)$ not only asymptotically but also during the transient, provided that the adaptive gain is selected sufficiently large. Thus, the control objective is reduced to designing K and $D(s)$ to ensure that the closed-loop reference system in (20)-(22) has the desired response specified in (3).

VI. Design of $\mathcal{L}_1$ Adaptive Controller

In this section, we address how to determine K and $D(s)$ to ensure that the reference system in (20)-(22) can achieve the control objective stated in (3). Consider the following *desired* system response $\dot{x}_{des}(t) = A_m x_{des}(t) + BK_g r(t)\,, y_{des}(t) = C^\top x_{des}(t)\,, x_{des}(0) = x_0\,,$ which is independent of uncertainties. Obviously

$$y_{des}(s) = H_o(s)K_g r(s) + C^\top(s\mathbb{I} - A_m)^{-1}x_0 \,. \quad (32)$$

First we prove that for a step input the steady state error between the reference system and this *desired system* is zero, if the unknown parameters are constant.

Lemma 3: For the reference system in (20)-(22) and the desired system in (32), subject to conditions (9)-(11), if $r(t)$, $\sigma(t)$, $\theta(t)$ are constant, then

$$\lim_{t\to\infty}(y_{ref}(t) - y_{des}(t)) = 0 \,. \quad (33)$$

Next, we characterize the performance bounds between the reference system and the desired system during the transient. Let $F_3(s) = H_o(s)(\mathbb{I} - F(s))$ and $F_4(s) = H_o(s)(\mathbb{I} - F(s))H_o^{-1}(s)C^\top(s\mathbb{I} - A_m)^{-1}$. We note that both $F_3(s)$ and $F_4(s)$ are proper and stable transfer functions. It follows from Lemma 2 that $\|x_{ref}\|_{\mathcal{L}_\infty}$ is finite for any bounded $r(t)$.

Lemma 4: For the reference system in (20)-(22) and the desired system in (32), subject to conditions (9)-(11), we have:

$$\|y_{ref} - y_{des}\|_{\mathcal{L}_\infty} \leq \|F_3(s)\|_{\mathcal{L}_1}(L\|x_{ref}\|_{\mathcal{L}_\infty} + \|K_g r\|_{\mathcal{L}_\infty}) + \|F_4(s)\|_{\mathcal{L}_1}\|\sigma\|_{\mathcal{L}_\infty} \,. \quad (34)$$

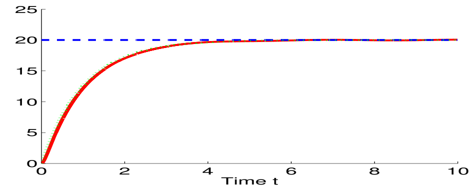
Theorem 1 and Lemma 4 imply that the output $y(t)$ of the system in (1) will follow $y_{des}(t)$ both in transient and steady state with quantifiable bounds, given in (28) and (34). Since $\lim_{F(s)\to\mathbb{I}}\|\mathbb{I} - F_3(s)\|_{\mathcal{L}_1} = 0$ and $\lim_{F(s)\to\mathbb{I}}\|\mathbb{I} - F_4(s)\|_{\mathcal{L}_1} = 0$, we have $\lim_{F(s)\to\mathbb{I}}\|y_{ref} - y_{des}\|_{\mathcal{L}_\infty} = 0$. Using (15), we notice that if we increase the bandwidth of $F(s)$ to achieve arbitrarily close approximation of the identity matrix $\mathbb{I}$, we can ensure that $y_{ref}(t)$ tracks $y_{des}(t)$ arbitrarily closely both in transient and steady state. However, increasing the bandwidth of $F(s)$ will lead to reduced time-delay margin and hurt the robustness of the closed-loop system as proved in [4].

Remark 1: We notice that Assumption 1 and the condition in (10) are new as compared to earlier results in [1]–[4]. These are required to compensate for the effects of the unmatched disturbance $\sigma(t)$ on the system output $y(t)$. If the disturbance $\sigma(t)$ can be factored as $B\sigma(t)$, where $\sigma(t) \in \mathbb{R}^m$ instead of $\sigma(t) \in \mathbb{R}^n$, both Assumption 1 and condition (10) can be dropped.

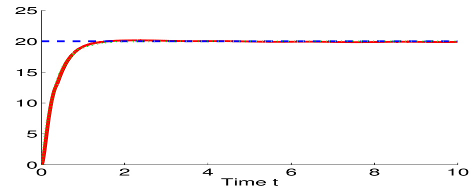
Remark 2: We notice that due to the unmatched nature of the disturbance, its effect on the performance of the system states cannot be compensated by the choice of the control signal. The control signal will only compensate for the effect of the disturbance on the system output. However, since the control signal is defined as an output of a low-pass system with appropriately banded frequencies, the large adaptive gain required in the analysis will not lead to adverse effects on the system states.

VII. Simulation Results

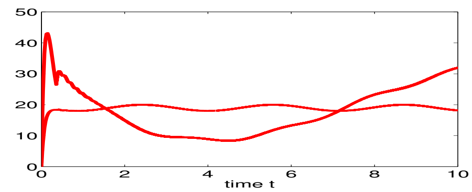
Consider the system: $\dot{x}(t) = A_m x(t) + B\Big(\begin{bmatrix} 0 \\ \theta(t)x(t)\end{bmatrix} + \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2\end{bmatrix}u(t)\Big) + \sigma(t),\ y(t) = C^\top x(t),\ x(0) = 0,$ where $A_m = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2\end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1\end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0\end{bmatrix}$. We note here that $\theta(t) = [\theta_1(t) \quad \theta_2(t) \quad \theta_3(t)]$ is a row vector. It can be derived straightforwardly that $H_o(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+3}\end{bmatrix}$. We notice that the desired system response is characterized by $y_{des}(s) = H_o(s)K_g r(s)$ with $K_g = -C^\top A_m^{-1}B = \begin{bmatrix} 1 & 0 \\ 0 & 3\end{bmatrix}$, $H_o(s)K_g = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{3}{s+3}\end{bmatrix}$. The $\mathcal{L}_1$ adaptive controller is designed based on (4), (5) and (6) as follows: $\dot{\hat{x}}(t) = A_m\hat{x}(t) + B\Big(\begin{bmatrix} 0 \\ \hat{\theta}(t)x(t)\end{bmatrix} + \begin{bmatrix} \hat{\omega}_1(t) & 0 \\ 0 & \hat{\omega}_2(t)\end{bmatrix}u(t)\Big) + \hat{\sigma}(t)$, $\dot{\hat{\omega}}_i(t) = \Gamma\text{Proj}(\hat{\omega}_i(t), [-(\tilde{x}^\top(t)PB)^\top u^\top(t))]_{\{i,i\}}$, $\dot{\hat{\theta}}(t) = \Gamma\text{Proj}(\hat{\theta}(t), [-(\tilde{x}^\top(t)PB)^\top x^\top(t))]_{\{2,:\}}$, $\dot{\hat{\sigma}}(t) = \Gamma\text{Proj}(\hat{\sigma}(t), -(\tilde{x}^\top(t)P)^\top)$, where $[X]_{\{i,i\}}$ indicates the the i^{th} row i^{th} column element of the matrix X, while $[X]_{\{2,:\}}$ indicates the 2^{nd} row vector of the matrix X. The control signal is given by: $\chi(s) = D(s)\bar{r}(s)\,,\ u(s) = -K\chi(s)$, where $D(s) = 1/s$, $K = k\mathbb{I}_{2\times 2}$, $k > 0$, $\bar{r}(t) = \begin{bmatrix} 0 \\ \hat{\theta}(t)x(t)\end{bmatrix} + \begin{bmatrix} \hat{\omega}_1(t) & 0 \\ 0 & \hat{\omega}_2(t)\end{bmatrix}u(t) + \bar{\sigma}(t) - K_g r(t)$, $\bar{\sigma}(s) = H_o^{-1}(s)C^\top(s\mathbb{I} - A_m)^{-1}\hat{\sigma}(s) =$



(a) $y_1(t)$ (solid), $r_1(t)$ (dashed) and $y_{des_1}(t)$ (dotted)

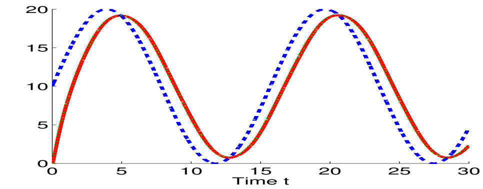


(b) $y_2(t)$ (solid), $r_2(t)$ (dashed) and $y_{des_2}(t)$ (dotted)

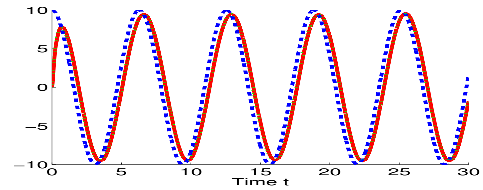


(c) Time-history of $u(t)$

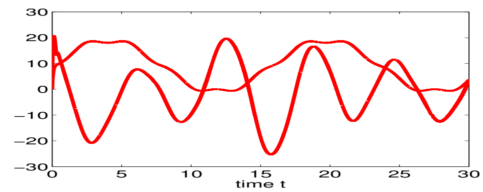
Fig. 1. Performance of $\mathcal{L}_1$ adaptive controller for $r = [20\ 20]$



(a) $y_1(t)$ (solid), $r_1(t)$ (dashed) and $y_{des_1}(t)$ (dotted)

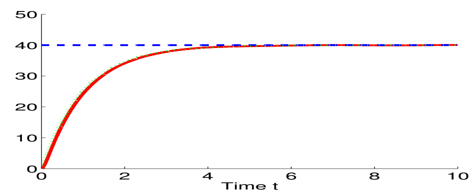


(b) $y_2(t)$ (solid), $r_2(t)$ (dashed) and $y_{des_2}(t)$ (dotted)

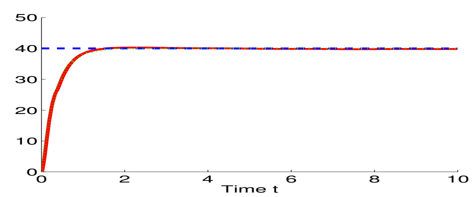


(c) Time-history of $u(t)$

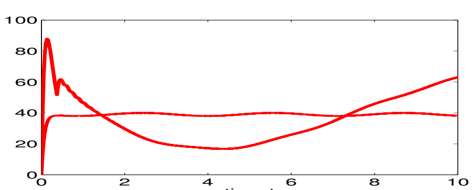
Fig. 3. Performance of $\mathcal{L}_1$ adaptive controller for $r(t) = [10 + 10\sin(0.4t) \quad 10\cos(t)]^\top$.



(a) $y_1(t)$ (solid), $r_1(t)$ (dashed) and $y_{des_1}(t)$ (dotted)

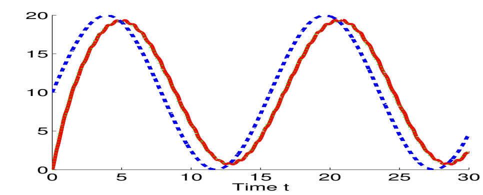


(b) $y_2(t)$ (solid), $r_2(t)$ (dashed) and $y_{des_2}(t)$ (dotted)

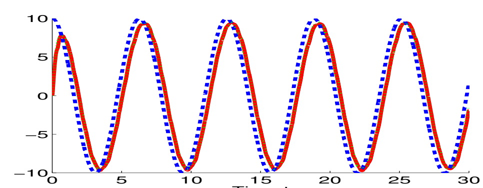


(c) Time-history of $u(t)$

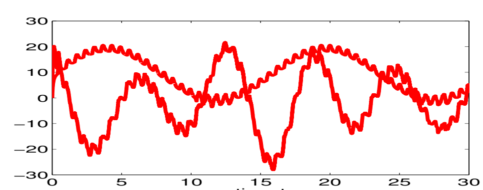
Fig. 2. Performance of $\mathcal{L}_1$ adaptive controller for $r = [40\ 40]$



(a) $y_1(t)$ (solid), $r_1(t)$ (dashed) and $y_{des_1}(t)$ (dotted)



(b) $y_2(t)$ (solid), $r_2(t)$ (dashed) and $y_{des_2}(t)$ (dotted)



(c) Time-history of $u(t)$

Fig. 4. Performance in the presence of disturbance defined in (35).

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \hat{\sigma}(s)$. Let $\omega_1 = 1$, $\omega_2 = 1.5$, and $\theta(t) = [0.5\sin(0.5t) \quad \sin(0.3t) \quad 0.5]$, $\sigma(t) = [1 + \sin(2t) \quad \sin(3t) \quad \sin(t)]^\top$. We assume the following conservative bounds for the unknown time-varying signals for the implementation of the projection operator: $\omega_i \in [1 \quad 1.5]$ for $i = 1, 2$, and $|\theta_i(t)| \leq 1$, $|\sigma_i(t)| \leq 10$, for $i = 1, 2, 3$. It can be derived straightforwardly that $L = 3$, while $\|\bar{G}(s)\|_{\mathcal{L}_1}$ can be calculated numerically. Letting $k = 15$, it can be verified numerically that the $\mathcal{L}_1$-gain upper bound is satisfied for all possible $\omega_i \in [1,\ 1.5]$.

The simulation results of the $\mathcal{L}_1$ adaptive controller are shown in Figures 1 and 2 for constant reference inputs $r = [20\ \ 20]$ and $r = [40\ \ 40]$, respectively. We note that it leads to scaled control inputs and scaled system outputs for scaled reference inputs. Figure 3 shows the system response and the control signal for the reference input $r(t) = [10 + 10\sin(0.4t) \quad 10\cos(t)]^\top$, without any retuning of the controller. In Fig. 4, we consider a different signal

$$\sigma(t) = [1 + 2\sin(10t) \quad 2\sin(10t) \quad 10\sin(10t)]^\top . \quad (35)$$

We note that the $\mathcal{L}_1$ adaptive controller guarantees smooth and uniform transient performance in the presence of different unknown time-varying disturbances without any retuning of the controller.

VIII. Conclusion

In this paper, we extend the $\mathcal{L}_1$ adaptive controller to multi–input multi–output systems in the presence of unmatched time-varying disturbances. By appropriate modification of the control architecture, the $\mathcal{L}_1$ adaptive controller compensates for the effects of the unmatched disturbances on the system outputs.

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Appendix

Proof of Lemma 2. The control law in (22) leads to the following closed-loop dynamics:

$$\begin{aligned} x_{ref}(s) &= G(s)r(s) + \bar{G}(s)\eta_1(s) - H(s)F(s)\bar{\sigma}_{ref}(s) \\ &\quad + (s\mathbb{I} - A_m)^{-1}\sigma(s) + (s\mathbb{I} - A_m)^{-1}x_0 , \end{aligned} \quad (36)$$

where $G(s) = H(s)F(s)K_g$, $\eta_1(s)$ is the Laplace transformation of signal $\eta_1(t) \triangleq \theta(t)x_{ref}(t)$. Since $\|\eta_{1_t}\|_{\mathcal{L}_\infty} \leq L\|x_{ref_t}\|_{\mathcal{L}_\infty}$, it follows from (23) and Lemma 1 that

$$\begin{aligned} &\|x_{ref_t}\|_{\mathcal{L}_\infty} \leq \|\bar{G}(s)\|_{\mathcal{L}_1} L \|x_{ref_t}\|_{\mathcal{L}_\infty} \\ &+ \|\eta_{2_t}\|_{\mathcal{L}_\infty} + \|F_1(s)(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1}\|\sigma_t\|_{\mathcal{L}_\infty} , \end{aligned} \quad (37)$$

where $\eta_2(t)$ has the following Laplace transformation: $\eta_2(s) = G(s)r(s) + (s\mathbb{I} - A_m)^{-1}\sigma(s) + (s\mathbb{I} - A_m)^{-1}x_0$. We note that the conditions (9)-(10) ensure that $\|F_1(s)(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1}$ is finite. The condition in (9) implies that $G(s)$ and $H(s)$ are stable, which further implies that $\|\eta_2\|_{\mathcal{L}_\infty}$ is finite, since $r(t)$, x_0 and $\sigma(t)$ are bounded. Hence, it follows from (37) that $\|x_{ref}\|_{\mathcal{L}_\infty} \leq \frac{\|\eta_2\|_{\mathcal{L}_\infty} + \|F_1(s)(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1}\|\sigma\|_{\mathcal{L}_\infty}}{1 - \|\bar{G}(s)\|_{\mathcal{L}_1} L}$, which implies that $x_{ref}(t)$ is bounded and completes the proof. □

Proof of Theorem 1. Letting $\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t)$, $\tilde{\sigma}(t) = \hat{\sigma}(t) - \sigma(t)$, $\tilde{\omega}(t) = \hat{\omega}(t) - \omega$, the following error dynamics can be derived from (1) and (4)

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + B(\tilde{\theta}(t)x(t) + \tilde{\omega}(t)u(t)) + \tilde{\sigma}(t) \quad (38)$$

with $\tilde{x}(0) = 0$. Consider the following candidate Lyapunov function: $V(\tilde{x}(t), \tilde{\theta}(t), \tilde{\omega}(t), \tilde{\sigma}(t)) = \tilde{x}^\top(t)P\tilde{x}(t) + \Gamma^{-1}\mathrm{tr}(\tilde{\theta}^\top\tilde{\theta}) + \Gamma^{-1}\mathrm{tr}(\tilde{\omega}^\top\tilde{\omega}) + \Gamma^{-1}(\tilde{\sigma}^\top\tilde{\sigma})$. Using the projection based adaptation laws from (5), one has the following upper bound:

$$\dot{V}(t) \leq -\tilde{x}^\top(t)Q\tilde{x}(t) + 2\Gamma^{-1}|\mathrm{tr}(\tilde{\theta}^\top(t)\dot{\theta}(t)) + \tilde{\sigma}^\top(t)\dot{\sigma}(t)| . \quad (39)$$

The projection algorithm ensures that $\hat{\theta}(t) \in \Theta$, $\hat{\omega}(t) \in \Omega$ and $\hat{\sigma}(t) \in \Delta$ for all $t \geq 0$, and therefore

$$\begin{aligned} &\max_{t\geq 0}\Big(\mathrm{tr}(\tilde{\theta}^\top(t)\tilde{\theta}(t)) + \mathrm{tr}(\tilde{\omega}^\top(t)\tilde{\omega}(t)) + \tilde{\sigma}^\top(t)\tilde{\sigma}(t)\Big) \leq \\ &4\Big(\max_{\theta\in\Theta}\mathrm{tr}(\theta^\top\theta) + \max_{\omega\in\Omega}\mathrm{tr}(\omega^\top\omega) + \max_{\sigma\in\Delta}(\sigma^\top\sigma)\Big) . \end{aligned} \quad (40)$$

If $V(t) \geq \theta_m/\Gamma$ at some t, then it follows from (40) that $\tilde{x}^\top(t)P\tilde{x}(t) \geq 4\frac{\lambda_{\max}(P)}{\Gamma\lambda_{\min}(Q)}\Big(d_\theta \max_{\theta\in\Theta}\mathrm{tr}\sqrt{\theta^\top\theta} + d_\sigma \max_{\sigma\in\Delta}\|\sigma\|_2\Big)$, and hence

$$\begin{aligned} &\tilde{x}^\top(t)Q\tilde{x}(t) \geq \lambda_{\min}(Q)\tilde{x}^\top(t)P\tilde{x}(t)/\lambda_{\max}(P) \geq \\ &4\Gamma^{-1}(d_\theta \max_{\theta\in\Theta}\mathrm{tr}\sqrt{\theta^\top\theta} + d_\sigma \max_{\sigma\in\Delta}\|\sigma\|_2) . \end{aligned} \quad (41)$$

The upper bounds in (2) along with the projection based adaptive laws (5) lead to the following upper bound:

$$\mathrm{tr}(\tilde{\theta}^\top(t)\dot{\theta}(t)) + \tilde{\sigma}^\top(t)\dot{\sigma}(t) \leq 2(d_\theta \max_{\theta\in\Theta}\mathrm{tr}\sqrt{\theta^\top\theta} + d_\sigma \max_{\sigma\in\Delta}\|\sigma\|_2) . \quad (42)$$

Hence, if $V(t) \geq \theta_m/\Gamma$, then it follows from (39), (41), (42) that

$$\dot{V}(t) \leq 0 . \quad (43)$$

Since we have set $\hat{x}(0) = x(0)$, we can verify that $V(0) \leq 4\Big(\max_{\theta\in\Theta}\mathrm{tr}(\theta^\top\theta) + \max_{\omega\in\Omega}\mathrm{tr}(\omega^\top\omega) + \max_{\sigma\in\Delta}(\sigma^\top\sigma)\Big)/\Gamma < \theta_m/\Gamma$. It follows from (43) that $V(t) \leq \theta_m/\Gamma$ for all $t \geq 0$. Since $\lambda_{\min}(P)\|\tilde{x}(t)\|^2 \leq \tilde{x}^\top(t)P\tilde{x}(t) \leq V(t)$, then $||\tilde{x}(t)||_2^2 \leq \dfrac{\theta_m}{\lambda_{\min}(P)\Gamma}$, which along with (24) implies that

$$||\tilde{x}(t)||_2 \leq \gamma_0 , \quad \forall\ t \geq 0 \quad (44)$$

and proves (26).

Let

$$\tilde{r}(t) = \tilde{\omega}(t)u(t) + \tilde{\theta}(t)x(t) , \ \xi_1(t) = \theta(t)x(t) . \quad (45)$$

It follows from (6) that $\chi(s) = D(s)(\omega u(s) + \bar{\sigma}(s) + \xi_1(s) - K_g r(s) + \tilde{r}(s))$, where $\tilde{r}(s)$ and $\xi_1(s)$ are the Laplace transformations of signals $\tilde{r}(t)$ and $\xi_1(t)$ respectively. Consequently $\chi(s) = (\mathbb{I} + D(s)\omega K)^{-1}D(s)(\xi_1(s) + \bar{\sigma}(s) - K_g r(s) + \tilde{r}(s))$, $u(s) = -K(\mathbb{I} + D(s)\omega K)^{-1}D(s)(\xi_1(s) + \bar{\sigma}(s) - K_g r(s) + \tilde{r}(s))$. Using the definition of $F(s)$ from (8), we can write

$$\omega u(s) = -F(s)(\xi_1(s) + \bar{\sigma}(s) - K_g r(s) + \tilde{r}(s)) , \quad (46)$$

and the system in (1) consequently takes the form:

$$\begin{aligned} x(s) &= G(s)r(s) + \bar{G}(s)\xi_1(s) - H(s)F(s)\tilde{r}(s) - \\ &H(s)F(s)\bar{\sigma}(s) + (s\mathbb{I} - A_m)^{-1}\sigma(s) + (s\mathbb{I} - A_m)^{-1}x_0 . \end{aligned} \quad (47)$$

Let $e(t) = x(t) - x_{ref}(t)$. Then, using (36) and (47), we have

$$e(s) = \bar{G}(s)\xi_2(s) - \xi_3(s) - \xi_4(s)\,, \quad (48)$$

where $\xi_3(s) = H(s)F(s)\tilde{r}(s)$, $\xi_4(s) = H(s)F(s)(\bar{\sigma}(s) - \bar{\sigma}_{ref}(s))$, and $\xi_2(s)$ is the Laplace transformation of the signal $\xi_2(t) = \theta(t)e(t)$. ¿From (7) and (23), we have $\xi_4(s) = F_1(s)(s\mathbb{I} - A_m)^{-1}\tilde{\sigma}(s)$, where $F_1(s)$ is defined in (18). The definition of $\xi_3(s)$ implies that $\xi_3(s) = H(s)F(s)(C^\top H(s))^{-1}C^\top H(s)\tilde{r}(s) = H(s)F(s)H_o^{-1}(s)C^\top H(s)\tilde{r}(s) = F_1(s)H(s)\tilde{r}(s)$. ¿From the relationship in (38) we have

$$\tilde{x}(s) = H(s)\tilde{r}(s) + (s\mathbb{I} - A_m)^{-1}\tilde{\sigma}(s)\,, \quad (49)$$

which leads to $\xi_3(s) + \xi_4(s) = F_1(s)\tilde{x}(s)$. Using (44), we can upper bound

$$\|\xi_3 + \xi_4\|_{\mathcal{L}_\infty} \le \|F_1(s)\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty} \le \|F_1(s)\|_{\mathcal{L}_1}\gamma_0\,. \quad (50)$$

For the error dynamics in (48), Lemma 1 implies that

$$\|e_t\|_{\mathcal{L}_\infty} \le \|\bar{G}(s)\|_{\mathcal{L}_1}\|\xi_{2t}\|_{\mathcal{L}_\infty} + \|(\xi_3 + \xi_4)_t\|_{\mathcal{L}_\infty}\,. \quad (51)$$

Using the definition of L in (12), one can verify easily that $\|\xi_{2t}\|_{\mathcal{L}_\infty} \le L\|e_t\|_{\mathcal{L}_\infty}$, which along with (50) and (51) implies that

$$\|e_t\|_{\mathcal{L}_\infty} \le \|\bar{G}(s)\|_{\mathcal{L}_1}L\|e_t\|_{\mathcal{L}_\infty} + \|F_1(s)\|_{\mathcal{L}_1}\gamma_0\,. \quad (52)$$

It follows from (11) and (52) that $\|e_t\|_{\mathcal{L}_\infty} \le \frac{\|F_1(s)\|_{\mathcal{L}_1}\gamma_0}{1-\|\bar{G}(s)\|_{\mathcal{L}_1}L}$ for all $t \ge 0$, which proves (27). The upper bound in (28) follows from Lemma 1 and the definition of $y(t)$ and $y_{ref}(t)$ directly.

To prove the bound in (29), we notice that from (22) and (46) that $u(s) - u_{ref}(s) = -\omega^{-1}F(s)(\xi_2(s) + \eta_4(s) + \tilde{r}(s))$, where $\tilde{r}(t)$ is defined in (45), $\eta_4(t) = \bar{\sigma}(t) - \bar{\sigma}_{ref}(t)$. Hence, we have

$$\|u - u_{ref}\|_{\mathcal{L}_\infty} \le \|\omega^{-1}F(s)\|_{\mathcal{L}_1}L\|e\|_{\mathcal{L}_\infty} + \|\eta_5\|_{\mathcal{L}_\infty}\,, \quad (53)$$

where $\eta_5(s) = -\omega^{-1}F(s)(\eta_4(s) + \tilde{r}(s))$. ¿From (7) and (23), we have

$$\begin{aligned}\eta_5(s) &= -\omega^{-1}F(s)\tilde{r}(s) - \\ &\quad \omega^{-1}F(s)H_o^{-1}(s)C^\top(s\mathbb{I} - A_m)^{-1}\tilde{\sigma}(s) \\ &= -\omega^{-1}F(s)(C^\top H(s))^{-1}C^\top H(s)\tilde{r}(s) - \\ &\quad \omega^{-1}F(s)H_o^{-1}(s)C^\top(s\mathbb{I} - A_m)^{-1}\tilde{\sigma}(s) \\ &= -\omega^{-1}F(s)H_o^{-1}(s)C^\top(H(s)\tilde{r}(s) \\ &\quad +(s\mathbb{I} - A_m)^{-1}\tilde{\sigma}(s))\end{aligned}$$

which along with (49) implies that

$$\eta_5(s) = -\omega^{-1}F(s)H_o^{-1}(s)C^\top\tilde{x}(s)\,. \quad (54)$$

It follows from (44) and (54) that

$$\|\eta_5\|_{\mathcal{L}_\infty} \le \|\omega^{-1}F(s)H_o^{-1}(s)C^\top\|_{\mathcal{L}_1}\gamma_0. \quad (55)$$

Hence, the relationships in (27), (53), (55) and the definition of $F_2(s)$ in (19) imply that

$$\|u - u_{ref}\|_{\mathcal{L}_\infty} \le \|\omega^{-1}F(s)\|_{\mathcal{L}_1}L\gamma_1 + \|F_2(s)\|_{\mathcal{L}_1}\gamma_0\,,$$

which proves (29). □

Proof of Lemma 3. If θ, r and σ are constant, it follows from (23) and (36) that

$$\begin{aligned}x_{ref}(s) &= G(s)r/s + \bar{G}(s)\theta x_{ref}(s) \\ &\quad -H(s)F(s)H_o^{-1}(s)C^\top(s\mathbb{I} - A_m)^{-1}\sigma/s \\ &\quad +(s\mathbb{I} - A_m)^{-1}\sigma/s + (s\mathbb{I} - A_m)^{-1}x_0,\end{aligned}$$

and hence

$$\begin{aligned}x_{ref}(s) &= (\mathbb{I} - \bar{G}(s)\theta)^{-1}\Big(G(s)r/s \\ &\quad -H(s)F(s)H_o^{-1}(s)C^\top(s\mathbb{I} - A_m)^{-1}\sigma/s \\ &\quad +(s\mathbb{I} - A_m)^{-1}\sigma/s + (s\mathbb{I} - A_m)^{-1}x_0\Big). \quad (56)\end{aligned}$$

It follows from the end value theorem that

$$\begin{aligned}\lim_{t\to\infty} y_{des}(t) &= \lim_{s\to 0} sy_{des}(s) \\ &= \lim_{s\to 0} sH_o(s)K_g r/s = H_o(0)K_g r\,, \quad (57) \\ \lim_{t\to\infty} y_{ref}(t) &= \lim_{s\to 0} sy_{ref}(s)\,. \quad (58)\end{aligned}$$

Eq. (56) and the definition of $y_{ref} = C^\top x_{ref}$ imply that

$$\begin{aligned}&\lim_{t\to\infty} y_{ref}(t) = \lim_{s\to 0} sy_{ref}(s) = C^\top(\mathbb{I} - \bar{G}(0)\theta)^{-1}\Big(G(0)r \\ &+(-A_m)^{-1}\sigma - H(0)F(0)H_o^{-1}(0)C^\top(-A_m)^{-1}\sigma\Big). \quad (59)\end{aligned}$$

Since (9) implies that $F(0) = \mathbb{I}$, we have $G(0) = H(0)K_g$ and $\bar{G}(0) = 0$. Using the definition of $H_o(0) = C^\top H(0)$ and the relationship in (59), we can write:

$$\begin{aligned}&\lim_{s\to 0} sy_{ref}(s) = C^\top H(0)K_g r + C^\top(-A_m)^{-1}\sigma - \\ &H_o(0)H_o^{-1}(0)C^\top(-A_m)^{-1}\sigma\Big) = H_o(0)K_g r. \quad (60)\end{aligned}$$

The limiting expression in (33) follows from (57) and (60) directly.

Proof of Lemma 4. It follows from (23) and (36) that

$$\begin{aligned}x_{ref}(s) &= G(s)r(s) + \bar{G}(s)\eta_1(s) \\ &\quad -H(s)F(s)H_o^{-1}(s)C^\top(s\mathbb{I} - A_m)^{-1}\sigma(s) \\ &\quad +(s\mathbb{I} - A_m)^{-1}\sigma(s) + (s\mathbb{I} - A_m)^{-1}x_0\,,\end{aligned}$$

and hence

$$\begin{aligned}y_{ref}(s) &= H_o(s)F(s)K_g r(s) + H_o(s)(\mathbb{I} - F(s))\eta_1(s) \\ &\quad -H_o(s)F(s)H_o^{-1}(s)C^\top(s\mathbb{I} - A_m)^{-1}\sigma(s) \\ &\quad +C^\top(s\mathbb{I} - A_m)^{-1}\sigma(s) + C^\top(s\mathbb{I} - A_m)^{-1}x_0\,, \\ &= y_{des}(s) - H_o(s)(\mathbb{I} - F(s))K_g r(s) + \\ &\quad H_o(s)(\mathbb{I} - F(s))\eta_1(s) + \\ &\quad H_o(s)(\mathbb{I} - F(s))H_o^{-1}(s)C^\top(s\mathbb{I} - A_m)^{-1}\sigma(s)\,.\end{aligned}$$

It follows from Lemma 1 and the definitions of $F_3(s)$, $F_4(s)$ that

$$\begin{aligned}\|y_{ref} - y_{des}\|_{\mathcal{L}_\infty} &\le \|F_3(s)\|_{\mathcal{L}_1}(L\|x_{ref}\|_{\mathcal{L}_\infty} + \|K_g r\|_{\mathcal{L}_\infty}) \\ &\quad +\|F_4(s)\|_{\mathcal{L}_1}\|\sigma\|_{\mathcal{L}_\infty}\,, \quad (61)\end{aligned}$$

which concludes the proof. □