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Adaptive Model Reference Control of a class of MIMO Discrete-time Systems with Compensation of Nonparametric Uncertainty

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Abstract— **In this paper, adaptive model reference control is investigated for a class of discrete-time multi-input-multi-output (MIMO) systems. Estimation of both unknown system parameters and nonparametric model uncertainty is constructed. Based on the estimation, a novel adaptive control is proposed which completely compensates the nonparametric model uncertainty. The boundedness of the closed-loop signals are guaranteed and the outputs are made asymptotically track the reference model outputs if there is no external disturbance. Simulation results are presented to demonstrate the effectiveness of the proposed control approach.**

I. INTRODUCTION

In recent years, adaptive control of discrete-time systems has been studied extensively and quite a number of adaptive schemes have been developed to deal with nonparametric model uncertainty. In contrast to continuous-time systems, nonparametric model uncertainty is very hard to deal for discrete-time systems. In most existing literature on discrete-time adaptive control, the nonparametric model uncertainty is assumed to be global bounded [1] such that some techniques, e.g., deadzone, projection and σ-modification, can be employed to guarantee the boundedness of parameter estimation as well as closed-loop signals [2], [3]. To improve control performance, some approximation based method such as neural network (NN) has been used to emulate and then compensate the nonparametric model uncertainty [4], where the nonparametric uncertainty is still required to be assumed globally bounded and due to NN approximation error, the model uncertainty cannot be completely compensated. On the other hand, in some other works [5], [6], [7], [8], [9], the nonparametric model uncertainty is assumed to be of linear growth of the system states or inputs and outputs. Then by using projection or deadzone in the parameter estimation, it is proved that when the growing rate of the nonparametric uncertainty is smaller than certain constant, the proposed adaptive control guarantees the closed-loop stability. However, it is noted that there is no compensation of the nonparametric uncertainty in these results. In [10], the nonparametric model uncertainty is assumed to belong to $L^{1+\alpha}$ with $\alpha \geq 1$ such that asymptotical tracking can be achieved using some robust control.

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Inspired by a recent paper [11] where an estimation of parametric and non-parametric uncertainties is proposed for a class of first order systems, we investigate adaptive control with completely compensation of the both uncertainties under assumption that the Lipschiz coefficient of the nonparametric uncertainty is smaller than some constant. The main idea is using past input output data in the adaptive control to estimate and then to cancel the nonparametric uncertainty in addition to the estimation of unknown parameters. In the parameter estimates update law, adaptive dead-zone [9] is adopted to guarantee the boundedness of estimations. The proposed adaptive control guarantees the boundedness of all the closed-loop signals and further, the nonparametric model uncertainty can be completely compensated in the closed-loop system such that the output tracking error is only affected by external disturbance.

The main contributions of the paper lie in:

(i) With the introduction of a notation l_k, a novel estimation of nonparametric model uncertainty is constructed.
(ii) Both unknown system parameters and nonparametric model uncertainty are estimated for adaptive control design.
(iii) In the presence of nonzero nonparametric model uncertainty, the proposed adaptive control guarantees the asymptotical output tracking if the system is free of external disturbance. model uncertainty .

Throughout this paper, the following notations are used.

- $\|\cdot\|$ denotes the Euclidean norm of vectors and induced norm of matrices.
- Z_0^+ represents the set of all nonnegative integers.
- $\mathbf{0}_{[\mathbf{p}]}$ stands for p-dimension zero vector.
- $(\hat{\cdot})$ and $(\tilde{\cdot})$ denote the estimate of unknown parameter and the estimation error, respectively.

II. PROBLEM FORMULATION

A. System Representation

Consider a class of multi-input-multi-output (MIMO) discrete-time systems with nonparametric uncertainty in the the following form:

$$A(q^{-1})y(k+\tau) = B(q^{-1})u(k) + \nu(\underline{z}(k)) + d(k) \qquad (1)$$

where $\tau \geq 1$ is system delay and $A(q^{-1})$ and $B(q^{-1})$ are polynomial matrices in terms of the unit back shift operator q^{-1} with $A(q^{-1})$ being diagonal, which can be represented in the following forms

$$A(q^{-1}) = I + A_1 q^{-1} + \cdots + A_{n_a} q^{-n_a}$$
$$B(q^{-1}) = B_0 + B_1 q^{-1} + \cdots + B_{n_b} q^{-n_b}$$

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where n_a and n_b denote the orders of $A(q^{-1})$ and $B(q^{-1})$ respectively, A_i and B_i are $n \times n$ constant matrices of q^{-i}, $u(k) = [u_1(k), \cdots, u_n(k)]^T \in R^n$ and $y(k) = [y_1(k), \cdots, y_n(k)]^T \in R^n$ are the system inputs and outputs, $d(k) \in R^n$ is bounded external disturbance satisfying $\|d(k)\| \leq \bar{d}$, with $\bar{d}$ a constant, and the nonparametric vector function $\nu(\underline{z}(k)) : R^{n(n_{\nu y}+n_{\nu u})} \to R^n$ denotes the nonparametric model uncertainty, with its arguments defined as

$$\begin{aligned}\underline{z}(k) &= [\underline{y}^T(k), \underline{u}^T(k-1)]^T \in R^{n(n_{\nu y}+n_{\nu u})} \qquad (2)\\ \underline{y}(k) &= [y^T(k), y^T(k-1), \ldots, y^T(k-n_{\nu y}+1)]^T\\ \underline{u}(k) &= [u^T(k), u^T(k-1), \ldots, u^T(k-n_{\nu u}+1)]^T\end{aligned}$$

Assumption 1: The nonlinear vector function of nonparametric uncertainty, $\nu(\cdot) : R^{n(n_{\nu y}+n_{\nu u})} \to R^n$, are Lipschiz function satisfying

$$\|\nu(\epsilon_1) - \nu(\epsilon_2)\| \leq L\|\epsilon_1 - \epsilon_2\|$$

where $L \leq L^*$ and L^* will be defined later in (37).

Assumption 2: The constants $n_{\nu u}$ and $n_{\nu y}$ are unknown but there is a know constant n_ν such that $n_{\nu u} \leq n_\nu$ and $n_{\nu y} \leq n_\nu$.

In the following of the paper, the nonparametric uncertainty $\nu(\underline{z}(k))$ will be denoted briefly as $\nu(k)$ without ambiguity.

Assumption 3: The system is in minimum phase, i.e., $B(q^{-1})$ is a stable operator.

Assumption 4: The system dynamics matrix $A(q^{-1})$ and $B(q^{-1})$ are unknown, but their orders n_a and n_b are known. In addition, the matrix B_0 is nonsingular and is known.

Consider a reference model as

$$P(q^{-1})y^*(k+\tau) = Rr(k) \qquad (3)$$

where $r(k)$ is bounded reference trajectory and

$$P(q^{-1}) = P_0 + P_1 q^{-1} + \cdots + P_{n_p} q^{-n_p}$$

is a diagonal weighting polynomial matrix and R is a diagonal weighting constant matrix. They are specified by the designer such that $P(q^{-1})$ is stable.

Given the reference model (3), the control objective is to design an adaptive control input $u(k)$, such that the output of system (1) tracks the output of the reference model, $y^*(k)$, while all closed loop signals remain bounded.

B. Preliminaries

The following lemmas will be used for adaptive control design and stability analysis in the remainder of the paper.

Lemma 1: [12] Given a bounded sequence $X(k) \in R^p$, i.e., $\sup\{\|X(k)\|\} < \infty$, and a fixed positive integer τ. Define

$$l_k = \arg\min_{l \leq k-\tau} \|X(k) - X(l)\| \qquad (4)$$

Then, we have $\lim_{k\to\infty} \|X(k) - X(l_k)\| = 0$.
Proof. See Appendix A. ■

Lemma 2: [13] For some given real scalar sequences $s(k)$, $b_1(k)$, $b_2(k)$ and vector sequence $\sigma(k)$, if the following conditions hold:

(i) $\lim_{k\to\infty} \frac{s^2(k)}{b_1(k)+b_2(k)\sigma^T(k)\sigma(k)} = 0$,
(ii) $0 < b_1(k) < K < \infty$ and $0 \leq b_2(k) < K < \infty$ for all $k \geq 1$,
(iii) $\|\sigma(k)\| \leq C_1 + C_2 \max_{0 \leq k' \leq k} |s(k')|$, where $0 < C_1 < \infty$ and $0 < C_2 < \infty$,

then, we have
(a) $\lim_{k\to\infty} s(k) = 0$, and (b)$\{\|\sigma(k)\|\}$ is bounded.

Lemma 3: The nonparametric uncertainty $\nu(k)$ satisfies

$$\|\nu(k)\| \leq L\|\underline{z}(k)\| + \|\nu(\mathbf{0}_{[n(n_{\nu y}+n_{\nu u})]})\|$$

Proof. It is obvious from Assumption 1. ■

III. Adaptive Control Design

A. System Transformation

Define a generalized output vector $y_p(k)$ as

$$y_p(k) = P(q^{-1})y(k) \qquad (5)$$

and introduce the following Diophantine equation

$$P(q^{-1}) = F(q^{-1})A(q^{-1}) + q^{-\tau}G(q^{-1}) \qquad (6)$$

where

$$\begin{aligned}F(q^{-1}) &= F_0 + F_1 q^{-1} + \ldots + F_{n_f} q^{-n_f}\\ G(q^{-1}) &= G_0 + G_1 q^{-1} + \ldots + G_{n_g} q^{-n_g}\end{aligned}$$

with $n_f = \tau - 1$, $n_g = \max\{n_a - 1, n_p - \tau\}$, F_i and G_i are $n \times n$ diagonal constant matrices.

Remark 1: It should be noted that $F_0 = P_0$ because $A(0) = I$.

Based on the Diophantine equation (6) and the definition of $y_p(k)$ in (5), we have

$$y_p(k+\tau) = G(q^{-1})y(k) + H(q^{-1})u(k) + \nu_F(k) + d_F(k) \qquad (7)$$

where

$$\begin{aligned}H(q^{-1}) &= F(q^{-1})B(q^{-1})\\ &= H_0 + H_1 q^{-1} + \cdots + H_{n_h} q^{-n_h} \qquad (8)\\ \nu_F(k) &= F(q^{-1})\nu(k), \quad d_F(k) = F(q^{-1})d(k) \qquad (9)\end{aligned}$$

with $n_h = \tau - 1 + n_b$. It should be noted that $H_0 = F_0 B_0 = P_0 B_0$ is known because P_0 is specified by designer and B_0 is known according to Assumption 4.

B. Estimation of Nonparametric Model Uncertainty

For convenience of analysis, denote Θ as

$$\Theta = [G_0, G_1, \ldots, G_{n_g}, H_1, \cdots, H_{n_h}]^T \qquad (10)$$

Then, equation (7) can be written as

$$y_p(k+\tau) = \Theta^T \Phi(k) + H_0 u(k) + \nu_F(k) + d_F(k) \qquad (11)$$

where

$$\begin{aligned}\Phi(k) &= [\bar{y}^T(k), \bar{u}^T(k-1)]^T \in R^{(n_p+n_b)\times n} \qquad (12)\\ \bar{y}(k) &= [y^T(k), y^T(k-1), \ldots, y^T(k-n_g)]^T\\ \bar{u}(k) &= [u^T(k), u^T(k-1), \ldots, u^T(k-n_h+1)]^T\end{aligned}$$

Let us define

$$\bar{z}(k) = [y^T(k), y^T(k-1), \ldots, y^T(k-m), u^T(k-1), \ldots, u^T(k-m)] \quad (13)$$

where $m \geq \max\{n_g, n_h, n_\nu\}$.

According to Lemma 1, we introduce the notation l_k, which is defined as

$$l_k = \arg\min_{l \leq k-\tau} \|\bar{z}(k) - \bar{z}(l)\| \quad (14)$$

This notation will be used later for estimation of nonparametric model uncertainty.

Based on the definition of $\bar{z}(k)$ in (13) and l_k in (14), we have the following lemma which will be used for analysis later.

Lemma 4: Consider the definition of l_k in (14). If

$$\lim_{k\to\infty} \|\bar{z}(k) - \bar{z}(l_k)\| = 0$$

then, we have

$$\lim_{k\to\infty} \|\Phi(k) - \Phi(l_k)\| = 0$$
$$\lim_{k\to\infty} \|\nu(k) - \nu(l_k)\| = 0$$

Proof. See Appendix B. ■

Define an auxiliary output $y_a(k)$ as

$$\begin{aligned} y_a(k) &:= \Theta^T\Phi(k) + \nu_F(k) + d_F(k) \\ &\equiv y_p(k+\tau) - H_0u(k) \end{aligned} \quad (15)$$

Then, we have

$$\begin{aligned} y_a(k) &= y_a(k) - y_a(l_k) + y_a(l_k) \\ &= \Theta^T\Phi(k) + \nu_F(k) + d_F(k) + y_a(l_k) \\ &\quad -\Theta^T\Phi(l_k) - \nu_F(l_k) - d_F(l_k) \\ &= \Theta^T[\Phi(k) - \Phi(l_k)] + y_a(l_k) \\ &\quad +\Delta\nu_F(k) + \Delta d_F(k) \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Delta\nu_F(k) &= \nu_F(k) - \nu_F(l_k) \\ \Delta d_F(k) &= d_F(k) - d_F(l_k) \end{aligned} \quad (17)$$

Denote $\hat{\Theta}(k)$ as the estimate of the unknown parameter Θ at the k-th step, and then we take

$$\hat{y}_a(k) = \hat{\Theta}^T(k)[\Phi(k) - \Phi(l_k)] + y_a(l_k) \quad (18)$$

as estimation of auxiliary output $y_a(k)$.

Remark 2: The nonparametric uncertainty $\nu(k)$ is included in the auxiliary output $y_a(k)$. Therefore, the estimation of $y_a(k)$ is a indirect way to estimation $\nu(k)$.

Remark 3: It should be noted that at k-th step, $l_k \leq k-\tau$ such that $y_a(l_k)$ can be calculated as $y_a(l_k) = y_p(l_k+\tau) - H_0u(l_k)$.

C. Estimation of Unknown System Parameters

Define the estimation error of auxiliary output as

$$\begin{aligned} \tilde{y}_a(k) &= \hat{y}_a(k) - y_a(k) = \tilde{\Theta}^T(k)[\Phi(k) - \Phi(l_k)] \\ &\quad -\Delta\nu_F(k) - \Delta d_F(k) \end{aligned} \quad (19)$$

where $\tilde{\Theta}(k) = \hat{\Theta}(k) - \Theta(k)$. According to Lemma 3 and the definition of $\nu_F(k)$ and $d_F(k)$ in (17), we have

$$\begin{aligned} \|\Delta\nu_F(k) + \Delta d_F(k)\| &\leq 2c_1L\max_{k'\leq k}\{\underline{z}(k')\} + 2c_2 \\ \|\Delta d_F(k)\| &\leq 2d_b \end{aligned} \quad (20)$$

where $d_b = \bar{d}\sum_0^{n_f}\|F_i\|$ and

$$c_1 = \sum_0^{n_f}\|F_i\|, \quad c_2 = d_b + \|\nu(\mathbf{0}_{[n(n_{\nu y}+\nu_u)]})\| \quad (21)$$

Because c_1 and c_2 are unknown, their corresponding estimations $\hat{c}_1(k)$ and $\hat{c}_2(k)$ can be used to construct the dead-zone. Now, define

$$\hat{c}(k) = 2\lambda\hat{c}_1(k)\max_{k'\leq k}\{\|\underline{z}(k)\|\} + 2\hat{c}_2(k) \quad (22)$$

where $L < \lambda \leq \lambda^*$ with λ^* defined later in (44). The dead-zone function is defined as

$$a(k) = \begin{cases} 1 - \frac{\hat{c}(k-\tau)}{\|\tilde{y}_a(k-\tau)\|} & \text{if } \|\tilde{y}_a(k-\tau)\| \geq \hat{c}(k-\tau) \\ 0 & otherwise \end{cases} \quad (23)$$

The update laws for $\hat{\Theta}(k)$, $\hat{c}_1(k)$ and $\hat{c}_2(k)$ are given as

$$\hat{\Theta}(k) = \hat{\Theta}(k-\tau) - \frac{a(k)\gamma[\Phi(k-\tau) - \Phi(l_{k-\tau})]\tilde{y}_a^T(k-\tau)}{D(k-\tau)} \quad (24)$$

$$\hat{c}_1(k) = \hat{c}_1(k-\tau) + \frac{a(k)\gamma\lambda\|\tilde{y}_a(k-\tau)\|}{D(k-\tau)}\max_{k'\leq k}\{\|\underline{z}(k'-\tau)\|\} \quad (25)$$

$$\hat{c}_2(k) = \hat{c}_2(k-\tau) + \frac{a(k)\gamma\|\tilde{y}_a(k-\tau)\|}{D(k-\tau)} \quad (26)$$

where $0 < \gamma < 1$ and

$$\begin{aligned} D(k) &= 1 + \max_{k'\leq k}\{\|\underline{z}(k)\|\} + \|[\Phi(k) - \Phi(l_k)]\|^2 \\ \hat{c}_1(0) &= 0, \quad \hat{c}_2(0) = 0 \\ \hat{\Theta}(0) &= \mathbf{0}_{[(n_p+n_b)\times n]} \end{aligned} \quad (27)$$

D. Control Law

The adaptive control law is given as

$$\hat{y}_a(k) + H_0u(k) = Rr(k) \quad (28)$$

or equivalently

$$u(k) = B_0^{-1}P_0^{-1}(Rr(k) - \hat{y}_a(k))$$

Combining (11), (15) and (28) together, the control can be described as

$$\begin{aligned} &G(q^{-1})y(k) + H(q^{-1})u(k) + \nu_F(k) + d_F(k) \\ &= Rr(k) - \tilde{y}_a(k) \end{aligned} \quad (29)$$

Substituting (29) into open loop system described in (7), we obtain the closed-loop system as following

$$y_p(k+\tau) = Py(k+\tau) = Rr(k) - \tilde{y}_a(k) \quad (30)$$

where $\tilde{y}_a(k)$ is the estimation error defined in (19). It is obvious that the closed-loop system (30) matches the reference model (3) if the estimation error $\tilde{y}_a(k)$ converge to zero.

The main results in this paper are summarized in the following theorem.

Theorem 1: Consider the closed-loop system consisting of system (1) under Assumptions 1, 2, 3 and 4, control (28) with parameter adaptation law (24), (25), and (26). Then there exist constant L^* and λ^*, such that for the nonparametric model uncertainty $\nu(k)$ in (1) with Lipschiz coefficient $L < L^*$, if the tuning parameter γ and λ in adaptation law (24), (25) and (26) are chosen to satisfy $0 < \gamma < 1$ and $L \le \lambda \le \lambda^*$, then all the signals in the closed-loop system are guaranteed to be bounded and the output tracking error satisfy

$$\lim_{k\to\infty} \sup\{P(q^{-1})e(k)\} = 2d_b, \quad e(k) = y(k) - y^*(k)$$

which implies that $y(k)$ will asymptotically converge to $y^*(k)$ in the absence of external disturbance $d(k)$, i.e., $d_b = 0$.

IV. Stability Analysis

To prove the boundedness of all the estimated parameters, we choose a Lyapunov function candidate including all the parameter estimation errors as following:

$$V(k) = \sum_{j=k-\tau+1}^{k} [\mathrm{tr}\{\tilde{\Theta}^T(j)\tilde{\Theta}(j)\} + 2\tilde{c}_1^2(j) + 2\tilde{c}_2^2(j)] \quad (31)$$

The difference of $V(k)$ is

$$\begin{aligned}
&\Delta V(k) = V(k) - V(k-1)\\
&= \mathrm{tr}\{\tilde{\Theta}^T(k)\tilde{\Theta}(k) - \tilde{\Theta}^T(k-\tau)\tilde{\Theta}(k-\tau)\}\\
&+2[\tilde{c}_1^2(k) - \tilde{c}_1^2(k-\tau)] + 2[\tilde{c}_2^2(k) - \tilde{c}_2^2(k-\tau)]\\
&= \frac{a^2(k)\gamma^2\|\tilde{y}_a(k-\tau)\|^2\|\Phi(k-\tau) - \Phi(l_{k-\tau})\|^2}{D^2(k-\tau)}\\
&-\mathrm{tr}\{\tilde{\Theta}^T(k-\tau)[\Phi(k-\tau)-\Phi(l_{k-\tau})]\tilde{y}_a^T(k-\tau)\}\frac{2a(k)\gamma}{D(k-\tau)}\\
&+\frac{2a^2(k)\gamma^2\lambda^2\|\tilde{y}_a(k-\tau)\|^2\max_{k'\le k}\{\|\underline{z}(k'-\tau)\|\}^2}{D^2(k-\tau)}\\
&+\frac{4a(k)\gamma\lambda\|\tilde{y}_a(k-\tau)\|\tilde{c}_1(k-\tau)\max_{k'\le k}\{\|\underline{z}(k'-\tau)\|\}}{D(k-\tau)}\\
&+\frac{2a^2(k)\gamma^2\|\tilde{y}_a(k-\tau)\|^2}{D^2(k-\tau)} + \frac{4a(k)\gamma\tilde{c}_2(k-\tau)\|\tilde{y}_a(k-\tau)\|}{D(k-\tau)}
\end{aligned}$$

From (19) and (20), we have

$$\begin{aligned}
&-\mathrm{tr}\{\tilde{\Theta}^T(k-\tau)[\Phi(k-\tau) - \Phi(l_{k-\tau})]\tilde{y}_a^T(k-\tau)\}\\
&= -\mathrm{tr}\{[\tilde{y}_a(k-\tau)+\Delta v_F(k-\tau)+\Delta d_F(k-\tau)]\tilde{y}_a^T(k-\tau)\}\\
&= -\|\tilde{y}_a(k-\tau)\|^2-\tilde{y}_a^T(k-\tau)[\Delta v_F(k-\tau)+\Delta d_F(k-\tau)]\\
&\le 2\|\tilde{y}_a(k-\tau)\|[c_1 L\max_{k'\le k}\|\underline{z}(k-\tau)\| + c_2]\\
&\quad - \|\tilde{y}_a(k-\tau)\|^2
\end{aligned} \quad (32)$$

From the definition of deadzone in (23), we have

$$\begin{aligned}
&a(k)[2\hat{c}(k-\tau)\|\tilde{y}_a(k-\tau)\| - 2\|\tilde{y}_a(k-\tau)\|^2]\\
&= -a(k)[2a(k)\|\tilde{y}_a(k-\tau)\|^2]
\end{aligned} \quad (33)$$

Noting (32), (33), and that $L \le \lambda$ and

$$1 + \max_{k'\le k}\{\|\underline{z}(k-\tau)\|\}^2 + \frac{1}{2}\|\Phi(k-\tau)-\Phi(l_{k-\tau})\| \le D(k-\tau)$$

Then, we have

$$\begin{aligned}
\Delta V(k) &\le \frac{2a^2(k)\gamma^2\|\tilde{y}_a(k-\tau)\|^2}{D(k-\tau)} - \frac{2a(k)\gamma\|\tilde{y}_a(k-\tau)\|^2}{D(k-\tau)}\\
&+\frac{4a(k)\gamma\|\tilde{y}_a(k-\tau)\|[c_1 L\max_{k'\le k}\|\underline{z}(k'-\tau)\| + c_2]}{D(k-\tau)}\\
&+\frac{4a(k)\gamma\lambda\tilde{c}_1(k-\tau)\|\tilde{y}_a(k-\tau)\|\max_{k'\le k}\{\|\underline{z}(k'-\tau)\|\}}{D(k-\tau)}\\
&+\frac{4a(k)\gamma\|\tilde{y}_a(k-\tau)\|\tilde{c}_2(k-\tau)}{D(k-\tau)}\\
&= \frac{2a^2(k)\gamma^2\|\tilde{y}_a(k-\tau)\|^2}{D(k-\tau)} - \frac{2a(k)\gamma\|\tilde{y}_a(k-\tau)\|^2}{D(k-\tau)}\\
&+\frac{2a(k)\gamma\|\tilde{y}_a(k-\tau)\|\hat{c}(k-\tau)}{D(k-\tau)}\\
&\le -\frac{2\gamma(1-\gamma)a^2(k)\|\tilde{y}_a(k-\tau)\|^2}{D(k-\tau)}
\end{aligned} \quad (34)$$

Noting that $0 < \gamma < 1$, we can see from (32) that the difference of Lyapunov function $V(k)$, $\Delta V(k)$, is non-positive and thus, the boundedness of $V(k)$ is guaranteed. It further results in the boundedness of $\hat{\Theta}(k)$, $\hat{c}_1(k)$ and $\hat{c}_2(k)$.

Taking summation on both hand sides of (32), we obtain

$$\sum_{k=0}^{\infty}\frac{2\gamma(1-\gamma)a^2(k)\|\tilde{y}_a(k-\tau)\|^2}{D(k-\tau)} \le V(0) - V(\infty)$$

which implies

$$\lim_{k\to\infty}\frac{a^2(k)\|\tilde{y}_a(k-\tau)\|^2}{D(k-\tau)} = 0 \quad (35)$$

Consider the closed-loop system in (30). According to the boundedness of reference signal $r(k)$ and Assumption 3, it can be concluded that

$$\begin{aligned}
\|\bar{z}(k)\| \le C_1 + C_2[&\max_{k'\le k}\{\|\tilde{y}_a(k')\|\}\\
&+ \max_{k'\le k}\{\|\nu(k')\|\}] + \max_{k'\le k}\{\|d(k')\|\}
\end{aligned}$$

Together with Lemma 3 and $\|\underline{z}(k)\| \le \|\bar{z}(k)\|$, we have

$$\|\bar{z}(k)\| \le C_3 + C_2\max_{k'\le k}\{\|\tilde{y}_a(k')\|\} + C_2 L\max_{k'\le k}\{\|\bar{z}(k')\|\} \quad (36)$$

where $C_3 = C_1 + C_2\|\nu(\mathbf{0})\| + \bar{d}$.

It is equivalent to

$$\max_{k'\le k}\|\bar{z}(k')\| \le C_3 + C_2\max_{k'\le k}\|\tilde{y}_a(k')\| + C_2 L\max_{k'\le k}\|\bar{z}(k')\|$$

Thus, there exists a positive constant

$$L^* = \min\{\lambda^*, \frac{1}{C_2}\} \quad (37)$$

where λ^* is defined in (44) such that

$$\max_{k'\le k}\|\bar{z}(k')\| \le \frac{C_3}{1 - C_2 L} + \frac{C_2}{1 - C_2 L}\max_{k'\le k}\|\tilde{y}_a(k')\| \quad (38)$$

holds for a class of nonparametric uncertainty with Lipschiz coefficient satisfying $L < L^*$.

Denoting $C_4 = \frac{C_3}{1-C_2L}$ and $C_5 = \frac{C_2}{1-C_2L}$, we have

$$\max_{k'\le k}\|\bar{z}(k')\| \le C_4 + C_5 \max_{k'\le k}\|\tilde{y}_a(k')\| \tag{39}$$

where C_4 and C_5 are some finite constants.

From (23), when $\|\tilde{y}_a(k-\tau)\| \ge \hat{c}(k-\tau)$ we have

$$a(k)\|\tilde{y}_a(k-\tau)\| = \|\tilde{y}_a(k-\tau)\| - \hat{c}(k-\tau) \ge 0$$

when $\|\tilde{y}_a(k-\tau)\| < \hat{c}(k-\tau)$, we have

$$a(k)\|\tilde{y}_a(k-\tau)\| = 0$$

which implies that

$$a(k)\|\tilde{y}_a(k-\tau)\| > \|\tilde{y}_a(k-\tau)\| - \hat{c}(k-\tau)$$

In summary, the following inequality always holds

$$a(k)\|\tilde{y}_a(k-\tau)\| \ge \|\tilde{y}_a(k-\tau)\| - \hat{c}(k-\tau) \tag{40}$$

Then, the following inequality can be derived from (39)

$$\begin{aligned}\max_{k'\le k}\|\bar{z}(k'-\tau)\| &\le C_5[\max_{k'\le k}\{\|\tilde{y}_a(k'-\tau)\|\}-\hat{c}(k'-\tau)\\ &\quad+\hat{c}(k'-\tau)] + C_4\\ &\le C_4 + C_5\max_{k'\le k}\{a(k')\|\tilde{y}_a(k'-\tau)\|\}\\ &\quad+C_4\max_{k'\le k}\{\hat{c}(k'-\tau)\}\end{aligned} \tag{41}$$

Due to the boundedness of $\hat{c}_1(k)$ and $\hat{c}_2(k)$, there exists constants C_6 and C_7 such that

$$\begin{aligned}\hat{c}(k-\tau) &\le C_6 + C_7\lambda\{\max_{k'\le k} \underline{z}(k'-\tau)\}\\ &\le C_6 + C_7\lambda\max_{k'\le k}\{\bar{z}(k'-\tau)\}\end{aligned} \tag{42}$$

which together with (41) yields

$$\begin{aligned}\max\nolimits_{k'\le k}\|\bar{z}(k'-\tau)\| &\le C_8\max\nolimits_{k'\le k}\{a(k')\|\tilde{y}_a(k'-\tau)\|\}\\ &+ C_9\lambda\max\nolimits_{k'\le k}\{\|\bar{z}(k'-\tau)\|\} + C_{10}\end{aligned} \tag{43}$$

where C_8, C_9 and C_{10} are some constants. It can be seen that there exist positive constant

$$\lambda^* = \frac{1}{C_9} \tag{44}$$

such that if $0 < \lambda < \lambda^*$, we have

$$\max_{k'\le k}\|\bar{z}(k'-\tau)\| \le C_{11}+C_{12}\max_{k'\le k}\{a(k')\|\tilde{y}_a(k'-\tau)\|\} \tag{45}$$

where $C_{11} = \frac{C_{10}}{1-\lambda C_9}$ and $C_{12} = \frac{C_8}{1-\lambda C_9}$. According to the definition of $\Phi(k)$ in (12) and definition of $\bar{z}(k)$ in (13), and $m > \max\{n_g, n_h\}$, we have

$$\begin{aligned}\max_{k'\le k}\|\Phi(k'-\tau)\| &\le \max_{k'\le k}\|\bar{z}(k'-\tau)\|\\ &\le C_{11} + C_{12}\{\max_{k'\le k} a(k')\|\tilde{y}_a(k'-\tau)\|\}\end{aligned}$$

which leads to

$$D(k-\tau) \le C_{13} + C_{14}\{\max_{k'\le k} a(k')\|\tilde{y}_a(k'-\tau)\|\}$$

according to the definition of $D(k)$ in (27), where C_{13} and C_{14} are some finite constants.

Then, applying Lemma 2 to (35), we have

$$\lim_{k\to\infty} a(k)\|\tilde{y}_a(k-\tau)\| = 0 \tag{46}$$

which together with (45) guarantees the boundedness of $\bar{z}(k)$ and $\Phi(k)$. Thus, the boundedness of $y(k)$ and $u(k)$ is obvious. According to Lemma 1, we see $\lim_{k\to\infty}\|\bar{z}(k) - \bar{z}(l_k)\| = 0$ and according to Lemma 4, we have $\lim_{k\to\infty}\|\Phi(k) - \Phi(l_k)\| = 0$ and $\lim_{k\to\infty}\Delta\nu_F(k) = 0$. Then, it can be seen from (19) that $\lim_{k\to\infty}\sup\{\tilde{y}_a(k)\} \le 2d_b$. From (30) and (3) it is easy to derive

$$\begin{aligned}P(q^{-1})e(k+\tau) &= P(q^{-1})y^*(k+\tau) - P(q^{-1})y(k+\tau)\\ &= \tilde{y}_a(k)\end{aligned}$$

which leads to $\lim_{k\to\infty}\sup\{P(q^{-1})e(k+\tau)\} = \lim_{k\to\infty}\sup\{\tilde{y}_a(k)\} \le 2d_b$. This completes the proof.

V. Simulation Results

The following system in [9] is used for simulation.

$$A(q^{-1})y(k+1) = B(q^{-1})u(k) + \nu(k) + d(k) \tag{47}$$

where

$$\begin{aligned}A(q^{-1}) &= I + q^{-1}I\\ B(q^{-1}) &= \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix} + \begin{bmatrix}0.5 & 0.5\\ 0 & 1\end{bmatrix}q^{-1}\end{aligned}$$

the model uncertainty $\nu(k)$ is given by

$$\begin{aligned}\nu_1(k) = \nu_2(k) &= 0.1\cos(0.05k)\times(0.5u_1(k) + u_2(k)\\ &\quad+u_1(k-1) + 0.5u_2(k-1) + y_1(k) + y_2(k))\end{aligned}$$

and the external disturbance is $d_1(k) = 0.01\cos(0.05k)$, $d_2(k) = 0.01\sin(0.05k)$.

The reference model is simply chosen as $y^*(k+1) = r(k)$, i.e., $P = R = I$. The reference input $r(k)$ is given as $r_1(k) = 0.5 + 0.25\cos(0.25\pi Tk) + 0.25\sin(0.5\pi Tk)$ and $r_2(k) = 0.5 + 0.25\sin(0.25\pi Tk) + 0.25\sin(0.5\pi Tk)$, where $T = 0.05$. Then, the control objective is to make the output $y(k)$ track the desired reference trajectory $y^*(k)$. The initial condition is $y(0) = [0,\ 0]^T$ and $u(-1) = [0,\ 0]^T$. The tuning factor $\gamma = 0.01$ and $\lambda = 0.1$. The simulation results are presented in Figures 1-3.

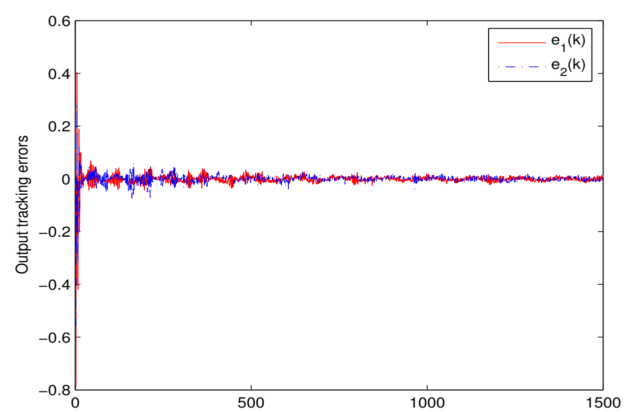

Fig. 1. Output tracking errors

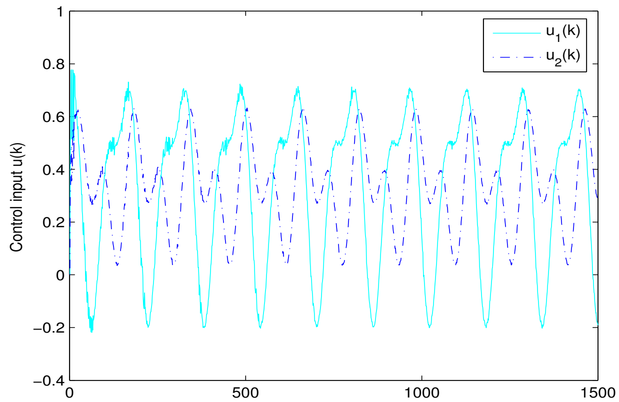

Fig. 2. Control input signals

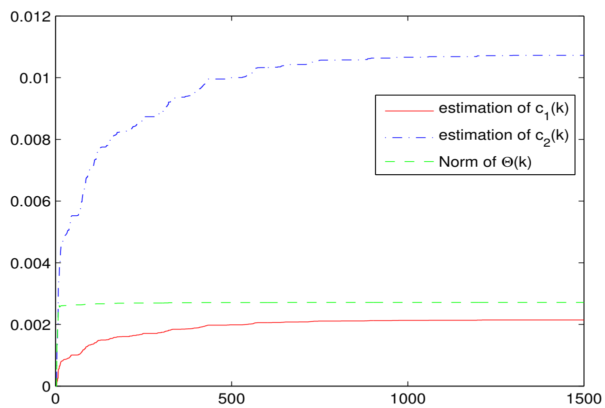

Fig. 3. Estimations $\hat{c}_1(k)$, $\hat{c}_2(k)$ and norm of $\hat{\Theta}(k)$

VI. Conclusion

In this paper, based on the estimation of both parametric and non-parametric uncertainties, a novel adaptive model reference control has been synthesized for a class of MIMO discrete-time systems with nonparametric model uncertainty and external disturbance. The adaptive control guarantees the boundedness of all the close-loop signals and completely compensate the uncertain nonlinearity. The system output exactly tracks the reference model output in the absence of external disturbance.

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Appendix A: Proof of Lemma 1

Proof. We will prove it by seeking a contradiction in a similar way as in [12]. Firstly, let us suppose that

$$\overline{\lim}_{k\to\infty}\|X(k)-X(l_k)\|=\epsilon>0 \tag{48}$$

where $\overline{\lim}$ denote the upper limit. Then we can take from $X(k)$ a subsequence $\{X(k_j), j\geq 1\}$ such that

$$\|X(k_j)-X(l_{k_j})\| \quad > \quad \frac{\epsilon}{2}, \quad k_j-l_{k_j}\geq\tau$$

According to the definition in (4), we have

$$\|X(k_j)-X(k')\|>\frac{\epsilon}{2}, \quad \forall 0\leq k'\leq k_j-\tau$$

Noting that $k_i\leq k_j-\tau$, $i<j$, we have $\|X(k_j)-X(k_i)\|>\frac{\epsilon}{2}$, or equivalently

$$\|X(k_j)-X(k_i)\|>\frac{\epsilon}{2}, \quad i\neq j$$

which means that $\{X(k_j), j\geq 1\}$ is unbounded. This contradicts to $\sup\{\|X(k)\|\}<\infty$. Consequently (48) cannot hold and thus we have

$$\underline{\lim}_{k\to\infty}\|X(k)-X(l_k)\|=\overline{\lim}_{k\to\infty}\|X(k)-X(l_k)\|=0$$

where $\underline{\lim}$ denotes the lower limit. Then, we have

$$\lim_{k\to\infty}\|X(k)-X(l_k)\|=0$$

This completes the proof. ■

Appendix B: Proof of Lemma 4

Proof. From the definition of $\bar{z}(k)$ in (13), the definition of $\underline{z}(k)$ in (2), the definition of $\Phi(k)$ in (12) and the definition of Euclidean norm, we see

$$\begin{aligned} 0\leq\|\underline{z}(k)-\underline{z}(l_k)\| &\leq \|\bar{z}(k)-\bar{z}(l_k)\| \\ 0\leq\|\Phi(k)-\Phi(l_k)\| &\leq \|\bar{z}(k)-\bar{z}(l_k)\| \end{aligned}$$

Then, according to squeeze rule, the lemma can be easily proved from $\lim_{k\to\infty}\|\bar{z}(k)-\bar{z}(l_k)\|=0$. ■

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