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Calculating the Terminal Region of NMPC for Lure Systems via LMIs

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Abstract— **One way to guarantee stability in nonlinear model predictive control (NMPC) is to calculate a suitable stabilizing terminal penalty term and terminal region. This is in general a difficult task. In this paper a structured approach to overcome this problem for the class of Lure systems subject to state and input constraints is provided. In particular, the terminal penalty term and the terminal region are computed by solving a set of LMIs. The example of a flexible joint robotic arm illustrates the results of this paper.**

I. INTRODUCTION

Due to its ability to handle control systems subject to state and input constraints model predictive control (MPC) has been applied in a wide area of practical control problems in recent years, see [8, 13] for an overview. The basic idea of predictive control is as follows: By solving online a finite horizon optimal control problem based on current measurements of the system, an optimal control input trajectory is obtained. The first part of this trajectory is applied to the system and the optimal control problem is solved again based on new measurements at the next sampling instant. Although MPC schemes often lead to good performance, closed-loop stability is not naturally guaranteed. By now, various approaches guaranteeing stability even for NMPC have been developed, see e.g. [2, 6, 7, 10, 12]. Several NMPC schemes use a terminal penalty term and a terminal region, both calculated off-line, to guarantee closed-loop stability, see e.g [3, 6]. However, the calculation of the terminal penalty term and the terminal region is not a trivial task. Various approaches have been proposed to tackle this problem. For example in [4] an LMI based approach is presented which uses linear differential inclusion (LDI) techniques [1]. However, this approach requires an LDI representation of the considered system which is conservative and in general hard to obtain. The goal of this paper is to derive a more structured way to obtain the terminal region and the terminal penalty term for the class of Lure systems.

The remainder of the paper is organized as follows: In Section II a brief introduction to stability of NMPC is given. Section III introduces the considered system class and provides the main results, namely a structured approach to derive a stabilizing terminal penalty term and terminal region for Lure systems via LMIs. Two different types of Lure systems are considered which differ by the sector conditions the appearing nonlinearities satisfy. The presented solution approaches differ in aspects of applicability on one side and solvability on the other side. In Section IV two examples illustrate the results. Conclusions are provided in Section V.

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II. STABILIZING PREDICTIVE CONTROL

Consider nonlinear systems of the form

$$\dot{x} = f(x,u),\ x(0) = x_0 \tag{1}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, subject to state and input constraints

$$u(t) \in \mathcal{U}\ \forall\ t \geq 0, \tag{2}$$

$$x(t) \in \mathcal{X}\ \forall\ t \geq 0. \tag{3}$$

Here $\mathcal{X} \subseteq \mathbb{R}^n$ is the state constraint set and $\mathcal{U} \subset \mathbb{R}^m$ is the set of feasible inputs. The control task is to stabilize the origin of system (1) in an optimal way while satisfying the constraints. One approach to achieve this is NMPC. This control method predicts the future behavior of the system. Therefore, we introduce predicted states and inputs, $\bar{x}$ and $\bar{u}$. The predicted states may differ from the real state x of the considered system (1). In general, the cost function J, that is minimized over the prediction horizon T_p, takes the form

$$\begin{aligned} J\big(\bar{x}(\cdot),\bar{u}(\cdot)\big) &= \int_{t_k}^{t_k+T_p} F\big(\bar{x}(\tau),\bar{u}(\tau)\big)d\tau \\ &+ E\big(\bar{x}(t_k+T_p)\big) \end{aligned} \tag{4}$$

with the stage cost F and the terminal penalty term E. The open-loop optimal control problem, that is solved repeatedly at the sampling instants $t_k = k\delta$, where δ is the sampling time between each optimization (assumed to be constant), is

$$\min_{\bar{u}(\cdot)} J\big(\bar{x}(\cdot),\bar{u}(\cdot)\big), \tag{5}$$

subject to

$$\begin{aligned} &\dot{\bar{x}}(\tau) = f\big(\bar{x}(\tau),\bar{u}(\tau)\big),\ \bar{x}(t_k) = x(t_k), \\ &\bar{x}(\tau) \in \mathcal{X},\ \bar{u}(\tau) \in \mathcal{U},\ \forall \tau \in \big[t_k, t_k+T_p\big], \\ &\bar{x}(t_k+T_p) \in \mathcal{E}. \end{aligned} \tag{6}$$

The solution of the optimization problem is the optimal input trajectory

$$\bar{u}^\star\big(t;x(t_k)\big) = arg \min_{\bar{u}(\cdot)} J\big(\bar{x}(\cdot),\bar{u}(\cdot)\big). \tag{7}$$

Note that in the stated version the system states are forced to lie within the terminal region $\mathcal{E}$ at the end of the prediction horizon T_p. The control input applied to system (1) is updated at each sampling instant t_k by the repeated solution of the open-loop optimal control problem (5), i.e. the applied control input is

$$u(t) = \bar{u}^\star(t;x(t_k)),\ t \in \big[t_k, t_k+\delta\big). \tag{8}$$

Each optimization at time t_k uses the measured state $x(t_k)$ as initial condition for the predicted system behavior $\bar{x}(t)$.

Therefore, the NMPC scheme presented provides state feedback at the sampling instants t_k.
To guarantee stability of the closed-loop system the following assumptions are typically required to hold [6]:

Assumption 1: $\mathcal{X} \subseteq \mathbb{R}^n$ is closed, connected, and the origin is contained in the interior of $\mathcal{X}$.
Assumption 2: $\mathcal{U} \subset \mathbb{R}^m$ is compact and the origin is contained in the interior of $\mathcal{U}$.
Assumption 3: The vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and satisfies $f(0,0) = 0$. In addition, it is locally Lipschitz in x.
Assumption 4: The system (1) has a unique continuous solution for any initial condition in the region of interest and any piecewise continuous and right-continuous input function $u(\cdot) : [0, T_p] \to \mathcal{U}$.
Assumption 5: $F : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is continuous in all arguments with $F(0,0) = 0$ and $F(x,u) > 0 \ \forall \ (x,u) \in \mathcal{X} \times \mathcal{U} \setminus \{0,0\}$.

It is shown in [5] and [7] that closed-loop stability can be guaranteed if the conditions on the terminal penalty term E and the terminal region $\mathcal{E}$ stated in Lemma 1 hold.

Lemma 1: If the open-loop optimal control problem (5) has a feasible solution at time $t = 0$, if the Assumptions 1-5 are satisfied, and if the terminal penalty term E and the terminal region $\mathcal{E}$ satisfy

1. E is $\mathcal{C}^1$, $E(x) \geq 0 \ \forall \ x \in \mathcal{E}$, and $E(0) = 0$,
2. $\mathcal{E} \subseteq \mathcal{X}$ is closed and connected, and the origin is contained in the interior of $\mathcal{E}$,
3. There exists a continuous local control law $k : \mathbb{R}^n \to \mathbb{R}^m$ with $k(0) = 0$, such that
$$\frac{\partial E}{\partial x} f\big(x, k(x)\big) + F\big(x, k(x)\big) < 0 \text{ and } k(x) \in \mathcal{U}, \quad (9)$$
for all $x \in \mathcal{E}$,

then the (nominal) closed-loop system is stable in the sense that $\|x\| \to 0$ as $t \to \infty$.

In general, it is hard to calculate the terminal penalty term and the terminal region, see e.g. [4, 6]. For certain subclasses of nonlinear systems there exist structured approaches to solve this problem. In the following we consider the class of Lure systems and derive a method to calculate the terminal penalty term and the terminal region via LMIs.

III. Main Results

In the following we limit the attention to nonlinear continuous time Lure systems, i.e. systems of the form
$$\begin{aligned} \dot{x} &= Ax + G\gamma(z) + Bu, \\ z &= Hx, \end{aligned} \quad (10)$$
where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $G \in \mathbb{R}^{n\times p}$ and $H \in \mathbb{R}^{p\times n}$ are constant linear matrices and $z \in \mathbb{R}^p$ denotes a linear combination of the states. The vector $\gamma(z) : \mathbb{R}^p \to \mathbb{R}^p$ consists of p nonlinear functions dependent on z such that the Assumptions 3 and 4 are satisfied. Furthermore, the signal z and the nonlinearities $\gamma(z)$ satisfy a so called sector condition of the form
$$\big(mz - \gamma(z)\big)^T \gamma(z) \geq 0, \quad (11)$$
where $m = \text{diag}(m_1, m_2, \ldots, m_p)$ with $m_i \in \mathbb{R}$, see [11]. The first part of this section considers Lure systems with a complete sector condition ($m \to \infty$), i.e. the nonlinearities $\gamma(z)$ lie either in the complete first or in the complete third quadrant. The resulting solution method is applicable to a broad class of Lure systems. However, in certain cases solvability of the obtained set of LMIs might be a problem since the approach requires that a restrictive equality condition holds. Thus, in the second part of this section, Lure systems with a more restrictive sector condition are considered, i.e. the nonlinearities are growth bounded in the first and third quadrant. This restriction on the nonlinearities is used to overcome the solvability problem of the LMIs.

Remark 1: The results of this paper can also be applied to nonlinear systems that can be transformed to Lure form (corresponding to the considered sector condition) via a regular transformation.

In the following, the control objective is to stabilize the origin of system (10) using the nonlinear model predictive controller defined by (1)-(8). For this, the stage cost F, the terminal penalty term E and the terminal region $\mathcal{E}$ are chosen to be
$$\begin{aligned} F(x,u) &= x^T Q x + u^T R u, & (12) \\ E(x) &= x^T P x, & (13) \\ \mathcal{E}(\alpha) &= \{x \in \mathcal{X} : x^T P x \leq \alpha\}, & (14) \end{aligned}$$
where $0 \leq Q = Q^T \in \mathbb{R}^{n\times n}$, $0 < R = R^T \in \mathbb{R}^{m\times m}$, $0 < P = P^T \in \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}^+$. With this choice $F(x,u)$ satisfies Assumption 5. For simplicity of presentation we consider linear constraints:

Assumption 6: The constraint sets $\mathcal{X}$ and $\mathcal{U}$ are described by linear inequalities:
$$\begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^n : c_{i_x} x \leq 1, i_x = 1, \ldots, r_x\}, & (15) \\ \mathcal{U} &= \{u \in \mathbb{R}^m : d_{i_u} u \leq 1, i_u = 1, \ldots, r_u\}, & (16) \end{aligned}$$
where $c_i \in \mathbb{R}^{1\times n}$ and $d_i \in \mathbb{R}^{1\times m}$ and r_x and r_u the number of state and input constraints.

The sets $\mathcal{X}$ and $\mathcal{U}$, the function $F(x,u)$, and the nonlinearity $\gamma(z)$ satisfy Assumptions 1-5 which are preconditions of Lemma 1. To simplify notation the constraint sets $\mathcal{X}$ and $\mathcal{U}$ are summarized by the constraint set
$$\mathcal{W} = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+m} : c_i x + d_i u \leq 1, i = 1, \cdots, r \right\}, \quad (17)$$
where $r = r_x + r_u$. Note that $d_i = 0$ for a state constraint and $c_i = 0$ for an input constraint. The remaining, in general

very challenging task is to derive a suitable terminal penalty term E and terminal region $\mathcal{E}(\alpha)$ such that the NMPC scheme (1)-(8) satisfies Lemma 1. For this, we consider a linear local feedback control

$$u = k(x) = Kx \tag{18}$$

with $K \in \mathbb{R}^{m\times n}$. As will be shown, Lemma 1 is fulfilled if a set of LMIs, depending on the linear feedback matrix K, is satisfied. Considering the linear feedback law (18) the constraint set $\mathcal{W}$ takes the form

$$\mathcal{W} = \{x \in \mathbb{R}^n : w_i x \leq 1, i = 1, \cdots, r\}, \tag{19}$$

with $w_i = c_i + d_i K$. From the definition of $\mathcal{W}$ it clearly follows that $\mathcal{W} \subseteq \mathcal{X}$. The following Lemma is needed to guarantee satisfaction of the constraints:

Lemma 2: The ellipsoid $\mathcal{E}(\alpha) = \{x \in \mathbb{R}^n : x^T P x \leq \alpha\}$ is contained in the set $\mathcal{W}$ if and only if

$$w_i(\alpha P^{-1})w_i^T \leq 1, i = 1, \cdots, r. \tag{20}$$

Proof: The proof can be found in [4] and [1]. ■

Based on this, in the following methods are derived which provide a structured way to calculate the terminal region and the terminal penalty term. In the first part, Lure systems satisfying a complete sector condition are considered. The second part provides an approach for Lure systems with growth bounded nonlinearities. The basic idea of both methods is to formulate the stability condition (9) as LMIs by taking the special structure of Lure systems into account.

A. Lure systems satisfying a complete sector condition

The approach presented in this subsection considers Lure systems (10) satisfying the sector condition

$$z^T \gamma(z) \geq 0. \tag{21}$$

This condition requires that the nonlinearities $\gamma(z)$ lie either in the complete first or in the complete third quadrant. The following theorem states for the considered class of systems stability conditions in terms of LMIs for P, K and α and thus provides a way to calculate the terminal region $\mathcal{E}(\alpha)$ and the terminal penalty term E defined in (12)-(14):

Theorem 1: If the open-loop optimal control problem (5) has a feasible solution at time $t = 0$, and if there exist matrices $0 < N_1 = N_1^T \in \mathbb{R}^{n\times n}$ and $N_2 \in \mathbb{R}^{m\times n}$, and a constant $\alpha \in \mathbb{R}^+$ such that the LMIs

$$\begin{bmatrix} -\Delta - \Delta^T & N_1 Q^{\frac{1}{2}} & N_2^T R^{\frac{1}{2}} \\ Q^{\frac{1}{2}} N_1 & I & 0 \\ R^{\frac{1}{2}} N_2 & 0 & I \end{bmatrix} > 0, \tag{22}$$

$$\begin{bmatrix} \frac{1}{\alpha} & c_i N_1 + d_i N_2 \\ (c_i N_1 + d_i N_2)^T & N_1 \end{bmatrix} \geq 0, \tag{23}$$

$$-HN_1 = G^T, \tag{24}$$

are satisfied for $i = 1, \ldots, r$ with $\Delta = [A\ B][N_1\ N_2^T]^T$, then the closed-loop of the NMPC scheme defined by (1)-(8) is stable in the sense that $\|x\| \to 0$ as $t \to \infty$, where $P = N_1^{-1}$ and $K = N_2 N_1^{-1}$.

Proof: Closed-loop stability of the NMPC scheme defined by (1)-(8) can be guaranteed if the conditions of Lemma 1 hold. Hence, one has to show that the conditions of Theorem 1 imply satisfaction of the conditions of Lemma 1. With the choice of $E(x) = x^T P x$, condition 1 of Lemma 1 is obviously satisfied $\forall x \in \mathcal{E}(\alpha)$ since $P = N_1^{-1} > 0$.
The stabilizing control law $k(x)$ in Lemma 1 is chosen to $u = k(x) = Kx$, where $K = N_2 N_1^{-1}$. Applying the Schur complement on (23), one obtains

$$\frac{1}{\alpha} - (c_i N_1 + d_i N_2) N_1^{-1} (c_i N_1 + d_i N_2)^T \geq 0, \tag{25}$$

$i = 1, \cdots, r$. Since $N_1 > 0$, $P = N_1^{-1}$, $K = N_2 N_1^{-1}$, and $w_i = c_i + d_i K$, this inequality is equivalent to

$$\frac{1}{\alpha} - w_i P^{-1} w_i^T \geq 0,\ i = 1, \ldots, r. \tag{26}$$

Thus, it follows from Lemma 2 that $\mathcal{E}(\alpha) \subseteq \mathcal{X}$ since $\mathcal{E}(\alpha) \subseteq \mathcal{W}$. Furthermore, the chosen terminal region $\mathcal{E}(\alpha) = \{x \in \mathcal{X} : x^T P x \leq \alpha\}$ is closed and connected and contains the origin. In addition, (26) implies that $u = k(x) = Kx$ satisfies the input constraints, and $k(x) = Kx$ clearly fulfills the requirement $k(0) = 0$. Thus, to proof condition 3 in Lemma 1 it remains to show that (9) holds $\forall\ x \in \mathcal{E}(\alpha)$. In the case of the considered class of Lure systems and the choice of $E(x)$ and $F(x, u)$, inequality (9) becomes

$$\begin{aligned} x^T(A^T P + PA + K^T B^T P + PBK)x \\ +x^T(Q + K^T RK)x \\ +\gamma^T(z) G^T P x + x^T P G \gamma(z) \quad < \quad 0. \end{aligned} \tag{27}$$

Using (24), it follows that

$$\gamma^T(z) G^T P x + x^T P G \gamma(z) \leq 0 \tag{28}$$

is satisfied since $G^T P = -H$ and $z^T \gamma(z) \geq 0$. Thus, inequality (27) holds if

$$\begin{aligned} x^T(A^T P + PA + K^T B^T P + PBK)x \\ +x^T(Q + K^T RK)x \quad < \quad 0 \end{aligned} \tag{29}$$

is fulfilled. This inequality holds for all x if

$$A^T P + PA + K^T B^T P + PBK + Q + K^T RK < 0. \tag{30}$$

Applying the Schur complement on (22), with $N_1 > 0$, $P = N_1^{-1}$, $K = N_2 N_1^{-1}$ one obtains (30). Hence, if (22) and (24) hold, condition (9) of Lemma 1 holds and therefore condition 3 of Lemma 1. Additionally, by assumption the open-loop optimal control problem (5) is feasible at $t = 0$. Therefore, all requirements for Lemma 1 are fulfilled and thus the closed-loop is stable in the sense that $\|x\| \to 0$ as $t \to \infty$. ■

One drawback of the approach presented in this subsection is the equality condition (24) which is required to guarantee closed-loop stability. For certain systems this condition may be too restrictive such that the set of LMIs (22)-(24) do not posses a feasible solution. Therefore, in the following subsection Lure systems with a more restrictive sector condition are considered. The conditions on the nonlinearities of such systems are used to overcome the solvability problem of the set of LMIs.

B. Lure systems with growth restricted nonlinearities

In the following we consider Lure systems which satisfy

$$\big(mz - \gamma(z)\big)^T \gamma(z) \geq 0, \tag{31}$$

where $0 < m < \infty$. Inequality (31) can be expressed in matrix form as

$$\begin{bmatrix} x \\ \gamma \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}H^T m^T \\ -\frac{1}{2}mH & I \end{bmatrix} \begin{bmatrix} x \\ \gamma \end{bmatrix} \leq 0. \tag{32}$$

As in the previous subsection, the nonlinearities $\gamma(z)$ lie either in the first or in the third quadrant, but are growth bounded. Loosely speaking, this means that the nonlinearities $\gamma(z)$ lie in a sector which is defined by the z-axis and the line mz. The following theorem provides a structured way to calculate a suitable terminal penalty term E and terminal region $\mathcal{E}(\alpha)$ as defined in (12)-(14) to guarantee closed-loop stability:

Theorem 2: If the open-loop optimal control problem (5) has a feasible solution at time $t = 0$, and if there exist matrices $0 < N_1 = N_1^T \in \mathbb{R}^{n\times n}$ and $N_2 \in \mathbb{R}^{m\times n}$, and constants $\tau \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$ such that the inequalities

$$\begin{bmatrix} -\Delta - \Delta^T & -\Gamma & N_1 Q^{\frac{1}{2}} & N_2^T R^{\frac{1}{2}} \\ -\Gamma^T & \tau I & 0 & 0 \\ Q^{\frac{1}{2}} N_1 & 0 & I & 0 \\ R^{\frac{1}{2}} N_2 & 0 & 0 & I \end{bmatrix} > 0, \tag{33}$$

$$\begin{bmatrix} \frac{1}{\alpha} & c_i N_1 + d_i N_2 \\ (c_i N_1 + d_i N_2)^T & N_1 \end{bmatrix} \geq 0, \tag{34}$$

are satisfied for $i = 1,\ldots,r$ with $\Delta = [A\ B][N_1\ N_2^T]^T$ and $\Gamma = G + \frac{\tau}{2} N_1 H^T m^T$, then the closed-loop of the NMPC scheme (1)-(8) is stable in the sense that $\|x\| \to 0$ as $t \to \infty$, where $P = N_1^{-1}$ and in (18) $K = N_2 N_1^{-1}$.

Proof: Using the results of the proof of Theorem 1 it follows that Theorem 2 guarantees stability of the closed-loop if it can be shown that inequality (27) holds for all x and γ satisfying (32). Writing (27) in matrix form as

$$\begin{bmatrix} x \\ \gamma \end{bmatrix}^T \begin{bmatrix} \begin{bmatrix} A^T P + PA + K^T B^T P \\ +PBK + Q + K^T RK \end{bmatrix} & PG \\ G^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ \gamma \end{bmatrix} < 0, \tag{35}$$

and applying the $\mathcal{S}$-Procedure, see e.g. [1], it follows that (35) respectively (27) holds for all x and γ satisfying (32) if there exists a non-negative $\tau \in \mathbb{R}^+$ such that the inequality

$$\begin{bmatrix} \begin{bmatrix} A^T P + PA + K^T B^T P \\ +PBK + Q + K^T RK \end{bmatrix} & PG + \frac{\tau}{2} H^T m^T \\ G^T P + \frac{\tau}{2} mH & -\tau I \end{bmatrix} < 0 \tag{36}$$

is satisfied. Multiplying (36) from both sides with the matrix $\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}$, substituting $N_1 = P^{-1}$, $N_2 = KP^{-1}$, and introducing Δ and Γ, as defined in Theorem 2, one obtains

$$\begin{bmatrix} \Delta + \Delta^T & \Gamma \\ \Gamma^T & -\tau I \end{bmatrix} + \begin{bmatrix} N_1 Q^{\frac{1}{2}} & N_2^T R^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Q^{\frac{1}{2}} N_1 & 0 \\ R^{\frac{1}{2}} N_2 & 0 \end{bmatrix} \leq 0. \tag{37}$$

Applying the Schur complement to (37) one obtains (33). Therefore, in combination with the results of the proof of Theorem 1 it is shown that Theorem 2 satisfies all requirements for Lemma 1 and thus, the closed-loop is stable in the sense that $\|x\| \to 0$ as $t \to \infty$. ∎

Stability of the closed-loop is already achieved if inequality (31), i.e. the sector condition of the considered system, is satisfied for all x in the state constraint set $\mathcal{X}$, since $\mathcal{E} \subseteq \mathcal{X}$ and (27) has to hold for all $x \in \mathcal{E}$. This means that also nonlinearities as for example $\gamma(z) = z^3$ can be considered as long as they satisfy (31) in the terminal region $\mathcal{E}$.

Remark 2: Inequality (33) does not represent an LMI if τ is considered as a free variable. However, one can apply e.g. a line search algorithm for τ, i.e. (33) is solved for several fixed values of τ as an LMI multiple times to obtain the desired solution.

One main advantage of the approaches presented is that the LMIs (22)-(24), and the inequalities (33)-(34), respectively, are easy to solve. Thus, the terminal region and the terminal penalty term can be calculated without facing computational problems. Usually, it is desirable to maximize the terminal region $\mathcal{E}(\alpha)$ in order to maximize the feasibility region of the NMPC scheme, see e.g. [4]. The procedure to compute a terminal region and a terminal penalty term for a stabilizing NMPC controller usually is the following: A terminal penalty matrix P is chosen and for the chosen P the constant α is computed such that the terminal region $\mathcal{E}(\alpha)$ is maximized, see e.g. [3, 6]. The LMIs (22)-(24) and the inequalities (33)-(34), respectively, provide the simultaneous calculation of P and α. Thus, both degrees of freedom can be used to maximize the terminal region. Since the volume of $\mathcal{E}(\alpha)$ is up to a constant $\alpha\det(P^{-1})$ the corresponding maximization problem is

$$\max_{\alpha,P,K} \alpha\ \det(P^{-1}), \tag{38}$$

subject to the LMIs (22)-(24) and the inequalities (33)-(34), respectively. This is a non-convex optimization problem, however, it can be transferred into a convex optimization problem, see [4] for details.

Remark 3: The presented approaches can also be used to design a linear controller which stabilizes the origin of the considered (nonlinear) system while satisfying the constraints. The nonlinearities do not have to be known exactly to obtain a stabilizing linear control law. Closed-loop stability is already obtained if it can be assured that the nonlinearities satisfy the required sector condition. Thus, the linear controller design is robust towards all nonlinearities satisfying the considered sector condition.

IV. Illustrative Examples

Two examples are presented to illustrate the derived methods to calculate the terminal region and the terminal penalty term. The approach presented in Section III-A is applied to an academic example which satisfies the complete sector condition. Furthermore, a model of a flexible joint robotic arm with growth bounded nonlinearities is used to illustrate the method introduced in Section III-B.

A. Academic Example

To illustrate the results of Section III-A, the Lure system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + x_3 + u \\ \dot{x}_3 &= -x_1 + 1.5x_2 - 2x_3 - 0.7x_3^3, \end{aligned} \tag{39}$$

is considered. Thus, the system is described by the matrices

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1.5 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\ G^T &= \begin{bmatrix} 0 & 0 & -0.7 \end{bmatrix}, \; H = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{40}$$

and the nonlinearity $\gamma(z) = z^3$, where $z = x_3$. The nonlinearity of the example system satisfies the complete sector condition $z^T\gamma(z) \geq 0$.

The constraint set $\mathcal{W}$ is defined by the input constraints $-3 \leq u \leq 3$ and the state constraints $-0.5 \leq x_1 \leq 0.5$ and $-0.75 \leq x_2, x_3 \leq 0.75$. The task is to calculate a terminal penalty term E and a terminal region $\mathcal{E}$ for the NMPC scheme (1)-(8), where E, $\mathcal{E}$ and the stage cost F are defined in (12)-(14), such that the NMPC controller stabilizes the origin of the example system (39) while satisfying the constraints. For the stage cost F defined in (12) the matrices Q and R are chosen to be identity matrices of suitable dimensions. The solution of the set of LMIs described in (22)-(24) delivers the linear feedback matrix

$$K = \begin{bmatrix} -2.5818 & -2.9678 & -0.6128 \end{bmatrix}, \tag{41}$$

and the terminal penalty matrix

$$P = \begin{bmatrix} 5.2109 & 2.7336 & 0 \\ 2.7336 & 3.7426 & 0 \\ 0 & 0 & 1.4286 \end{bmatrix}, \tag{42}$$

and thus, with $\alpha = 0.8036$, the terminal penalty term $E(x) = x^TPx$ and the terminal region $\mathcal{E}(\alpha) = \{x \in \mathcal{W} : x^TPx \leq \alpha\}$. These matrices satisfy the conditions of Lemma 1 and thus the NMPC scheme is stable in the sense that $\|x\| \to 0$ as $t \to \infty$. Figure 1 shows the calculated terminal region $\mathcal{E}(\alpha) = \{x \in \mathcal{W} : x^TPx \leq \alpha\}$ in the constraint set $\mathcal{W}$.

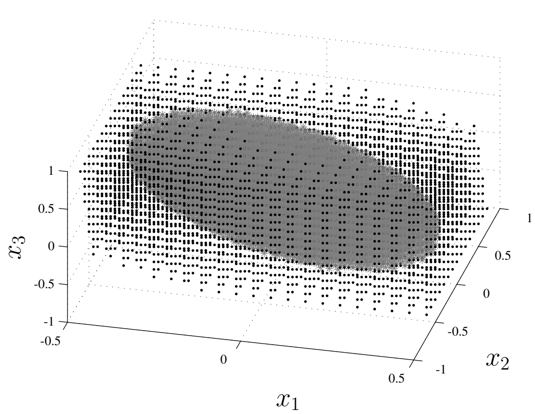

Fig. 1. Terminal region $\mathcal{E}(\alpha)$ in the constraint set $\mathcal{W}$.

B. Flexible Joint Robotic Arm

The approach presented in Section III-B is applied to a flexible joint robotic arm as shown in Figure 2, see e.g. [9]. The dynamics of the robotic arm are given by the matrices

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -16.17 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, \\ G^T &= \begin{bmatrix} 0 & 0 & 0 & -3.33 \end{bmatrix}, \; H = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}. \end{aligned} \tag{43}$$

and the nonlinearity

$$\gamma(z) = \sin z + z. \tag{44}$$

One can observe that the nonlinearities contain, in contrast to the model in [9], a linear part. This term was added in order to satisfy the sector condition and is compensated in the linear dynamics matrix A. Thus, the dynamics of the robotic arm correspond to those in [9]. To fulfill the sector condition (31) the following inequality has to hold

$$\big(mz - \sin(z) - z\big)^T \big(\sin(z) + z\big) \geq 0. \tag{45}$$

This is obviously the case for all $m \geq 2$. In the following the constant m is chosen to be $m = 2$. The constraint set $\mathcal{W}$ is defined by the input constraints $-2 \leq u \leq 2$ and the state constraints $-\frac{\pi}{2} \leq x_1, x_3 \leq \frac{\pi}{2}$ and $-5 \leq x_2, x_4 \leq 5$. The task is to calculate the terminal region $\mathcal{E}$ and the terminal penalty term E via the inequalities (33)-(34), where $\mathcal{E}$, E and F (respectively Q and R) are defined as in the previous example. Using a line search algorithm and solving $(33)-(34)$ with fixed values for τ (see Remark 2) as

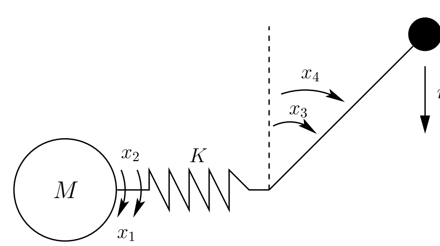

Fig. 2. Flexible joint robotic arm.

LMIs, the largest terminal region $\mathcal{E}$ is obtained for $\tau = 7.7$. The corresponding solution of the LMIs (33)-(34) delivers the linear feedback matrix

$$K = \begin{bmatrix} -1.8582 & -0.4410 & 0.6562 & -0.5066 \end{bmatrix}, \quad (46)$$

and the terminal penalty matrix

$$P = \begin{bmatrix} 17.6206 & 1.0311 & -10.8430 & 3.3873 \\ 1.0311 & 0.1626 & -0.6668 & 0.2020 \\ -10.8430 & -0.6668 & 15.4697 & -1.1346 \\ 3.3873 & 0.2020 & -1.1346 & 1.1990 \end{bmatrix}, \quad (47)$$

and thus, with $\alpha = 2.5503$, the terminal penalty term $E(x) = x^T P x$ and the terminal region $\mathcal{E}(\alpha) = \{x \in \mathcal{W} : x^T P x \leq \alpha\}$. Interestingly, for the robotic arm a solution of the set of LMIs derived in Section III-A does not exist due to the equality constraint (24).

V. Conclusions

In the first part of this paper a brief overview of some aspects of stabilizing predictive control has been given. To satisfy the conditions for NMPC stability, a suitable terminal penalty term and terminal region have to be calculated. This is in general a nontrivial task. Therefore, in the second part a subclass of nonlinear systems, namely Lure systems, has been considered and two methods have been introduced which both provide a structured approach to calculate the terminal penalty term and the terminal region via LMIs. The first method is applicable to Lure systems satisfying a complete sector condition and thus it is applicable to a broader class of systems. However, it requires that a rather restrictive equality constraint is satisfied which can lead to solvability problems of the obtained set of LMIs. Therefore, in a second step Lure systems with more restrictive, growth bounded nonlinearities have been considered. The conditions on these nonlinearities have been used to overcome the solvability problem of the first method. In the last part two examples, one for each solution approach, have illustrated the results of this paper.

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