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A State Observer for LTI Systems with Delayed Outputs: Time-Varying Delay

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***Abstract*— This paper analyzes the stability properties of a state observer estimating the system states from delayed measurements for a linear time invariant plant. The delay is assumed to be a known piecewise constant function of time. The observer construction is a two step procedure and has a "chain-like" structure, consisting of two cascaded dynamical systems. The manifestation of the time-varying delayed output on the observer stability is analyzed at both the "zeroth" and the "first" links in the chain of observers.**

I. Introduction

In several real-time applications, all the state variables may not be available for direct measurements. Hence a state observer is generally employed to reconstruct the unmeasurable state variables. The observation problem is complicated if the system output is available after a delay interval. Studies have shown that stable "time-delay" observers can be realized in practice under suitable conditions [1]-[4]. Careful scrutiny of research work reveal though that while the delays associated with states [10]-[12] have been studied in great detail, the setbacks due to output delay ramifications are rarely addressed.

Notable research to estimate the states in the presence of delayed outputs include constant-gain observer design [9] and chain observer for observing the states from delayed outputs [6]-[8]. In the latter, the chain of state observers rebuild the systems states at different time-delay instants within the delay window. The main assumption is that the delay is known and a constant. It is seen that if the delay is known but time-varying the observation errors are quite large.

In this paper, the chain observer with a cascade of two systems [6] estimating the states from the delayed output is considered. The delay is modeled as a piecewise constant. We analyze the impact of the time varying nature of the delay on the stability of observer system. In the process, we derive conditions under which exponential stability of the observer error dynamics is guaranteed.

The paper is organized as follows. In section 2, the mathematical description of the single-input single-output (SISO) linear systems with delayed output is presented. In section 3, the dynamics and the stability concepts of the observer are discussed. In section 4, the theoretical results are verified through simulation studies performed on a simple linear system and finally the conclusions are presented.

II. Linear systems subject to output delays

The class of single-input-single-output (SISO) systems considered in this paper are represented as,

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \quad t \geq \Delta, \quad \boldsymbol{x}(-\Delta_1) = \bar{\boldsymbol{x}} \\ \bar{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t - \Delta(t))\end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector. $u(t) \in \mathbb{R}$ is the system input. $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant matrices with appropriate dimensions. $\bar{y}(t)$ represents the delayed output while the undelayed output is represented as $y(t) = \boldsymbol{C}\boldsymbol{x}(t)$.

It is assumed that the system in (1) is observable i.e., the pair $(\boldsymbol{A}, \boldsymbol{C})$ is stabilizable. Alternately, there exists a suitably dimensioned gain matrix K such that $\mathrm{Re}\left(\lambda_i(\boldsymbol{A} - \boldsymbol{K}C)\right) < 0,\ i = 1, \ldots, N$, where $\lambda_i(\cdot)$ denotes the i^{th} eigenvalue of $(\cdot)$. We denote, $\boldsymbol{A}_m = \boldsymbol{A} - \boldsymbol{K}C$.

Δ is a function of time, i.e, its magnitude varies randomly with time and $0 < \Delta(t) \leq \tilde{\Delta}$. $\tilde{\Delta}$ is the upper bound on all the delays. The output delay is piecewise constant, i.e, the value Δ_i is a constant in the interval $(t_{i-1},\ t_i)$. For the next interval $(t_i,\ t_{i+1})$, it assumes a different value. The time intervals $(t_i - t_{i-1}), \quad \forall i = 1 \ldots N.$ are assumed to be of unequal widths of size Δt_i. We assume that this delay profile is known (see Fig. (1)).

III. Main Result

A chain observer for state estimation of output delayed LTI systems (described above) is derived. The observer structure is motivated by a construction similar to the one in [5]. We propose a cascade structure involving two dynamical systems for the observer dynamics for estimating the states using the delayed output measurements.

$$\dot{\hat{\boldsymbol{x}}}_0(t) = \boldsymbol{A}\hat{\boldsymbol{x}}_0(t) + \boldsymbol{B}u_0(t) + \boldsymbol{K}(\bar{y}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}_0(t)) \tag{2}$$

$$\dot{\hat{\boldsymbol{x}}}_1(t) = \boldsymbol{A}\hat{\boldsymbol{x}}_1(t) + \boldsymbol{B}u_1(t) + e^{\boldsymbol{A}\Delta(t)}\boldsymbol{K}(\bar{y}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}_1(t)) \tag{3}$$

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The variable $\hat{\boldsymbol{x}}_0(t)$ denotes the estimate of the states at a time $t-\Delta(t)$ and $\hat{\boldsymbol{x}}_1$ denotes the estimate of the states at the current time t. Likewise, $u_0(t)$ is the system input at time $t-\Delta(t)$ and u_1 denotes the estimate of the states at the current time t. The observer states are initialized as follows

$$\begin{aligned}\hat{\boldsymbol{x}}_0(0) &= \hat{\boldsymbol{x}}(-\Delta_1)\\ \hat{\boldsymbol{x}}_1(\tau) &= \hat{\boldsymbol{x}}(0)\end{aligned} \tag{4}$$

Δ_1 is the output delay magnitude in the interval $t_0 \leq t < t_1$ and τ is the initial time, t_0

Referring to the cascade structure above, we denote the observer dynamics in (2) as the zeroth-observer and the dynamics in (3) as the first-observer.

Theorem 3.1: Given the SISO system in (1) subject to time-varying, piece-wise constant measurement delays, the observer system in (2) and (3) with initial conditions specified in (4) ensures stable observation error dynamics, i.e., $\|\boldsymbol{x}(t)-\hat{\boldsymbol{x}}_1(t)\| \leq \epsilon,\ \epsilon > 0$. For the specific case when the measurement delays are constant, the observation errors, $\|\boldsymbol{x}(t)-\hat{\boldsymbol{x}}_1(t)\| \rightarrow 0$

Proof: The proof is detailed in the sections below and is organized as follows. We first prove the stability of the *zeroth-observer*. It is shown that the errors associated with observing the states at $\boldsymbol{x}(t-\Delta(t))$ are bounded. The second part of the proof shows the boundedness of the errors associated with observing the states at $\boldsymbol{x}(t)$, namely the stability of the *first-observer*. ■

A. Stability analysis of the zeroth-observer

In this section, the effects of the change in the delay at specific instants of time on the observer error dynamics is analyzed. Firstly over any interval $[t_{i-1}\ t_i]$ where the delay is constant (say Δ_c), the observation error is defined as

$$\boldsymbol{\eta}_0(t) = \boldsymbol{x}(t-\Delta_c) - \hat{\boldsymbol{x}}(t-\Delta_c) \tag{5}$$

The time derivative of $\boldsymbol{\eta}_0(t)$ in this interval over which the delay is a constant is

$$\begin{aligned}\dot{\boldsymbol{\eta}}_0(t) &= \dot{\boldsymbol{x}}(t-\Delta_c) - \dot{\hat{\boldsymbol{x}}}(t-\Delta_c) \qquad (6)\\ &= \boldsymbol{A}\boldsymbol{x}(t-\Delta_c) + \boldsymbol{B}u(t-\Delta_c)\\ &\quad -\{\boldsymbol{A}\hat{\boldsymbol{x}}(t-\Delta_c) + \boldsymbol{B}u(t-\Delta_c) + \boldsymbol{K}\boldsymbol{C}\boldsymbol{\eta}_0(t)\}\\ \dot{\boldsymbol{\eta}}_0(t) &= \boldsymbol{A}_m\boldsymbol{\eta}_0(t) \qquad (7)\end{aligned}$$

where, as mentioned before $\boldsymbol{A}_m = \boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ is Hurwitz.

Next, the error in observation due to changes in the delay magnitude is analyzed.

- At $t = t_1$ the zeroth observer estimates the states at the instant $t_1 - \Delta_1$, while the first observer estimates the states at t_1.

- At the same instant, the output delay changes to a new value, say Δ_2.

- This implies that the *zeroth-observer* would now have to start estimating the states from $t_1 - \Delta_2$. This is due to the fact that, with the change in Δ, the initial condition for the *zeroth-observer* changes correspondingly for the interval over which this new delay value remains a constant i.e, at $t = 0$, the initial condition for the *zeroth-observer* is $\hat{\boldsymbol{x}}(-\Delta_1)$ in the interval $[0,\ t_1]$ and with the change in delay at $t = t_1$, the initial condition for the *zeroth-observer* is $\hat{\boldsymbol{x}}(t_1-\Delta_2)$ in the interval $[t_1,\ t_2]$.

- The estimated values for this interval $[t_1,\ t_2]$ need not be specified explicitly because it is available from the output of the *first-observer*.

- The *zeroth-observer* now starts estimating the states as though they are propagating from t_1 with an initial condition starting from $t-\Delta_2$.

- Hence there is a finite change in the observer states from $t_1-\Delta_1$ to $t_1-\Delta_2$. This contributes to the error in the observation which is equal to $\hat{\boldsymbol{x}}(t_1-\Delta_2)-\hat{\boldsymbol{x}}(t_1-\Delta_1)$.

Differentiating $\hat{\boldsymbol{x}}(t_1-\Delta_2)-\hat{\boldsymbol{x}}(t_1-\Delta_1)$, substituting the observer dynamics and after simplifying one obtains,

$$\begin{aligned}&\dot{\hat{\boldsymbol{x}}}(t_1-\Delta_2)-\dot{\hat{\boldsymbol{x}}}(t_1-\Delta_1) =\\ &\qquad = (\boldsymbol{A}-\boldsymbol{K}\boldsymbol{C})(\hat{\boldsymbol{x}}(t_1-\Delta_2)-\hat{\boldsymbol{x}}(t_1-\Delta_1))\\ &\qquad +\boldsymbol{B}(u(t_1-\Delta_2)-u(t_1-\Delta_1))\\ &\qquad = \boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}1}(t) + \boldsymbol{B}\boldsymbol{\delta}_{\boldsymbol{u}1}(t)\end{aligned} \tag{8}$$

In (8), $\boldsymbol{\delta}_{\hat{\boldsymbol{x}}1}(t)$ represents the finite error between the observer states due to change in the delay value at $t = t_i$. Similarly, $\boldsymbol{\delta}_{u1}(t)$ denotes the error in the inputs due to the change in the delay. Thus, generalizing (8), at any instant $t = t_i$, when the magnitude of the output delay changes, the error in observer due to the change in initial conditions can be expressed as

$$\dot{\hat{\boldsymbol{x}}}(t_i-\Delta_{i+1})-\dot{\hat{\boldsymbol{x}}}(t_i-\Delta_i) = \boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}i}(t) + \boldsymbol{B}\boldsymbol{\delta}_{\boldsymbol{u}i}(t) \tag{9}$$

where Δ_i is the delay in the interval $t_{i-1} \leq t < t_i$ and Δ_{i+1} is the delay in the next interval $t_i \leq t < t_{i+1}$. $\boldsymbol{\delta}_{\hat{\boldsymbol{x}}i}(t) = \hat{\boldsymbol{x}}(t_i-\Delta_{i+1})-\hat{\boldsymbol{x}}(t_i-\Delta_i)$ and $\boldsymbol{\delta}_{ui}(t) = \boldsymbol{u}(t_i-\Delta_{i+1})-\boldsymbol{u}(t_i-\Delta_i)$. Note, that this error exists because, the *first-observer* has not yet converged to the true states at $t = t_i - \Delta_{i+1}$. Substituting (9) in (6), for any $t = t_i$ and simplifying,

$$\dot{\boldsymbol{\eta}}_0(t_i) = \boldsymbol{A}_m\boldsymbol{\eta}_0(t_i) + \boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}i}(t) + \boldsymbol{B}\boldsymbol{\delta}_{ui}(t) \tag{10}$$

Therefore, the observation error dynamics can be written as

$$\dot{\boldsymbol{\eta}}_0(t) = \begin{cases} \boldsymbol{A}_m\boldsymbol{\eta}_0(t) + \boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}i}(t) + \boldsymbol{B}\boldsymbol{\delta}_{ui}(t), & \text{at } t > t_i\\ \boldsymbol{A}_m\boldsymbol{\eta}_0(t) & \text{for } t \in [t_{i-1},\ t_i)\end{cases} \tag{11}$$

The delay profile assumed at the observer is shown in Fig. (1).

Next, a general expression for $\boldsymbol{\eta}_0(t)$ for any instant $t = t_N$, $N = 0,\ 1,\ 2, \ldots, \infty$ at which the delay magnitude changes

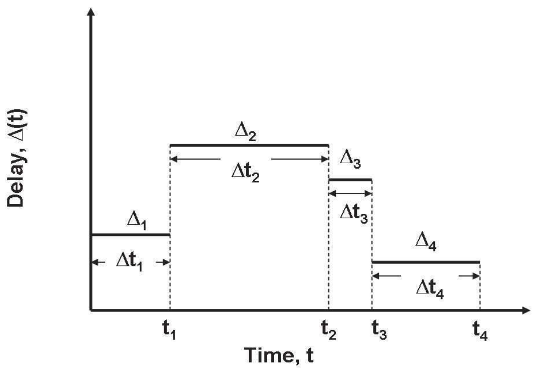

Fig. 1. Profile of time-varying delay.

is derived. The procedure is as follows:

The solution of the dynamic equation (11) at $t = t_1$ is

$$\begin{aligned}
&\boldsymbol{\eta}_0(t_1) = \exp(\boldsymbol{A}_m(t_1 - 0))\boldsymbol{\eta}_0(0) \\
&\quad + \int_0^{t_1} \exp(\boldsymbol{A}_m(t_1 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}1}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u1}(s)\right\} ds \\
&\|\boldsymbol{\eta}_0(t_1)\| \leq \|\exp(\boldsymbol{A}_m \Delta t_1)\boldsymbol{\eta}_0(0)\| \\
&+ \left\| \int_0^{t_1} \exp(\boldsymbol{A}_m(t_1 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}1}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u1}(s)\right\} ds \right\| \\
&\leq \|\exp(\boldsymbol{A}_m \Delta t_1)\|.\|\boldsymbol{\eta}_0(0)\| \\
&+ \left\| \int_0^{t_1} \exp(\boldsymbol{A}_m(t_1 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}1}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u1}(s)\right\} ds \right\|
\end{aligned} \tag{12}$$

Since $\boldsymbol{A}_m$ is Hurwitz, we have $\|\exp(\boldsymbol{A}_m t)\| \leq \exp(-mt)$, where m is the real part of the minimum eigenvalue of $\boldsymbol{A}_m$

Denoting

$$\bar{\boldsymbol{\delta}}_1 = \left\| \int_0^{t_1} \exp(\boldsymbol{A}_m(t_1 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}1}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u1}(s)\right\} ds \right\|$$

(12) can be re-written as

$$\|\boldsymbol{\eta}_0(t_1)\| \leq \exp(-m\Delta t_1) \|\boldsymbol{\eta}_0(0)\| + \bar{\boldsymbol{\delta}}_1 \tag{13}$$

Next, at $t = t_2$, the error vector is given as

$$\begin{aligned}
&\boldsymbol{\eta}_0(t_2) = \exp(\boldsymbol{A}_m \Delta t_2)\boldsymbol{\eta}_0(t_1) \\
&\quad + \int_{t_1}^{t_2} \exp(\boldsymbol{A}_m(t_2 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}2}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u2}(s)\right\} ds
\end{aligned} \tag{14}$$

and similarly as before we obtain,

$$\begin{aligned}
&\|\boldsymbol{\eta}_0(t_2)\| \leq \|\exp(\boldsymbol{A}_m \Delta t_2)\boldsymbol{\eta}_0(t_1)\| \\
&+ \left\| \int_{t_1}^{t_2} \exp(\boldsymbol{A}_m(t_2 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}2}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u2}(s)\right\} ds \right\| \\
&\text{i.e., } \|\boldsymbol{\eta}_0(t_2)\| \leq \exp(-m\Delta t_2)\|\boldsymbol{\eta}_0(t_1)\| + \bar{\boldsymbol{\delta}}_2
\end{aligned} \tag{15}$$

Now substituting for $\|\eta_0(t_1)\|$ from (13), we get

$$\begin{aligned}
\|\boldsymbol{\eta}_0(t_2)\| &\leq \exp(-m\Delta t_2) \exp\{-m\Delta t_1\} \|\boldsymbol{\eta}_0(0)\| \\
&\quad + \exp(-m\Delta t_2)\bar{\boldsymbol{\delta}}_1 + \bar{\boldsymbol{\delta}}_2 \\
&\leq \exp(-m(\Delta t_1 + \Delta t_2)) \|\boldsymbol{\eta}_0(0)\| \\
&\quad + \exp(-m\Delta t_2)\bar{\boldsymbol{\delta}}_1 + \bar{\boldsymbol{\delta}}_2
\end{aligned} \tag{16}$$

where

$$\bar{\boldsymbol{\delta}}_2 = \left\| \int_{t_1}^{t_2} \exp(\boldsymbol{A}_m(t_2 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}2}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u2}(s)\right\} ds \right\|$$

Continuing as above, the error vector at any instant $t = t_N$ can be written as

$$\begin{aligned}
\|\boldsymbol{\eta}_0(t_N)\| &\leq \exp\left(-m\sum_{i=1}^{N} \Delta t_i\right) \|\boldsymbol{\eta}_0(0)\| \\
&\quad + \sum_{j=1}^{N-1} \exp\left(-m \sum_{i=j+1}^{N} \Delta t_i\right) \bar{\boldsymbol{\delta}}_j + \bar{\boldsymbol{\delta}}_N
\end{aligned} \tag{17}$$

where for $i = 1,\ 2, \ldots, N$

$$\bar{\boldsymbol{\delta}}_i = \left\| \int_{t_{i-1}}^{t_i} \exp(\boldsymbol{A}_m(t_i - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}i}(s) + \boldsymbol{B}\boldsymbol{\delta}_{ui}(s)\right\} ds \right\|$$

Remark 1: In (17), the term $\bar{\boldsymbol{\delta}}_i$ represents the error in observation due to the delay variations. This error is multiplied by an exponential term, which decays to zero and hence as $t \to \infty$, these cumulative delay errors also decay to zero. If the delay changes very fast, i.e, Δt_i is very small, then the cumulative error due the delay changes will be larger. Similarly, if the delay changes at a very slow rate, then the cumulative error as time increases will be of smaller magnitude and the observer will be able to track the system states much better.
Remark 2: From (17), it can be seen that at any $t = t_N$, there will be a residual non-decaying error term $\bar{\boldsymbol{\delta}}_N$. This implies that the observation error will be equal to a finite non-zero value given by the magnitude of the term $\bar{\boldsymbol{\delta}}_N$. It also implies that the observation error can be expected to lie in a residual set of magnitude defined by the error term $\bar{\boldsymbol{\delta}}_N$. The residuals at $t = t_N$ due to earlier delay changes would have decayed to very small values due to the multiplying exponential term.
Remark 3: In (17), if there were no delay changes, i.e the measurement delay was a constant over the entire time horizon, then there would be no residual errors, ie. $\bar{\boldsymbol{\delta}}_i = 0$. and (17) reduces to

$$\|\boldsymbol{\eta}_0(t_N)\| \leq \exp(-mt_N) \|\boldsymbol{\eta}_0(0)\| \tag{18}$$

which is the expression for observation error with constant delays.
Remark 4: If all the delays intervals were of equal width, i.e $\Delta t_i = \Delta t_j, \quad i \neq j$, then the observation error at $t = t_N$

is given as

$$\|\boldsymbol{\eta}_0(t_N)\| \leq \exp\left(-mN\Delta t\right)\|\boldsymbol{\eta}_0(0)\| + \sum_{j=1}^{N} \exp\left(-m(N-j)\Delta t\right)\bar{\boldsymbol{\delta}}_j \tag{19}$$

B. Stability analysis of the first-observer

Using the state transition matrix, the system state at t i.e, $\boldsymbol{x}_1(t)$ can be written as

$$\boldsymbol{x}_1(t) = \exp(\boldsymbol{A}\Delta(t))\boldsymbol{x}_0(t) + \int_{t-\Delta(t)}^{t} \exp(\boldsymbol{A}(t-s))\boldsymbol{B}u(s)ds \tag{20}$$

Similarly, the observer states estimating the system states at t can be expressed as:

$$\hat{\boldsymbol{x}}_1(t) = \exp(\boldsymbol{A}\Delta(t))\hat{\boldsymbol{x}}_0(t) + \int_{t-\Delta(t)}^{t} \exp(\boldsymbol{A}(t-s))\boldsymbol{B}u(s)ds \tag{21}$$

Subtracting (21) from (20) we obtain,

$$\boldsymbol{\eta}_1(t) = \exp(\boldsymbol{A}\Delta(t))\boldsymbol{\eta}_0(t) \tag{22}$$

Next, a general expression for the observation error $\boldsymbol{\eta}_1(t)$ at any $t = t_i, i = 1,\ 2, \ldots, N$ is derived as follows:

At $t = t_1$, (22) can be written as

$$\|\boldsymbol{\eta}_1(t_1)\| \leq \|\exp(\boldsymbol{A}\Delta(t))\|.\|\boldsymbol{\eta}_0(t)\| \tag{23}$$

Substituting for $\|\boldsymbol{\eta}_0(t)\|$,

$$\begin{aligned} \|\boldsymbol{\eta}_1(t_1)\| &\leq \alpha_1\{\exp\left(-m\Delta t_1\right)\|\boldsymbol{\eta}_0(0)\| + \bar{\boldsymbol{\delta}}_1\} \\ &\leq \alpha_1 \exp\left\{-m\Delta t_1\right)\|\boldsymbol{\eta}_0(0)\| + \alpha_1\bar{\boldsymbol{\delta}}_1 \end{aligned} \tag{24}$$

where $\|\exp(\boldsymbol{A}\Delta_1)\| \leq \alpha_1$. Similarly, at $t = t_2$, the observation error is

$$\begin{aligned} \|\boldsymbol{\eta}_1(t_2)\| &\leq \alpha_2\{\exp\left(-m(\Delta t_1 + \Delta t_2)\right)\|\boldsymbol{\eta}_0(0)\| \\ &\quad + \exp\left(-m\Delta t_2\right)\bar{\boldsymbol{\delta}}_1 + \bar{\boldsymbol{\delta}}_2\} \\ &\leq \alpha_2 \exp\left(-m(\Delta t_1 + \Delta t_2)\right)\|\boldsymbol{\eta}_0(0)\| \\ &\quad + \alpha_2 \exp(m_1\Delta_1)\bar{\boldsymbol{\delta}}_1 + \alpha_2\bar{\boldsymbol{\delta}}_2 \end{aligned} \tag{25}$$

where $\|\exp(\boldsymbol{A}\Delta_2)\| \leq \alpha_2$. Hence a general expression for the observation error at any instant $t = t_N$ is

$$\begin{aligned} \|\boldsymbol{\eta}_1(t_N)\| &\leq \alpha_N \exp\left(-m\sum_{i=1}^{N}\Delta t_i\right)\|\boldsymbol{\eta}_0(0)\| \\ &\quad + \alpha_N \sum_{j=1}^{N-1} \exp\left(-m\sum_{i=j+1}^{N}\Delta t_i\right)\bar{\boldsymbol{\delta}}_j + \alpha_N\bar{\boldsymbol{\delta}}_N \end{aligned} \tag{26}$$

where $\|\exp(\boldsymbol{A}\Delta_N)\| \leq \alpha_N$.

(26) implies that there will be a finite non-zero observation error at any instant $t = t_N$. This is the error that occurs in the zeroth observer due to the delay changes. Additionally, in the *first-observer*, each of these residual errors is multiplied by a scaling constant $\|\exp(\boldsymbol{A}\Delta_i)\| \leq \alpha_i, \quad i = 1, 2, \ldots, N$. For a non-Hurwitz matrix A, each $\boldsymbol{\alpha}_i$ will amplify the residual error further.

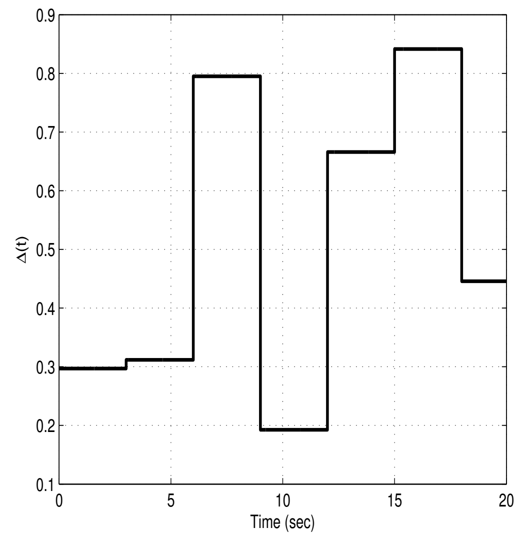

Fig. 2. Profile of time varying delay

If matrix A is Hurwitz then $\|\exp(\boldsymbol{A}\Delta_N)\|$, aids in diminishing the observation errors. Thus, in this case the estimated states will be closer to the true states. Note, m_1 is the smallest eigenvalue of $\boldsymbol{A}$,

IV. Simulation results

The plant model for the actual system and the estimator are in the canonical form with the state space representation as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ \bar{y}(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t - \Delta(t)) \end{aligned}$$

The gain matrix $\boldsymbol{K}$ is chosen such that the eigen values of the matrix $\boldsymbol{A} - \boldsymbol{KC}$ are at -10 and -15 respectively. The input is $u(t) = \sin(t)$. The initial conditions for the linear system and the observer are

$$\boldsymbol{x}(\tau) = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix},\ \hat{\boldsymbol{x}}_0(\tau) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \hat{\boldsymbol{x}}_1(\tau) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \tau \in [-\Delta(t), 0]$$

The chain observer is simulated for a time-delayed output, with the delay, $\Delta(t)$ varying in time. The value of $\Delta(t)$ varies randomly in the range $\Delta \in (0.1, 1)$ seconds and $\Delta t = 3$ seconds.

Figure (2) shows the profile of $\Delta(t)$ w.r.t time. Figure (3) shows the observation error, both transient and steady state responses. The observation error plots are as expected. If there was no change in the delay, then there would have been a peak in the beginning and the errors would have gradually decayed to zero. But the time varying delay

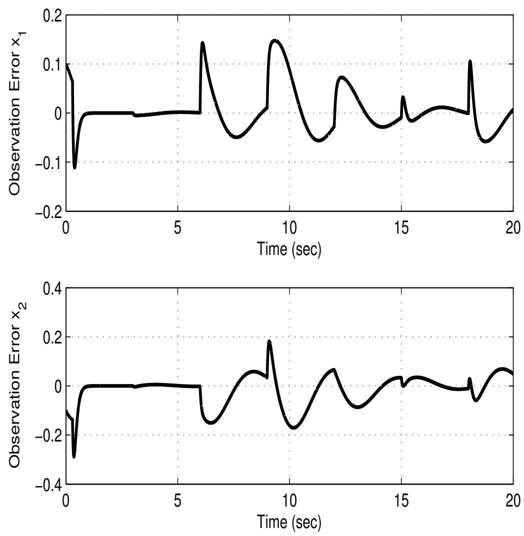

Fig. 3. Observation Error

changes the observation errors significantly, as can been seen from the plots.

Between $t = 0$ and $t = 6$, the change in the delay is small, hence the error reaches the steady state value quickly. But after 6 seconds, the differences in the two consecutive delay magnitudes is larger. This results in larger peak overshoots in the transient response of the observer. The overshoot in the observer states is to accommodate the change in the system output, due to change in the delay magnitude. The observer then settles down to a steady state. But, the changing delays constantly trigger the observer dynamics at every 3 seconds. The observer is able to track the true states fairly accurately.

V. Conclusions and future work

In this paper we have relaxed the assumption that the output delay is constant at all times. This is closer to a practical situation, for example, cooperating dynamical systems that exchange outputs over finite low bandwidth communication channels. The uncertainties in these communication channels contribute to changes in the delay. We have analyzed observer responses to this time-varying delay and in the process derived conditions for stable observer error dynamics. The simulation results also confirm the theoretical results. Current research is underway to extend these results for a situation when the observer may not have a perfect knowledge of the delay interval or the exact magnitude of the delay itself.

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