2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

$\mathcal{D}$-Dissipativity Analysis for descriptor systems

O. Rejichi and O. Bachelier and M. Chaabane and D. Mehdi and T. Al-chaer

Abstract— **The paper is concerned with the $\mathcal{D}$-dissipativity of descriptor systems, that is disspativity with respect to a region in the complex plane. The region can be a half plane or a disc. It naturally covers continuous-time and discrete-time descriptor systems as special cases. Necessary and sufficient LMI conditions are provided to check the $\mathcal{D}$-dissipativity. The constraints associated to the proposed conditions are much weaker than the existing ones.**

I. Introduction

Singular systems also known as descriptor systems are an important class of dynamic system models from both a theoretical and practical point of view due to their capacity in describing algebraic constraints between physical variables [1], [2]. Among basic notions of state-space systems generalized to descriptor systems, dissipativity, which includes positive and bounded realness as special cases, is one of the most important properties of dynamical systems and plays a crucial role in various problems of analysis and synthesis of control systems. In the literature, dissipativity concept was initially introduced by Willems in his seminal two-part papers [3] in terms of an inequality involving a storage function and a supply rate. In many mechanical and electrical engineering applications, dissipativity is related to the notion of energy. Dissipative systems are characterized by the following property: at any moment of time, the amount of the energy the system can supply to its environment can not exceed the amount of energy that has been supplied to it [3]. This type of inequalities have major advantages. The most distinguished advantage in using dissipation inequalities is probably the fact that the investigation of a possibly large number of differential equations, given by the control system description, boils down to a small number of algebraic inequalities. Hence, the complexity of the analysis and the design tasks are usually substantially reduced. Furthermore, dissipation inequalities have immediate connections to stability, optimal control, and robustness. The theory of dissipative systems, or in other terms the Kalman-Popov-Yakuvobich (KYP) Lemma, contains some basic tools, such as passivity theorem, bounded real lemma [4].

For state-space systems, the KYP Lemma is one of the most fundamental results in the field of dynamical systems analysis, feedback control, and signal processing. Various properties of dynamical systems can be characterized by a set of inequality constraints in the frequency domain. The KYP Lemma establishes equivalence between such frequency domain inequality (FDI) for a transfer function and a linear matrix inequality (LMI) for its state space realization [5], [6].

Miss Rejichi and Mr Chaabane are with Université de Sfax, ENIS, Electrical Engineering Department, Route de Soukra, km 3.5, 3018 Sfax, Tunisia, (e-mail : ouiem.rejichi@etu.univ-poitiers.fr)

Mr. Bachelier, Mr. Mehdi and Mr El-chaer are with Université de Poitiers, ESIP-LAII, 40 Av. du Recteur Pineau, 86022 Poitiers, (e-mails : Olivier.Bachelier / Driss.Mehdi@univ-poitiers.fr)

Viewing the importance of the theory of dissipativity notion and the generality of singular systems models, development of dissipativity analysis for descriptor systems becomes an important branch of research in the control community. For descriptor systems there are several contributions related to dissipativity from which we quote [7], [8], [9]. However, most of the existing results require a certain assumption or restriction on the realization of descriptor systems, while the KYP Lemma for state-space systems is valid independently of the choice of the realization. On the other hand, a new matrix inequality condition has been proposed recently in [10], [11] that is necessary and sufficient for dissipativity of a descriptor system irrespective of the realization of the considered descriptor system.

Even if the dissipativity is an important requirement, there is no doubt that a level of performance is usually needed. For this reason, it is important to define criteria that allow to evaluate the performances especially the transient ones. These performances are strongly influenced by the location of the state matrix spectrum in the complex plane, in terms of settling time, damping ratio. Hence it can be useful to check if the closed-loop eigenvalues lie inside a region which must be discerningly specified to guarantee satisfactory transient behavior [12], [13]. Recently, the characterization of pole clustering via LMI has been extended to descriptor systems as in [14] for instance.

In the present paper, using the notion of disspativity theory, we propose a matrix inequality condition that is necessary and sufficient for dissipativity of descriptor systems guaranteeing eigenvalue-clustering in a subregion of a the complex plane, this is defined here as the $\mathcal{D}$-dissipativity.

The paper is organized as follows. Section 2 presents some preliminaries. In section 3, necessary and sufficient conditions for the $\mathcal{D}$-dissipativity of descriptor systems are established.

Notation: $\mathbb{C}$ and $\mathbb{R}$ stand, respectively, for complex and real numbers sets. We denote by $X^\top$ the transpose (conjugate if in $\mathbb{C}^{nxn}$) of matrix X, by the Hermitian expression $\mathsf{Sym}\{\cdot\}$:

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

$\text{Sym}\{X\} = X + X^\top$. Matrix inequalities are considered in the sense of Löewner *i.e.* "< 0"("≤ 0") means negative (semi-)definite and "> 0" ("≥ 0") positive (semi-)definite. HPD stands for Hermitian Positive Definite. I_n is the identity matrix of order n, 0 is a null matrix of suitable dimension.

II. Preliminaries

As a shorthand notation for descriptor system, we write (E, A, B, C, D) where A, B, C and D are matrices with appropriate dimensions. The matrix E may be singular, we shall assume that $rank(E) = r \leq n$. In particular a continuous time system will be described by the set of equation $\dot{x}(t) = Ax(t) + Bw(t)$ and $z(t) = Cx(t) + Dw(t)$ where $x(\cdot)$, $w(\cdot)$ and $z(\cdot)$ stand respectively for the state, the input and the output of the system.

A. Clustering region $\mathcal{D}$

The general definition of EMI-region is given:

Definition 1: : Let $R \in \mathbb{C}^{2d\times 2d}$ be a Hermitian matrix defined by

$$\begin{cases} R = R^\top = \begin{bmatrix} r_2 & r_1^* \\ r_1 & r_0 \end{bmatrix}, \\ r_2 \geq 0. \end{cases}$$

The region is defined as the set

$$\mathcal{D}_R = \{s \in \mathbb{C} \,|\, f(s) = \begin{bmatrix} s \\ 1 \end{bmatrix}^\top R \begin{bmatrix} s \\ 1 \end{bmatrix} < 0\} \tag{1}$$

The set of regions includes, for instance, shifted and half planes or discs.

We define also the frontier of the region as

$$\partial\mathcal{D}_R = \{s \in \mathbb{C} \,|\, \begin{bmatrix} s \\ 1 \end{bmatrix}^\top R \begin{bmatrix} s \\ 1 \end{bmatrix} = 0\}. \tag{2}$$

B. Classical definitions

As defined in [1], [2], the system (E, A, B, C, D), with the associated matrix pencil $(sE - A)$, is said to be

- stable if $\det(sE - A) \neq 0, \ \forall s \in \mathcal{D}_R$
- regular if $\det(sE - A)$ is not identically zero,
- impulse free if $\deg\{\det(sE - A)\} = rank(E)$

Provided (E, A) is regular there exist two non singular matrices U and V such that [1]

$$\bar{E} = UEV = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} = UAV = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$$

$$\bar{B} = UBV = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}.$$

Let also $E^\perp$ be defined as

$$E^\perp = V(I - UEV)\,U.$$

III. Dissipativity of descriptor systems

Let $S = S^T \in \mathbb{R}^{(m+p)\times(m+p)}$ and consider the following quadratic form of (w, z)

$$s(w, z) = \begin{bmatrix} w \\ z \end{bmatrix}^\top S \begin{bmatrix} \omega \\ z \end{bmatrix}$$

which defines a supply rate. where w and z are, respectively, the input and the output of the system.

Definition 2: The descriptor system (E, A, B, C, D) is said to be dissipative with respect to the supply rate $s(\cdot, \cdot)$ if the matrix pencil $(sE - A)$ is regular, the descriptor system has no impulsive modes and for any $T \geq 0$ and for $w \in L_2[0, T]$

$$\int_0^t S(w(t), z(t))dt \leq 0 \tag{3}$$

provided $x(0) = 0$, where $x(\cdot)$ is the state of the system.
The time-domain condition (3) is equivalent to the following frequency-domain condition:

$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^\top S \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \forall\omega \in \mathbb{R} \cup \{\infty\} \tag{4}$$

where $G(i\omega) = C(i\omega E - A)^{-1}B + D$. By setting

$$M = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\top S \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \tag{5}$$

the inequality (4) is written as

$$\begin{bmatrix} (i\omega E - A)^{-1}B \\ I \end{bmatrix}^\top M \begin{bmatrix} (i\omega E - A)^{-1}B \\ I \end{bmatrix} < 0 \quad \forall\omega \in \mathbb{R} \cup \{\infty\} \tag{6}$$

where i is the complex number satisfying $i^2 = -1$.

A. Analysis of $\mathcal{D}$-dissipativity for descriptor systems

This contribution aims at developing a condition for $\mathcal{D}$-admissibility using dissipativity inequalities. We recall that a version of the KYP lemma for descriptor systems is given, for instance, in [10] but here we focus on equivalent conditions in a subregion of the complex plan.

In this section, we will give a strict dissipative condition for the system (E, A, B, C, D) which is stated in the following theorem

Theorem 3: Suppose that the following assumptions hold

- $det(sE - A) \neq 0 \quad \forall s \in \partial\mathcal{D}$,
- $lim_{s\to\infty}(sE - A)^{-1}$ exists.

Then the two conditions below are equivalent:
i) $\forall s \in \partial\mathcal{D}_R$ we have

$$\begin{bmatrix} (sE - A)^{-1}B \\ I \end{bmatrix}^\top M \begin{bmatrix} (sE - A)^{-1}B \\ I \end{bmatrix} < 0 \tag{7}$$

ii) there exist matrices $X = X^\top \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$ et $W \in \mathbb{R}^{n\times m}$ satisfying the inequality

$$M + \begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix}^\top \mathcal{U}(E, X, Y) \begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix} + \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix}^\top \text{Sym}\left\{\left(\mathbb{J} \otimes WE^\perp\right)\right\} \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix} < 0 \tag{8}$$

with

$$\mathcal{U}(E, X, Y) = \left(\begin{bmatrix} I_n & 0 \\ 0 & E^\top \end{bmatrix} (R \otimes X) \begin{bmatrix} I_n & 0 \\ 0 & E \end{bmatrix}\right) + \text{Sym}\left\{\mathbb{J} \otimes YE^\perp\right\} \tag{9}$$

A similar condition to *ii)* is given in [15] from which the regularity and the impulse freeness are obtained. However, for simplicity, in the present note we did not work out this aspect and these constraints are taken as afore-given assumption in theorem 3.

Proof

The basic idea is to transform the considered region to the left half of the complex plane or to the unit circle. In order to derive a $\mathcal{D}$-dissipativity condition we will distinguish two cases

1) $r_2 = 0$
2) $r_2 \neq 0$

1) Case $r_2 = 0$: In this case, matrix R can be written as:

$$\begin{bmatrix} 0 & r_1^\top \\ r_1 & r_0 \end{bmatrix} = \begin{bmatrix} r_1 I & \frac{r_0}{2} I \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 I & \frac{r_0}{2} I \\ 0 & I \end{bmatrix} \tag{10}$$

Hence, we transform s into σ as follows $s = \dfrac{1}{r_1}\left(\sigma - \dfrac{r_0}{2}\right)$ with σ belonging to the the frontier of the left half complex plane then we have

$$\begin{bmatrix} (sE - A)^{-1} B \\ I_m \end{bmatrix} = \begin{bmatrix} \left(\sigma E - (r_1 A + \frac{r_0}{2} E)\right)^{-1} (r_1 B) \\ I_m \end{bmatrix}. \tag{11}$$

At this step, let us consider the two matrices

$$\Sigma = \begin{bmatrix} I & 0 \\ -\bar{A}_{22}^{-1} \bar{A}_{21} & I \end{bmatrix} \text{and} \quad \Gamma = \begin{bmatrix} I & -\bar{A}_{12} \bar{A}_{22}^{-1} \\ 0 & I \end{bmatrix},$$

that transform matrix $\bar{E}$, $\bar{A}$ and $\bar{B}$ respectively as

$$\bar{\bar{E}} = \Gamma \bar{E} \Sigma = \bar{E} \quad \bar{\bar{A}} = \Gamma \bar{A} \Sigma = \begin{bmatrix} \bar{\bar{A}}_{11} & 0 \\ 0 & \bar{A}_{22} \end{bmatrix}$$

$$\bar{\bar{Y}} = \Gamma \bar{Y} \Gamma^\top \quad \bar{\bar{B}} = \Gamma \bar{B} = \begin{bmatrix} \bar{\bar{B}}_1 \\ \bar{\bar{B}}_2 \end{bmatrix}$$

with which equation (11) is transformed into

$$\begin{bmatrix} V\Sigma & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} (\sigma I_r - (r_1 \bar{\bar{A}}_{11} + \frac{r_0}{2} I_r))^{-1} r_1 \bar{\bar{B}}_1 \\ -\bar{A}_{22}^{-1} \bar{\bar{B}}_2 \\ I_m \end{bmatrix} \tag{12}$$

which can be rewritten as

$$\begin{bmatrix} V\Sigma & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} -\bar{A}_{22}^{-1} \bar{\bar{B}}_2 \\ I \end{bmatrix} \end{bmatrix} \begin{bmatrix} \sigma I_r - (r_1 \bar{\bar{A}}_{11} + \frac{r_0}{2} I_r)^{-1} r_1 \bar{\bar{B}}_1 \\ I_m \end{bmatrix}. \tag{13}$$

Let $\hat{A}_1 = (r_1 \bar{\bar{A}}_{11} + \frac{r_0}{2} I_r)$; $\hat{B}_1 = r_1 \bar{\bar{B}}_1$, For more clarity we define respectively two matrices $\tilde{M}$ and Θ as follows

$$\tilde{M} = \begin{bmatrix} V\Sigma & 0 \\ 0 & I_m \end{bmatrix}^\top M \begin{bmatrix} V\Sigma & 0 \\ 0 & I_m \end{bmatrix},$$

$$\Theta = \begin{bmatrix} I & 0 \\ 0 & -\bar{\bar{B}}_2 \\ 0 & I \end{bmatrix}^\top \tilde{M} \begin{bmatrix} I & 0 \\ 0 & -\bar{\bar{B}}_2 \\ 0 & I \end{bmatrix}.$$

Hence, inequality (7) is expressed as

$$\begin{bmatrix} (\sigma I - \hat{A}_1)^{-1} \hat{B}_1 \\ I \end{bmatrix}^\top \Theta \begin{bmatrix} (\sigma I - \hat{A}_1)^{-1} \hat{B}_1 \\ I \end{bmatrix} < 0. \tag{14}$$

From the KYP-Lemma for conventional systems [4], the above inequality holds if and only if there exists a Hermitian matrix $\bar{\bar{X}}_{11}$ such as

$$\Theta + \begin{bmatrix} \hat{A}_1 & \hat{B}_1 \\ I & 0 \end{bmatrix}^\top \begin{bmatrix} 0 & \bar{\bar{X}}_{11} \\ \bar{\bar{X}}_{11} & 0 \end{bmatrix} \begin{bmatrix} \hat{A}_1 & \hat{B}_1 \\ I & 0 \end{bmatrix} < 0.$$

Using [16, Fact 2.2.3] we get easily from above

$$\tilde{M} + \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I_r & 0 & 0 \end{bmatrix}^\top \begin{bmatrix} 0 & \bar{\bar{X}}_{11} \\ \bar{\bar{X}}_{11} & 0 \end{bmatrix} \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I_r & 0 & 0 \end{bmatrix} \tag{15}$$
$$+\text{Sym}\left\{ \begin{bmatrix} \bar{\bar{X}}_{12} + r_1^{-1} \bar{\bar{Y}}_{12} \\ r_1^{-1} \bar{\bar{Y}}_{22} \\ r_1^{-1} \bar{\bar{W}}_{12} \end{bmatrix} [\, 0 \quad r_1 \bar{A}_{22} \quad r_1 \bar{\bar{B}}_2 \,] \right\} < 0$$

which can be written alternatively as

$$\tilde{M} + \text{Sym}\left\{ \begin{bmatrix} \bar{\bar{X}}_{11} & \bar{\bar{X}}_{12} \\ 0 & 0 \\ 0 & r_1^{-1} \bar{\bar{W}}_{12} \end{bmatrix} \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ 0 & r_1 \bar{A}_{22} & r_1 \bar{\bar{B}}_2 \end{bmatrix} \right\} \tag{16}$$
$$+\text{Sym}\left\{ \begin{bmatrix} 0 & r_1^{-1} \bar{\bar{Y}}_{12} \\ 0 & r_1^{-1} \bar{\bar{Y}}_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ 0 & r_1 \bar{A}_{22} & r_1 \bar{\bar{B}}_2 \end{bmatrix} \right\} < 0$$

or as

$$\tilde{M} + \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I_n & 0 \end{bmatrix}^\top \begin{bmatrix} 0 & r_1^* \bar{\bar{E}}^\top \bar{\bar{X}} \\ r_1 \bar{\bar{X}} \bar{\bar{E}} & r_0 \bar{\bar{E}}^\top \bar{\bar{X}} \bar{\bar{E}} \end{bmatrix} \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I_n & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I_n & 0 \end{bmatrix}^\top \begin{bmatrix} 0 & \left(\bar{\bar{Y}} \bar{\bar{E}}^\perp\right)^\top \\ \bar{\bar{Y}} \bar{\bar{E}}^\perp & 0 \end{bmatrix} \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I_n & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ 0 & I_m \end{bmatrix}^\top \begin{bmatrix} 0 & \left(\bar{\bar{W}} \bar{\bar{E}}^\perp\right)^\top \\ \bar{\bar{W}} \bar{\bar{E}}^\perp & 0 \end{bmatrix} \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ 0 & I_m \end{bmatrix} < 0$$

with $\bar{\bar{X}} = \begin{bmatrix} \bar{\bar{X}}_{11} & \bar{\bar{X}}_{12} \\ \bar{\bar{X}}_{21} & \bar{\bar{X}}_{22} \end{bmatrix}$, $\bar{\bar{Y}} = \begin{bmatrix} \bar{\bar{Y}}_{11} & \bar{\bar{Y}}_{12} \\ \bar{\bar{Y}}_{21} & \bar{\bar{Y}}_{22} \end{bmatrix}$, and $\bar{\bar{W}} = [\, \bar{\bar{W}}_{11} \quad \bar{\bar{W}}_{12} \,]$.

2) Case $r_2 \neq 0$: In this case we have

$$\sigma = \frac{1}{\alpha}\left(s + \frac{r_1^*}{r_2}\right)$$

with $\alpha = \dfrac{r_1^* r_1}{r_2^2} - \dfrac{r_0}{r_2}$.

In a similar way as in the previous case, we have :

$$\begin{bmatrix} I_r & 0 \\ 0 & -\bar{A}_{22}^{-1} \bar{\bar{B}}_2 \\ 0 & I_m \end{bmatrix}^\top \left(\beta \tilde{M} + \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I & 0 & 0 \end{bmatrix}^\top \right.$$
$$\left. \begin{bmatrix} \bar{\bar{X}}_{11} & 0 \\ 0 & -\bar{\bar{X}}_{11} \end{bmatrix} \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I_r & 0 \\ 0 & -\bar{A}_{22}^{-1} \bar{\bar{B}}_2 \\ 0 & I_m \end{bmatrix} < 0.$$

with $\beta = \left(r_2 \alpha^2\right)^{-1}$.

In order to get similar result as in the case $r_2 = 0$ we have to add some null terms as follows

$$\begin{bmatrix} I_r & 0 \\ 0 & -\bar{A}_{22}^{-1} \bar{\bar{B}}_2 \\ 0 & I_m \end{bmatrix}^\top \left(\beta \tilde{M} + \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I & 0 & 0 \end{bmatrix}^\top \begin{bmatrix} \bar{\bar{X}}_{11} & 0 \\ 0 & -\bar{\bar{X}}_{11} \end{bmatrix} \right.$$
$$\begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I & 0 & 0 \end{bmatrix} + \text{Sym}\left\{ \left(\begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I & 0 & 0 \end{bmatrix}^\top \begin{bmatrix} \bar{\bar{X}}_{12} \\ 0 \end{bmatrix} \right.\right.$$
$$\left.\left.\left. + \begin{bmatrix} \alpha\beta \bar{\bar{Y}}_{12} \\ \alpha\beta \bar{\bar{Y}}_{22} \\ \alpha\beta \bar{\bar{W}}_{12} \end{bmatrix} \right) [\, 0 \quad \alpha^{-1} I \quad \alpha^{-1} \bar{\bar{B}}_2 \,] \right) \begin{bmatrix} I_r & 0 \\ 0 & -\bar{A}_{22}^{-1} \bar{\bar{B}}_2 \\ 0 & I_m \end{bmatrix} \right\} < 0.$$

Applying [16, Theorem 2.3.14] to the condition above we get easily that

$$
\begin{aligned}
&\beta\tilde{M} + \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I & 0 & 0 \end{bmatrix}^{\top} \begin{bmatrix} \bar{\bar{X}}_{11} & 0 \\ 0 & -\bar{\bar{X}}_{11} \end{bmatrix} \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I & 0 & 0 \end{bmatrix} \\
&+\mathrm{Sym}\left\{ \left(\begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I & 0 & 0 \end{bmatrix}^{\top} \begin{bmatrix} \bar{\bar{X}}_{12} \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha\beta\,\bar{\bar{Y}}_{12} \\ \alpha\beta\,\bar{\bar{Y}}_{22} \\ \alpha\beta\,\bar{\bar{W}}_{12} \end{bmatrix} \right) \begin{bmatrix} 0 \\ \alpha^{-1}\bar{A}_{22}^{\top} \\ \alpha^{-1}\,\bar{\bar{B}}_2^{\top} \end{bmatrix}^{\top} \right\} \\
&+ \begin{bmatrix} 0 \\ \alpha^{-1}\bar{A}_{22}^{\top} \\ \alpha^{-1}\,\bar{\bar{B}}_2^{\top} \end{bmatrix} \bar{\bar{X}}_{22} \begin{bmatrix} 0 \\ \alpha^{-1}\bar{A}_{22}^{\top} \\ \alpha^{-1}\,\bar{\bar{B}}_2^{\top} \end{bmatrix}^{\top} < 0.
\end{aligned}
$$

The condition above can also be written as

$$
\begin{aligned}
&\beta\tilde{M} + \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I & 0 & 0 \\ 0 & \alpha^{-1}\bar{A}_{22} & \alpha^{-1}\,\bar{\bar{B}}_2 \\ 0 & 0 & 0 \end{bmatrix}^{\top} \begin{bmatrix} \bar{\bar{X}}_{11} & 0 & \bar{\bar{X}}_{12} & 0 \\ 0 & -\bar{\bar{X}}_{11} & 0 & 0 \\ \bar{\bar{X}}_{12}^{\top} & 0 & \bar{\bar{X}}_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\begin{bmatrix} \hat{A} & 0 & \hat{B} \\ I & 0 & 0 \\ 0 & \alpha^{-1}\bar{A}_{22} & \alpha^{-1}\,\bar{\bar{B}}_2 \\ 0 & 0 & 0 \end{bmatrix} \\
&+\mathrm{Sym}\left\{ \begin{bmatrix} 0 & \alpha\beta\,\bar{\bar{Y}}_{12} \\ 0 & \alpha\beta\,\bar{\bar{Y}}_{22} \\ 0 & \alpha\beta\,\bar{\bar{W}}_{12} \end{bmatrix} \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ 0 & \alpha^{-1}\bar{A}_{22} & \alpha^{-1}\,\bar{\bar{B}}_2 \end{bmatrix} \right\} < 0.
\end{aligned}
$$

Here, notice that performing permutations on rows and columns, the condition above can be transformed into

$$
\begin{aligned}
&\begin{bmatrix} \alpha^{-1}\left(\bar{\bar{A}} + \frac{r_1^{\top}}{r_2}\,\bar{\bar{E}}\right) & \alpha^{-1}\,\bar{\bar{B}} \\ I & 0 \end{bmatrix}^{\top} \begin{bmatrix} \bar{\bar{X}} & 0 \\ 0 & -\,\bar{\bar{E}}^{\top}\,\bar{\bar{X}}\,\bar{\bar{E}} \end{bmatrix} \\
&\quad \begin{bmatrix} \alpha^{-1}\left(\bar{\bar{A}} + \frac{r_1^{\top}}{r_2}\,\bar{\bar{E}}\right) & \alpha^{-1}\,\bar{\bar{B}} \\ I & 0 \end{bmatrix} \\
&= \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I & 0 \end{bmatrix}^{\top} \begin{bmatrix} \alpha^{-1}I & \alpha^{-1}\frac{r_1^{\top}}{r_2}\,\bar{\bar{E}} \\ 0 & I \end{bmatrix}^{\top} \begin{bmatrix} \bar{\bar{X}} & 0 \\ 0 & -\,\bar{\bar{E}}^{\top}\,\bar{\bar{X}}\,\bar{\bar{E}} \end{bmatrix} \\
&\quad \begin{bmatrix} \alpha^{-1}I & \alpha^{-1}\frac{r_1^{\top}}{r_2}\,\bar{\bar{E}} \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I & 0 \end{bmatrix} \\
&= \left(r_2\alpha^2\right)^{-1} \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I & 0 \end{bmatrix}^{\top} \begin{bmatrix} r_2\,\bar{\bar{X}} & r_1^{\top}\,\bar{\bar{X}}\,\bar{\bar{E}} \\ r_1\,\bar{\bar{E}}^{\top}\,\bar{\bar{X}} & r_0\,\bar{\bar{E}}^{\top}\,\bar{\bar{X}}\,\bar{\bar{E}} \end{bmatrix} \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I & 0 \end{bmatrix}
\end{aligned}
$$

with $\bar{\bar{X}}$ defined as for case 1.

Note also that

$$
\begin{aligned}
&\mathrm{Sym}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \alpha\beta\,\bar{\bar{W}}_{12} \end{bmatrix} \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ 0 & \alpha^{-1}\bar{A}_{22} & \alpha^{-1}\,\bar{\bar{B}}_2 \end{bmatrix} \right\} \\
&= \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ 0 & \alpha^{-1}\bar{A}_{22} & \alpha^{-1}\,\bar{\bar{B}}_2 \\ 0 & 0 & I_m \end{bmatrix}^{\top} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha\beta\,\bar{\bar{W}}_{12}^{\top} \\ 0 & \alpha\beta\,\bar{\bar{W}}_{12} & 0 \end{bmatrix} \\
&\quad \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ 0 & \alpha^{-1}\bar{A}_{22} & \alpha^{-1}\,\bar{\bar{B}}_2 \\ 0 & 0 & I_m \end{bmatrix} \\
&= \beta \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ 0 & I_m \end{bmatrix}^{\top} \begin{bmatrix} 0 & \left(\bar{\bar{W}}\,\bar{\bar{E}}^{\perp}\right)^{\top} \\ \bar{\bar{W}}\,\bar{\bar{E}}^{\perp} & 0 \end{bmatrix} \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ 0 & I_m \end{bmatrix}
\end{aligned}
$$

with $\bar{\bar{W}}$defined as in case 1. Similarly

$$
\begin{aligned}
&\mathrm{Sym}\left\{ \begin{bmatrix} 0 & \alpha\beta\,\bar{\bar{Y}}_{12} \\ 0 & \alpha\beta\,\bar{\bar{Y}}_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{A} & 0 & \hat{B} \\ 0 & \alpha^{-1}\bar{A}_{22} & \alpha^{-1}\,\bar{\bar{B}}_2 \end{bmatrix} \right\} \\
&= \beta \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I_n & 0 \end{bmatrix}^{\top} \begin{bmatrix} 0 & \left(\bar{\bar{Y}}\,\bar{\bar{E}}^{\perp}\right)^{\top} \\ \bar{\bar{Y}}\,\bar{\bar{E}}^{\perp} & 0 \end{bmatrix} \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I_n & 0 \end{bmatrix}
\end{aligned}
$$

with $\bar{\bar{Y}}$ defined as in case 1.

Therefore combining the previous expressions we get

$$
\begin{aligned}
&\tilde{M} + \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I & 0 \end{bmatrix}^{\top} \begin{bmatrix} r_2\,\bar{\bar{X}} & r_1^{\top}\,\bar{\bar{X}}\,\bar{\bar{E}} \\ r_1\,\bar{\bar{E}}^{\top}\,\bar{\bar{X}} & r_0\,\bar{\bar{E}}^{\top}\,\bar{\bar{X}}\,\bar{\bar{E}} \end{bmatrix} \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I & 0 \end{bmatrix} \\
&+ \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I_n & 0 \end{bmatrix}^{\top} \begin{bmatrix} 0 & \left(\bar{\bar{Y}}\,\bar{\bar{E}}^{\perp}\right)^{\top} \\ \bar{\bar{Y}}\,\bar{\bar{E}}^{\perp} & 0 \end{bmatrix} \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ I_n & 0 \end{bmatrix} \\
&+ \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ 0 & I_m \end{bmatrix}^{\top} \begin{bmatrix} 0 & \left(\bar{\bar{W}}\,\bar{\bar{E}}^{\perp}\right)^{\top} \\ \bar{\bar{W}}\,\bar{\bar{E}}^{\perp} & 0 \end{bmatrix} \begin{bmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ 0 & I_m \end{bmatrix} < 0
\end{aligned}
$$

which is similar to the result obtained in the case $r_2 = 0$.

Hence, inequality (8) holds after congruent transformations and setting

$$
\mathbb{J} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad X = U^{\top}\Gamma^{\top}\,\bar{\bar{X}}\Gamma U
$$

$$
Y = V^{-\top}\Sigma^{-\top}\,\bar{\bar{Y}}\Sigma^{-1}V^{-1}, \quad W = \bar{\bar{W}}\Sigma^{-1}V^{-1}
$$

which completes the proof.

Remark 4: In the case $E = I_n$ and consequentely $E^{\perp} = 0$, the singular system reduces to the conventional state-space model, then taking $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which corresponds to the left half complex plane, equation (8) coincides with the KYP Lemma for continuous-time system. Similarly taking $R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which corresponds to the unit circle, equation (8) coincides with the KYP Lemma for discrete-time systems. From this point of view, Theorem 3 naturally extends existing results on the dissipativity of state-space systems to singular ones. In [6], a version of the KYP lemma for descriptor systems is given with additional frequency range constraints. In our opinion, these frequency constraints allow the strict inequality to hold thanks to the sign constraint on the associated matrix Q, [6, Theorem 3]. In the present version it is the introduction of the additional matrices Y and W that allows to insure the strict inequality.

Remark 5: It is worth noting that in the particular case of continuous-time system, condition (8) is similar to [10, Inequality 4] where we have removed the algebraic constraint on matrices W and X. Matrix X here is assumed to be symmetric.

IV. Conclusion

In this paper, we have established new strict LMI condition for checking the $\mathcal{D}$-dissipativity for descriptor systems. The considered regions denoted by $\mathcal{D}$ covers that of continuous-time and discrete descriptor systems as special cases. The proposed result can be viewed as a strict inequality version of the KYP lemma for descriptor systems with no frequency range constraints.

References

[1] L. Dai, *Singular control systems*. New-York: Springer-Verlag, 1989.
[2] F. L. Lewis, "A survey of linear singular systems," *Circuits systems signal processing*, vol. 5, pp. 266–36, 1986.
[3] J. Willems, "Dissipative dynamical systems, part1: general theory; part2: linear systems with quadratic supply rate," *Archives of Rational Mechanical Analysis*, vol. 45, pp. 321–393, 1972.
[4] A. Rantzer, "On the kalman-yakubovich-popov lemma," *Systems & Control Letters*, vol. 28, pp. 7–10, 1996.
[5] T. Iwasaki, G. Meinsma, and M. Fu, "Generalized S procedure and finite frequency KYP lemma," *Mathematical Problemes in Engineering*, vol. 6, pp. 305–320, 2000.
[6] T. Iwasaki and S. Hara, "Generalized KYP lemma: unified characterization of frequency domain inequalities with design applications," *IEEE Transactions on Automatic Control*, vol. 50, pp. 612–633, 2005.
[7] K. Takaba, N. Morihira, and T. A. Katayama, "Generalized Lyapunov theorem for descriptor system," *Systems & Control Letters*, vol. 40, pp. 24, 49–51, 1995.
[8] I. Masubuchi and Y. Kamitane, "H_∞-control for descriptor systems: A matrix inequality approach," *Automatica*, vol. 33, pp. 669–673, 1997.
[9] A. Rehm and F. Allgwer, "Self-scheduled h8 output feedback control of descriptor systems," *Computers and Chemical Engineering*, vol. 24, p. 279284, 2000.
[10] I. Masubuchi, "Dissipativity inequalities for continuous time descriptor systems with applications to synthesis of control," *Systems & Control Letters*, vol. 55, p. 158 164, 2006.
[11] ——, "Output feedback controller synthesis for descriptor systems satisfying closed-loop dissipativity," *Automatica*, vol. 43, p. 339 345, 2007.
[12] M. Chilali, P. Gahinet, and P. Apkarian, "Robust pole placement in LMI regions," *IEEE transactions on Automatic Control*, vol. 44, pp. 2257–2270, 1999.
[13] O. Bachelier and D. Mehdi, "Robust D_U-stability analysis," *International Journal of Robust and Nonlinear Control*, vol. 13, pp. 533–558, 2003.
[14] K. L. Hsiung and L. Lee, "Pole clustering characterization via LMI for descriptor systems," *Conference on Decision and Control*, pp. 1313–1314, 1997.
[15] M. Chaabane, O. Bachelier, M. Souissi, and D. Mehdi, "Stability and stabilization of continuous descriptor systems : An lmi approach," *Mathematical Problems in Engineering*, vol. 2006, pp. ID–39 367, 2006.
[16] R. E. Skelton, T. Iwasaki, and K. Grigoriadis, *A unified approach to linear control design*. Bristol-USA: Taylor and Francis series in Systems and Control, 1997.