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A sub-optimal second order sliding mode controller for systems with saturating actuators

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***Abstract*— In this paper, the problem of the possible saturation of the continuous control variable in the sub-optimal second order sliding mode controller applied to relative degree one systems with saturating actuators is addressed. It is proved that during the sliding phase, if basic assumptions are made, the continuous control variable never saturates, while, during the reaching phase, the presence of saturating actuators can make the steering of the sliding variable to zero in finite time not always guaranteed. In the present paper, the original algorithm is modified in order to solve this problem: a new strategy is proposed, which proves to be able to steer the sliding variable to zero in a finite time in spite of the presence of saturating actuators.**

I. INTRODUCTION

Saturation is a particular kind of nonlinearity which affects many dynamical systems [1]. In particular, when a controller is designed, one must take into account that the generated control signal can be in fact upper and lower-bounded by the actuator saturation. If the saturation bounds are never reached in practice, the designer can avoid considering their effect. In contrast, if they are reached, the actual control signal acting on the system is different from the one generated by the controller. A very common case when this happens is the so-called ‘wind-up’ effect that affects PID controllers [2], and, in general, every controller which takes into account the integral of the error signal. Strategies to solve this problem have been proposed, in order to avoid undesired behaviors (typically, longer settling times and higher overshoots).

Sliding mode control is a particular kind of robust control (see [3], [4] or [5]), which allows the controller to completely reject any disturbance acting on the control channel (i.e. the so-called ‘matched disturbances’, which also include the parameter uncertainties) when the ‘sliding variables’ (i.e the variables which are directly steered to zero in a finite time by the controller action) are equal to zero, and the so-called ‘sliding mode’ is enforced. In conventional sliding mode control, the generated control signal is discontinuous, which can generate the so-called ‘chattering’ effect, that is, a very high frequency oscillation of the sliding variables around zero (see [6], [7] and [8]). To reduce this undesired effect, many solutions have been proposed. Among them, one of the most promising is based on the generation of ‘higher order sliding modes’ (for a detailed description of the topic, see [9] and [10]). When such a solution is used, the control variable acts discontinuously on a high order derivative of the considered sliding variable, forcing to zero both the sliding variable and a certain number of its time derivatives. The resulting control which acts on the first-order time derivative of the sliding variable is continuous, because it is obtained by integrating one or more times the discontinuous signal generated by the controller.

In particular, in this paper, a second order sliding mode controller [11], designed according to the so-called ‘sub-optimal’ approach (see [12] and [13]), is taken into account. This controller acts discontinuously on the second-order time derivative of the sliding variable, so that the control variable which acts on the system is the integral of this discontinuous control. If the continuous control signal is affected by a saturation, the behavior of the system can be different from that expected, and even the stabilization of the sliding variable to zero cannot be assured any more. To solve the problem, a desaturation strategy is proposed, which produces appreciable results, allowing to maintain the finite time convergence to zero of the sliding variable and the consequent attainment of the control objective in spite of the presence of the actuator saturation.

The paper is organized as follows: Section II is devoted to the formulation of the control problem. Section III analyzes in which cases a saturation of the continuous control signal can occur and what can be its effect on the controlled system, when the sub-optimal algorithm is applied. In Sections IV a strategy to solve the problem is proposed; simulation examples relating to this modification of the original sub-optimal control law are shown in Section V, while some conclusions are gathered in Section VI.

II. PROBLEM FORMULATION

Consider a class of uncertain nonlinear system, described by the state equations

$$\dot{x}(t) = \phi(x(t),t) + \gamma(x(t),t)u(t) \tag{1}$$

where $x \in \mathscr{X} \subset I\!R^n$ is the state vector, while

$$u \in [-U;U] \tag{2}$$

is the control variable, $\phi : I\!R^{n+1} \to I\!R^n$ and $\gamma : I\!R^{n+1} \to I\!R^n$ are uncertain and sufficiently smooth vector fields. The control objective is to steer to zero a scalar output variable, called ‘sliding variable’, defined as

$$\sigma_1 = \sigma_1(x(t)). \tag{3}$$

This variable has relative degree $r = 1$ with respect to the control variable $u(t)$; moreover, a diffeomorphism $\Omega : I\!R^{n-1} \times I\!R \to I\!R^n$ is defined, such that the dynamics of the

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internal state $\xi(t) \in I\!R^{n-1}$ is BIBO stable. System (1) can be transformed into the normal form

$$\begin{aligned}\dot{\sigma}_1(t) &= f(\xi(t),\sigma_1(t),t)+g(\xi(t),\sigma_1(t),t)u(t) \quad (4)\\ \dot{\xi}(t) &= \psi(\xi(t),\sigma_1(t)) \quad (5)\end{aligned}$$

where $f(\xi(t),\sigma_1(t),t) \in I\!R$ and $g(\xi(t),\sigma_1(t),t) \in I\!R$ are uncertain and sufficiently smooth functions, while $\psi(\xi(t),\sigma_1(t),t) \in I\!R^{n-1}$ is a known and sufficiently smooth vector field. Note that, if the uncertain parts of the system model can be expressed in this way, they are called 'matched' uncertainties. The above mentioned diffeomorphism is such that

$$x(t) = \Omega(\xi(t),\sigma_1(t)), \quad (6)$$

and so it is possible to refer to the functions which depend on $\xi(t)$ and $\sigma_1(t)$ as functions of $x(t)$. Now, assume that the first order input-output dynamics (4) is globally bounded, and that the sign of $g(x(t),t)$ is positive.

Assume also that the following bounds are defined

$$|f(x(t),t)| \leq F \quad (7)$$

$$0 < G_1 \leq g(x(t),t) \leq G_2 \quad (8)$$

where F, G_1 and G_2 are positive scalars, and that U in (2) is such that

$$U > \frac{F}{G_1}. \quad (9)$$

Given system (4)-(5) with the bounds (2), (7), and (8), the problem dealt with in this paper is to design a continuous control law capable of making the system state evolve from a certain finite time instant t_r onward on the manifold $\sigma_1(t) = 0$. When $\sigma_1(t) = 0$ the system exhibits the so-called sliding mode, in which the internal state evolves according to

$$\dot{\xi}(t) = \psi(\xi(t),0). \quad (10)$$

The properties of this motion depend on the choice of the sliding variable $\sigma_1(t)$. Note that assumption (9) implies that a first order control law

$$u(t) = -U sign(\sigma_1(t)) \quad (11)$$

is capable of making $\sigma_1(t) = 0$ in finite time. Yet, this is a discontinuous control law, which can be a limit to its applicability in many engineering contexts.

If the relative degree of the system with respect to the control variable is $r = 1$, like in the previous system, second order sliding mode control can be used to reduce the so-called 'chattering' phenomenon, due to the high frequency switchings of the control variable when the sliding manifold $\sigma_1(t)$ is steered to zero [11]. Indeed, using the so called 'sub-optimal' control law [12], [13], a new control variable is defined as

$$w(t) = \dot{u}(t) \quad (12)$$

so that the auxiliary control signal $w(t)$ turns out to be discontinuous while $u(t)$, the actual control, is continuous. The design is performed by relying on equation (4), which can be differentiated with respect to time, obtaining the following auxiliary system, the state of which consists of the sliding variable and its first time derivative, i.e.

$$\begin{aligned}\dot{\sigma}_1(t) &= f(x(t),t)+g(x(t),t)u(t)\\ &= \sigma_2(t) \quad (13)\\ \dot{\sigma}_2(t) &= \dot{f}(x(t),t)+\dot{g}(x(t),t)u(t)+g(x(t),t)\dot{u}(t)\\ &= h(x(t),u(t),t)+g(x(t),t)w(t) \quad (14)\end{aligned}$$

where $h(x(t),u(t),t) = \dot{f}(x(t),t)+\dot{g}(x(t),t)u(t)$.

Consider now the classical 'sub-optimal' control law

$$w(t) = -\alpha W sign[\sigma_1(t) - 0.5\sigma_M] \quad (15)$$

where, if $(\sigma_1(t) - 0.5\sigma_M)(\sigma_M - \sigma_1(t)) > 0$ then $\alpha = \alpha^* \in (0,1] \cap (0, 3G_1/G_2)$, else $\alpha = 1$, while σ_M is the last extremal value of σ_1, set equal to $\sigma_1(t_0)$ at the starting time instant, and

$$W > \max\left(\frac{H}{\alpha^* G_1}; \frac{4H}{3G_1 - \alpha^* G_2}\right). \quad (16)$$

This control law solves the problem of steering both $\sigma_1(t)$ and $\sigma_2(t)$ to zero in finite time, provided that the unknown disturbances are such that (8) holds and, in addition

$$|h(x(t),u(t),t)| \leq H \quad (17)$$

where H is a positive scalar.

Note that in the conventional design of the sub-optimal second order sliding mode control law the presence of the bounds (2) on the control amplitude is not taken into account. The actual control $u(t)$ is obtained by integrating the discontinuous auxiliary control signal $w(t)$ and, a-priori, it is not possible to assess if $u(t)$ will exceed the saturation bounds $[-U, U]$.

III. THE SATURATION PROBLEM

Now consider the case in which the sub-optimal algorithm is used in a control system with actuator saturation. In particular, assume that the plant model belongs to the class described in Section II. As previously mentioned, since the system relative degree is one, the actual control signal $u(t)$ is continuous and given by $u(t) = u(t_0) + \int_{t_0}^{t} w(\tau)d\tau$.

The scheme of the control system is reported in Fig. 1: the sub-optimal controller has the sliding variable $\sigma_1(t)$ (which is regarded as the output of the plant) as input signal, and generates, as output, the signal $w(t)$, which is then integrated. The dashed line wraps all the blocks that are part of the controller. The control signal $u(t)$ is then fed into to the process (grey rectangle) as input; the process is characterized by the presence of a saturation on $u(t)$, which generates the signal $u_S(t)$ with $-U$ and U as lower and upper-bounds, respectively. The saturated signal is the actual input to the plant. To analyze the effects of the saturation of the control variable $u(t)$, let us take into account two different phases: the reaching phase and the sliding phase. During the reaching phase the controller steers $\sigma_1(t)$ and $\sigma_2(t)$ from their initial values to zero in a finite time; then, during the sliding phase, the auxiliary system state is kept to the origin, and the controlled system evolves in sliding mode. The sliding phase, by virtue of (9), will be shown to be insensitive to the presence of the saturation of $u(t)$, and for this reason

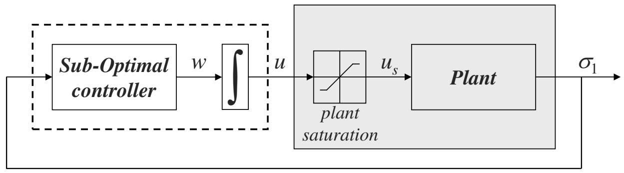

Fig. 1. The scheme of the control system using the original sliding mode controller

it is analyzed first; then, the problems occurring during the reaching phase are briefly discussed.

A. The sliding phase

Once the system is in sliding mode, the following result can be proved:

THEOREM 1: *Consider the auxiliary system (13)-(14), with bounds (7), (8) and (17); and $w(t)$ as in (15). If $\sigma_1(t) = \sigma_2(t) = 0$ for all $t \geq t_r$, t_r being the time instant when the sliding manifold is reached*

$$|u(t)| < U \quad \forall t \geq t_r. \tag{18}$$

Proof: The actual control signal $u(t)$ is the time integral of a bounded function $w(t)$, and so it is a continuous function. From (14), $h(x,u(t),t) = \dot{f}(x(t),t) + \dot{g}(x(t),t)u(t)$ and $h(x(t),u(t),t)$ is bounded (see (17)), then $f(x(t),t)$ and $g(x(t),t)$ are continuous functions as well, and so is their ratio, being $g(x(t),t) \neq 0$. After these considerations, $\sigma_2(t) = f(x(t),t) + g(x(t),t)u(t) = 0$ implies

$$u(t) = -\frac{f(x(t),t)}{g(x(t),t)} \tag{19}$$

that is, signal $u(t)$ coincides with the so-called equivalent control defined in [3] for the conventional first order sliding mode case. Taking into account the bounds on the uncertain terms and (9), it follows that

$$|u(t)| = \frac{|f(x(t),t)|}{|g(x(t),t)|} \leq \frac{F}{G_1} < U \tag{20}$$

which concludes the proof.

□

Some considerations should be noted regarding this theorem:

Remark 1: The fact that the continuous control signal generated by the second order sliding mode controller integrating the discontinuous auxiliary control signal coincides with the equivalent control has already been noted in other works (see, e.g. [10]). The focus of Theorem 1 is, in fact, on the fact that the presence of the actuator saturation does not affect the performance while in sliding mode, rather than on the continuity properties of the generated control signal. Anyway, one can note that, in case one between $f(x(t),t)$ or $g(x(t),t)$ is a known quantity, the availability of a measurement of $u(t)$ permits to identify the other signal, which is a-priori unknown.

Remark 2: This result is independent of the use of a sub-optimal controller, since the only property of the controller that has been exploited is the capability of keeping $\sigma_2(t)$ to zero after a finite period of time, which is true for any second order sliding mode controller.

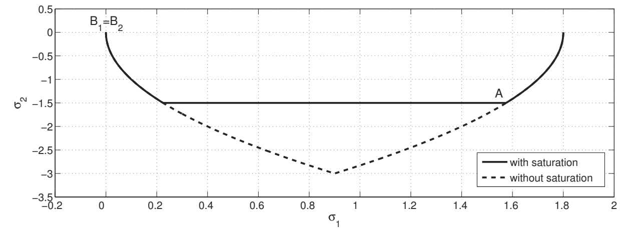

Fig. 2. The trajectory of the auxiliary system in first example without disturbances

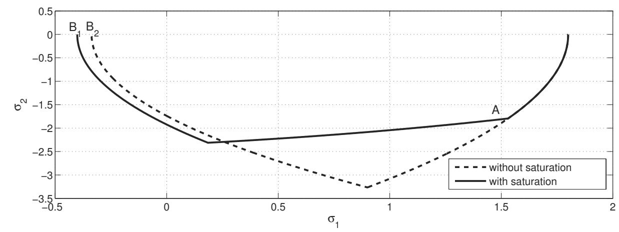

Fig. 3. The trajectory of the auxiliary system in the first example with disturbances

Remark 3: Theorem 1 means in practice that if a second order sliding mode is enforced, then, during this mode, the control signal $u(t)$ is always within the saturation bounds. So, in the sliding phase, the auxiliary control signal (15) can be used neglecting the presence of the actuator saturation.

B. The reaching phase

During the reaching phase, since $w(t)$ can be constant for a relatively long time interval, $u(t)$ can reach one of the saturation bounds. As a first example, consider an auxiliary system as in (13)-(14), starting from $\sigma_1(t_0) = 1.8$, $\sigma_2(t_0) = 0$, with $u(t_0) = 0$. The value of the controller gain is set to $W = 5$, with $\alpha^* = 1$; now, two cases are considered. In the first one (Fig. 2) $f(x(t),t) = h(x(t),t) = 0$ and $g(x(t),t) = 1$: in the figure, the trajectory of the auxiliary system without saturation (i.e. $U \rightarrow +\infty$) is depicted as a dashed line, while the trajectory of the same system, but with $u(t)$ saturated to ± 1.5, is reported as a continuous line. At point A, the two trajectories split, but, later, they still join together, and they reach the σ_1-axis at the same point $B_1 \equiv B_2$, which, in this particular case (no uncertain terms), coincides with the origin. In the second case (Fig. 3) all the conditions are the same as in the first one, except for the value of the uncertain terms. The term $f(x(t),t)$ is equal to $sin(t)$, while $g(x(t),t) = 1$, and, consequently, $h(x(t),t) = \dot{f}(x(t),t) = cos(t)$. It is easy to see that parameter $W = 5$ complies with (16), while $U = 1.5$ is large enough to satisfy (9). Looking at Fig. 3, it is apparent that, after departing from each other at point A, the two trajectories does not join any more, and points B_1 and B_2 are different. This fact puts into evidence that the sequence of the extremal values of variable $\sigma_1(t)$ in the saturated case differs from the one attainable when the saturation is not present. This implies that, in principle, relying on the assumptions of the sub-optimal algorithm, the convergence to the origin is no longer assured.

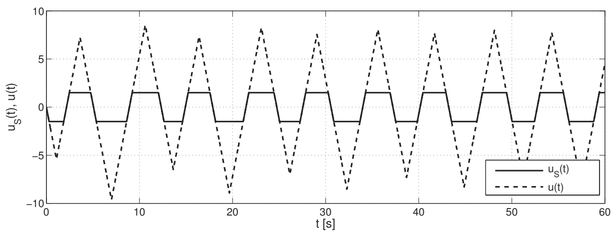

Fig. 4. The time evolutions of the actual control signal $u_s(t)$ and of $u(t)$ in the second example

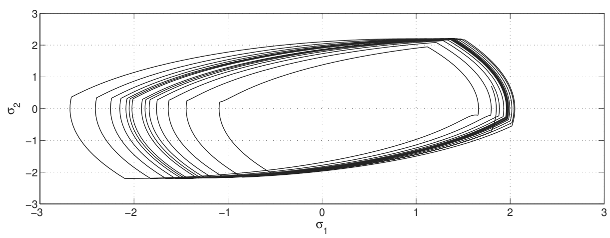

Fig. 5. The trajectory of the auxiliary system in the second example

This is better illustrated in the following example. Consider the same auxiliary system of the previous example, starting from $\sigma_1(t_0) = 1.8$, $\sigma_2(t_0) = \sqrt{2}/2$, $u(t_0) = 0$ with $F = 1$, $H = 1$, $G_1 = 0.75$, $G_2 = 1.25$. The terms which are regarded as uncertain and which respect these bounds are $f(t) = sin(t + 3\pi/4)$, $g(t) = 0.8$, and, consequently, $h(t) = cos(t + 3\pi/4)$. The controller is designed with $\alpha^* = 1$, $W = 5$, according to (16), while $U = 1.5$ satisfies (9). Fig. 4 shows the time evolutions of the actual control signal $u_S(t)$ (continuous line), and of $u(t)$ (dashed line). Fig. 5 shows the behavior of the system in this example: the auxiliary state trajectory never reaches the origin of the σ_1-σ_2 plane.

To counteract the undesired effects of the actuator saturation, an original desaturation strategy is described in the following section.

IV. THE DESATURATION STRATEGY

To face the problem previously discussed, a modification of the sub-optimal second order sliding mode controller based scheme in Fig. 1 is hereafter proposed. The idea is to modify the part of the control scheme enclosed in the dashed rectangle in Fig. 1. The modified part of the scheme is illustrated in Fig. 6. This part will be called in the reminder of the paper 'modified sub-optimal controller'. Assuming that $u(t)$ is not directly measurable, the saturation due to the actuator is replicated in the modified sub-optimal controller, so that the variable $u(t)$ now exiting the controller coincides with $u_S(t)$. Then, the control law (15) is modified as follows:

$$\begin{cases} u(t) &= \mathrm{sat}_{[-U;+U]}\left\{u(t_0) + \int_{t_0}^{t} w(\tau)d\tau\right\} \\ w(t) &= -\alpha W sign\left(\sigma_1(t) - \frac{1}{2}\sigma_M\right) \end{cases} \quad (21)$$

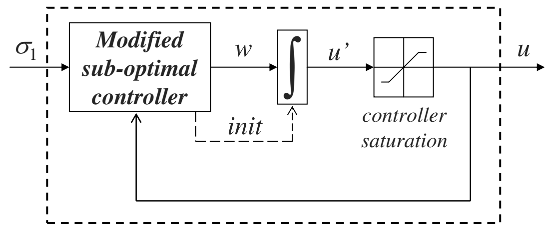

Fig. 6. The modified sub-optimal controller

where the function 'sat' means that, at any switching time t_{c_i} (that is, when $\sigma_1(t_{s_i}) = 0.5\sigma_M$) the signal *init* becomes active re-initializing the state of the integrator as follows

$$\begin{cases} \text{if } u'(t_{c_i}^-) > U \text{ then } u'(t_{c_i}^+) = U; \\ \text{if } u'(t_{c_i}^-) < -U \text{ then } u'(t_{c_i}^+) = -U; \\ \text{else } u'(t_{c_i}^+) = u'(t_{c_i}^-). \end{cases} \quad (22)$$

The following theorem proves that the proposed modified sub-optimal control law guarantees the reaching the origin of the σ_1-σ_2 plane in a finite time and the consequent enforcement of a second order sliding mode.

THEOREM 2: *Consider the auxiliary system (13)-(14), with the bounds (7), (8), (9) and (17), and the auxiliary control law in (21), with the 're-initialization strategy' (22). Then, if (16) holds, the state of (13)-(14) converges to the origin in a finite time, through the generation of a sequence of states with coordinates $(\sigma_{M_i}, 0)$, the first component being the i-th extremal value of σ_1 featuring the following contraction property:*

$$|\sigma_{M_{i+1}}| < |\sigma_{M_i}|, \quad i = 1, 2, ... \quad (23)$$

Proof-Part 1: Contraction property.

Case 1.1: Consider that the auxiliary system starts at t_0 in $\sigma_1(t_0) = \sigma_M > 0$, $\sigma_2(t_0) = 0$. It is crucial to note that at this point the system cannot be in saturation, because if $u(t_0) = U$, then from (2) and (13) it would follow that $\sigma_2(t_0) > 0$ and, for the same reason, $u(t_0) = -U$ would imply $\sigma_2(t_0) < 0$. The generated control signal is $w(t_0) = -\alpha^* W$, and $u(t)$ could reach its lower bound $-U$ before the switching time t_{c_1}. Consider two different controllers starting from the same initial condition: the only difference between them is that the first one has a saturation value $U_1 > F/G_1$ that is reached before t_{c_1}, while for the second one $U_2 \to +\infty$, and is never reached. Using the second controller, when the commutation occurs at time instant t_{c_1} such that $\sigma_1^{(2)}(t_{c_1}) = 0.5\sigma_M$ (the superscript (i), $i = 1, 2$, denotes the considered system), the corresponding value of $\sigma_2^{(2)}(t_{c_1})$ belongs to the interval

$$\left[-\sqrt{\sigma_M(\alpha^* G_2 W + H)}; -\sqrt{\sigma_M(\alpha^* G_1 W - H)}\right] \quad (24)$$

but, according to (13), is also true that

$$\sigma_2^{(2)}(t_{c_1}) = f(x(t_{c_1}), t_{c_1}) - g(x(t_{c_1}), t_{c_1})\left(U_1 + \alpha^* W(t_{c_1} - t_{s_1})\right) \quad (25)$$

where $t_{s_1} < t_{c_1}$ is the time instant when the first controller has saturated. Using the first controller instead, it is easy to verify that, when $\sigma_1^{(1)}(t_{c_1}) = 0.5\sigma_M$, one has that

$$\sigma_2^{(1)}(t_{c_1}) = f(x(t_{c_1}), t_{c_1}) - g(x(t_{c_1}), t_{c_1})U_1. \quad (26)$$

The value of $\sigma_2^{(1)}(t_{c_1})$ is obviously strictly negative, since $U = U_1$ in this case and (7), (8) and (9) hold; $\sigma_2^{(1)}(t_{c_1})$ is lower-bounded by $\sigma_2^{(2)}(t_{c_1})$, independently from the time instant when the saturation has occurred, that is, $\sigma_2^{(1)}(t_{c_1})$ belongs to the interval

$$\left[-\sqrt{\sigma_M(\alpha^* G_2 W + H)}, 0\right). \quad (27)$$

Now, in case the second controller is used, starting from any point in interval (24), for $t_1^* > t_{c_1}$, the the state trajectory crosses the abscissa axis $\sigma_2^{(2)}(t_1^*) = 0$ when $\sigma_1^{(2)}(t_1^*)$ belongs to the interval

$$\left[-\frac{1}{2}\frac{(\alpha^* G_2 - G_1)W + 2H}{G_1 W - H}\sigma_M; \frac{1}{2}\frac{(G_2 - \alpha^* G_1)W + 2H}{G_2 W + H}\sigma_M\right]. \quad (28)$$

According to Theorem 1 in [13], by virtue of (16), all the points in this interval are nearer to the origin than σ_M.

Using the second controller, starting from a point in interval (27), being $u(t_c) = -U$, one can obtain that, for $t_1^* > t_{c_1}$, when $\sigma_2(t_1^*) = 0$,

$$\sigma_1^{(1)}(t_1^*) \in \left[-\frac{1}{2}\frac{(\alpha^* G_2 - G_1)W + 2H}{G_1 W - H}\sigma_M, \frac{1}{2}\sigma_M\right). \quad (29)$$

All points in this interval are nearer to the origin than σ_M.

If the initial condition is $\sigma_1(t_0) = \sigma_M < 0$, $\sigma_2(t_0) = 0$ the proof is the same as in the considered case, with reversed extremes of the intervals.

Case 1.2: For a starting condition with $\sigma_1(t_0) > 0$ and $\sigma_2(t_0) > 0$ one has that $w(t_0) = -\alpha^* W$, and this value is kept until the first switching time instant t_{c_1} is reached. From (8), (14), (16) and (17) follows that

$$\dot{\sigma}_2(t) = h(x(t), t) - \alpha^* g(x(t), t)W < 0, \quad \forall t \in [t_0, t_{c_1}] \quad (30)$$

that is, $\sigma_2(t)$ decreases, and the trajectory of the auxiliary system moves toward the σ_1-axis. Considering that t_{c_1} cannot occur before the time instant when the a saturation bound is reached (or, in case a saturation bound is not reached, before the time instant when $\sigma_1(t_{c_1}) = \sigma_1(t_0)/2 = \sigma_M/2 < 0$), then t_{c_1} cannot occur before the σ_1-axis is crossed, for two reasons: first, no saturation can occur before the σ_1-axis is crossed, since at any time instant when $\sigma_2(t) > 0$, from (7), (8), (9) and (13) it follows that $u(t) > -U$; moreover $\dot{u}(t) < 0$ implies that U cannot be reached as well; second, being $\sigma_2(t_0) > 0$, the value of $\sigma_1(t)$, before the time instant when the σ_1-axis is crossed, increases, and it cannot happen that $\sigma_1(t) = \sigma_1(t_0)/2$ is reached during this time interval. The value $\sigma_1(t_{c_1}) = \sigma_1(t_0)/2$ will be reached after $\sigma_1(t)$ starts decreasing, and this implies that the σ_1-axis has been already crossed.

This means that, starting with $\sigma_2(t_0) > 0$, the auxiliary system state reaches the σ_1-axis in a finite time interval, and, from this moment on, the contraction of the extremal values will take place as described in Case 1.1. If the initial condition has $\sigma_1(t_0) < 0$ and $\sigma_2(t_0) < 0$ the proof is analogous to the one already seen for $\sigma_1(t_0) > 0$ and $\sigma_2(t_0) > 0$.

Case 1.3: For a starting condition with $\sigma_1(t_0) < 0$ and $\sigma_2(t_0) > 0$, the initial value of the auxiliary control variable is $w(t_0) = \alpha^* W$. For considerations analogous to those made in Case 1.2, both $\sigma_1(t)$ and $u(t)$ increase, and a switching time instant is reached after a finite time interval that is maximum if $\sigma_1(t)$ is allowed to reach the value $\sigma_1(t) = \sigma_1(t_0)/2$. At the switching time instant, the control variable $u(t)$ could have reached the saturation value U or not; in both cases, after the switching occurs, the behavior of the auxiliary system state is described by Case 1.2, even if the state is still verifying $\sigma_1(t_0) < 0$ and $\sigma_2(t_0) > 0$. If $\sigma_1(t_0) > 0$ and $\sigma_2(t_0) < 0$ the proof is analogous to the one already seen for $\sigma_1(t_0) < 0$ and $\sigma_2(t_0) > 0$.

Part 2: Finite time convergence. Considering that for any initial condition the auxiliary state reaches the σ_1-axis after a finite time interval, from (29) it is easy to obtain that

$$|\sigma_{M_j}| < \left|\frac{1}{2}\frac{(\alpha^* G_2 - G_1)W + 2H}{G_1 W - H}\right|^{j-1} |\sigma_{M_1}| \quad (31)$$

and then, using (16),

$$\lim_{j \to \infty} \sigma_{M_j} = 0. \quad (32)$$

The finite time convergence is not assured yet, because σ_{M_j} could tend to zero only asymptotically. Let us consider a generic element of the sequence $\{\sigma_{M_j}\}$, such that $\sigma_1(t_j) = \sigma_{M_j}$ and $\sigma_2(t_j) = 0$. The minimum time necessary for the controller to saturate (note that at point $(\sigma_1(t_j), \sigma_2(t_j))$ it is not in saturation), starting from t_j, is

$$T_{sat}^{min} = \frac{\Delta}{W} \quad (33)$$

where $\Delta = U - F/G_1$, while the maximum time that can elapse before the subsequent switching time instant is

$$T_{c_j}^{max} = \sqrt{\frac{|\sigma_{M_j}|}{\alpha^* G_1 W - H}}. \quad (34)$$

The value of T_{sat}^{min} is a constant value which does not depend on the iteration index j, while, from (32), one can see that

$$\lim_{j \to \infty} T_{c_j}^{max} = 0 \quad (35)$$

as σ_{M_j}, and $\{\sigma_{M_j}\}$ is a contractive sequence, as proved in Part 1. So, the sequence $\{T_{c_j}^{max}\}$ is monotonically decreasing. For this reason, after a finite time, there will be an iteration $\hat{j}$, when

$$T_{c_{\hat{j}}}^{max} < T_{sat}^{min} \quad (36)$$

and this will remain true for any $j > \hat{j}$. After this iteration, the control signal $u(t)$ will not saturate any more. From $t_{\hat{j}}$ onward, the proposed auxiliary control law (21) generates a sequence of time instants $\{t_{M_j}\}$ when an extremal value of $\sigma_1(t)$ occurs. Each term of this sequence is upper-bounded by the corresponding term of the sequence

$$\hat{t}_{M_{j+1}} = \hat{t}_{M_j} + \frac{(G_1 + \alpha^* G_2)W}{(G_1 W - H)\sqrt{\alpha^* G_2 W + H}}\sqrt{|\sigma_{M_j}|}. \quad (37)$$

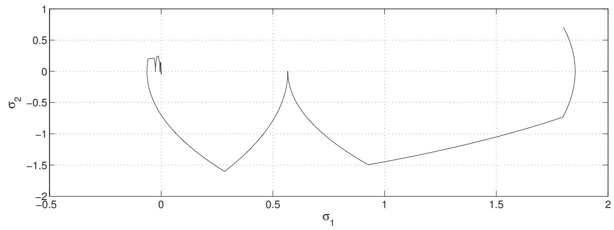

Fig. 7. Trajectory of the auxiliary system

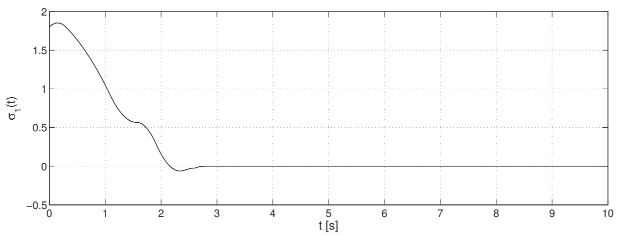

Fig. 8. Time evolution of the sliding variable $\sigma_1(t)$

From (37), recursively

$$\begin{aligned} \hat{t}_{M_{j+1}} &= \frac{(G_1+\alpha^* G_2)W}{(G_1 W - H)\sqrt{\alpha^* G_2 W + H}} \sum_{i=1}^{j} \sqrt{|\sigma_{M_i}|} + t_{M_1} \\ &= \theta \sum_{i=1}^{j} \sqrt{|\sigma_{M_i}|} + t_{M_1}. \qquad (38) \end{aligned}$$

Then, from (31), with implicit definition of the symbols,

$$\begin{aligned} t_{M_{j+1}} &< \theta \sum_{i=1}^{j} \gamma^{i-1} \sqrt{|\sigma_{M_1}|} + t_{M_1} \\ &= \theta' \sum_{j=1}^{k} \gamma^{j-1} + t_{M_1}. \qquad (39) \end{aligned}$$

Considered that assumption (16) holds and that it is trivial to see that $\gamma < 1$, one can easily obtain that

$$\lim_{j\to\infty} t_{M_j} < \frac{\theta'}{1-\gamma} + t_{M_1} \qquad (40)$$

which concludes the proof.

□

V. Simulation results

Consider again the example of Section III, in which the origin of the σ_1-σ_2-plane was not reached using the conventional algorithm (see, Figs. 4 and 5), and apply the control strategy (21)-(22). The auxiliary system state converges to the origin in a finite time as shown in Fig. 7, while the time evolution of σ_1 and of $u(t)$ are shown in Figs. 8 and 9, respectively. Note that, after about 2.7 seconds, once the sliding manifold is reached, $u(t)$ features a sinusoidal time evolution, which coincides with the equivalent control (as expected from Theorem 1). From this time instant on, saturation will not occur any more.

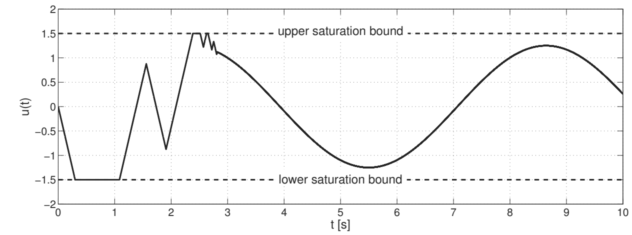

Fig. 9. Time evolution of the control signal

VI. Conclusions

The paper addresses the problem of saturation of the continuous control variable when the sub-optimal second order sliding mode controller is used in order to reduce the chattering effect. The reaching phase and the sliding phase are analyzed, showing that actuator saturation can occur only during the first one. Nevertheless, when saturation occurs, problems can arise as far as the convergence to zero of the sliding variable is concerned. In this paper, a modified sub-optimal algorithm is proposed. The modification is oriented to avoid the delay in the controller switching caused by actuator saturation. The proposed controller proves to guarantee the convergence of the sliding variable and of its first time derivative to zero in a finite time, in spite of the presence of uncertain terms affecting the system model and of the saturating actuators.

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