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# Model Predictive Control of Uncertain Continuous-Time Systems with Piecewise Constant Control Input: A Convex Approach

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*Abstract*— **In this paper two model predictive control (MPC) approaches, an on-line and an off-line MPC approach, for constrained uncertain continuous-time systems with piecewise constant control input are presented. Both MPC approaches guarantee robust asymptotic stability by taking into account the hybrid dynamics of the closed-loop system, resulting from the continuous-time system dynamics and the discrete-time model predictive controller dynamics. Examples are presented to demonstrate the applicability of the proposed MPC approaches.**

## I. INTRODUCTION

In the last decades many model predictive control (MPC) approaches have been developed for continuous-time or discrete-time systems, see e.g. [6, 7, 9, 10]. However, in many practical applications a continuous-time system is controlled via a model predictive controller that operates in a discrete-time environment, i.e. the control signal is piecewise constant and generated by a digital computer. In this paper two new MPC approaches are proposed that account for the hybrid nature of the closed-loop system resulting from the continuous-time system dynamics and from the discrete-time controller dynamics. In the on-line MPC approach, that is based on [6], the piecewise constant control signal of the model predictive controller is obtained, at each sampling instant, by minimizing an upper bound of a “worst-case” objective function using convex optimization techniques. To reduce the on-line computations, for example to apply MPC to fast systems, an off-line MPC approach is proposed as well. This model predictive controller, that is based on [9, 10], obtains a suboptimal piecewise constant control signal from an off-line created look-up table. The stability properties of both proposed MPC approaches are studied via the sampled-data control techniques of [8]. Note that unlike existing model predictive controllers with piecewise constant control signal for continuous-time systems, see e.g. [4, 5], the proposed ones can cope with state and input constraints as well as with time-varying sampling intervals.

The remainder of the paper is organized as follows: The considered MPC design problem is described in Section II. In Section III and IV the proposed on-line and off-line MPC approaches are presented. Simulation results are shown in Section V and Section VI concludes the paper.

### A. Notation

The transpose of a matrix $X$ is denoted by $X^T$, $X > 0$ (or $X \geq 0$) is a symmetric positive definite (or positive semidefinite) matrix, and $\left[\begin{smallmatrix} X & Y \\ \star & Z \end{smallmatrix}\right]$ stands for $\left[\begin{smallmatrix} X & Y \\ Y^T & Z \end{smallmatrix}\right]$ . Furthermore, $I$ (0) is the identity (zero) matrix of appropriate dimension.

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## II. PROBLEM FORMULATION

Consider the continuous-time system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \;\; x(t_0) = x_0 \qquad (1)$$

where $x \in \mathbb{R}^{n_x}$ is the system state, $u \in \mathbb{R}^{n_u}$ is the control input, $x_0 \in \mathbb{R}^{n_x}$ is the initial condition at time instant $t_0 \geq 0$, and $S(t) := [A(t) \;\; B(t)]$ is the system matrix. The system matrix $S(t)$ belongs to a polytope described by $\mathcal{P} := \{\sum_{l=1}^{n_s} \alpha_l(t)S_l, \; \sum_{l=1}^{n_s} \alpha_l(t) = 1, \; 0 \leq \alpha_l(t) \leq 1\}$, where $S_l = [A_l \;\; B_l]$ and $A_l \in \mathbb{R}^{n_x \times n_x}, B_l \in \mathbb{R}^{n_x \times n_u}$ are constant matrices. Hence, any system matrix $S(t)$ can be written as a convex combination of $n_s$ vertices $S_l$. Furthermore, the system state and control input are restricted to fulfill constraints described by the set

$$\mathcal{C} := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n_x+n_u} : |z_j(t)| \leq 1, \; j = 1, \ldots, n_c \right\}, \quad (2)$$

where $z_j = c_j x + d_j u$ with $c_j \in \mathbb{R}^{1 \times n_x}$ and $d_j \in \mathbb{R}^{1 \times n_u}$. Note that one can describe the state constraints by $d_j = 0$ and the input constraints by $c_j = 0$. The control task is to robustly stabilize the origin of system (1) with a piecewise constant control input in an optimal way while satisfying the constraints. One approach to achieve this is MPC. In MPC an optimization problem is solved at the sampling instants $t_k$, where $t_k$ is a sequence satisfying $0 \leq t_0 < \ldots < t_k < t_{k+1} < \ldots$ with $\lim_{k\to\infty} t_k = \infty$, $\delta_k = t_{k+1} - t_k$, and $0 < \delta_{\min} \leq \delta_k \leq \delta_{\max}$, to compute a piecewise constant control input. The MPC optimization problem considered in this paper is the following one:

$$\min_{K(t_k)} \; \max_{\bar{S}(\cdot)\in\mathcal{P}} J(t_k) \qquad (3)$$

subject to

$$\dot{\bar{x}}(\tau) = \bar{A}(\tau)\bar{x}(\tau) + \bar{B}(\tau)\bar{u}(\tau), \;\; \bar{x}(\tau_{k_0}) = x(t_k), \qquad (4a)$$

$$\bar{u}(\tau) = K(t_k)\bar{x}(\tau_{k_i}), \; \tau \in [\tau_{k_i}, \tau_{k_{i+1}}), i = 0, 1, \ldots, \infty, \quad (4b)$$

$$1 \geq |\bar{z}_j(\tau)|, \qquad j = 1, \ldots, n_c, \qquad (4c)$$

where the bar denotes predicted variables, e.g. $\bar{u}(\cdot)$ is the predicted piecewise constant control input based on the measured state $x(t_k)$, and $\tau_{k_i}$ is an arbitrary sequence of “predicted” sampling instants satisfying $\tau_{k_0} = t_k$, $\lim_{i\to\infty} \tau_{k_i} = \infty$, and $0 < \tau_{k_{i+1}} - \tau_{k_i} \leq \delta_{\max}$. Furthermore, the performace objective in (3) is given by

$$J(t_k) = \int_{t_k}^{\infty} (\bar{x}(s)^T Q \bar{x}(s) + \bar{u}(s)^T R \bar{u}(s)) ds, \qquad (5)$$

where $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are symmetric positive definite matrices. Hence, the goal of the MPC optimization

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problem is to compute, at time instant $t_k$, the feedback matrix $K(t_k)$ such that the robust performance objective $\max_{\bar{S}(\cdot)\in\mathcal{P}} J(t_k)$ is minimized subject to constraints and time-varying sampling intervals. As usual in MPC, the solution of the optimization problem is used only in one sampling interval, i.e. $u(t) = K(t_k)x(t_k)$ for $t \in [t_k, t_{k+1})$. At the next sampling instant $t_{k+1}$ a new feedback matrix $K(t_{k+1})$ is computed based on the new measured system state $x(t_{k+1})$. Thus, the piecewise constant control signal of the proposed model predictive controller is

$$u(t) = K(t_k)x(t_k),\ t \in [t_k, t_{k+1}),\ k = 0, \ldots, \infty. \quad (6)$$

However, the optimization problem (3)-(4c) is not attractive from a computational point of view because it is a min-max optimization problem. Therefore, the problem is relaxed by minimizing an upper bound of the robust performance objective $\max_{\bar{S}(\cdot)\in\mathcal{P}} J(t_k)$, that is derived in the next section.

### A. An upper bound of the robust performance objective

In the following an upper bound of the robust performance objective is derived that can be minimized via the feedback control law (4b). Consider the closed-loop system consisting of system (4a) and feedback control law (4b). This closed-loop system, that is of hybrid nature because the system state is described in continuous-time while the feedback control law (4b) is only updated at sampling instants $\tau_{k_i}$, is described via the hybrid (impulsive) system

$$\begin{aligned} \dot{\bar{x}}(\tau) &= \bar{A}(\tau)\bar{x}(\tau) + \bar{B}(\tau)K(t_k)\bar{\chi}(\tau), \\ \dot{\bar{\chi}}(\tau) &= 0, && \tau_{k_i} \le \tau < \tau_{k_{i+1}}, \\ \bar{x}(\tau_{k_{i+1}}) &= \bar{x}(\tau_{k_{i+1}}^-), && \tau = \tau_{k_{i+1}}, \\ \bar{\chi}(\tau_{k_{i+1}}) &= \bar{x}(\tau_{k_{i+1}}^-), && i = 0, 1, \ldots, \infty, \end{aligned} \quad (7)$$

where system state $\bar{\chi} \in \mathbb{R}^{n_x}$ describes the hybrid behavior of (4b). For system (7), an upper bound of the robust performance objective can be derived via the function [8]

$$\begin{aligned} \bar{V}(\tau) &= \bar{x}(\tau)^T P(t_k)\bar{x}(\tau) \\ &+ \int_{\tau-\mu(\tau)}^{\tau} (\delta_{\max} - \tau + s)\dot{\bar{x}}(s)^T P(t_k)\dot{\bar{x}}(s)\, ds \\ &+ (\delta_{\max} - \mu(\tau))(\bar{x}(\tau) - \bar{\chi}(\tau))^T P(t_k)(\bar{x}(\tau) - \bar{\chi}(\tau)), \end{aligned} \quad (8)$$

where $P(t_k)$ is a symmetric positive definite matrix and $\mu(\tau) = \tau - \tau_{k_i},\ \tau \in [\tau_{k_i}, \tau_{k_{i+1}})$, for $i = 0, 1, \ldots, \infty$. At sampling instants $\tau_{k_i}$, the function (8) does not increase because the second and third term are non-negative before the sampling instants $\tau_{k_i}$ and they are zero right afterwards, i.e. function (8) satisfies

$$\bar{V}(\tau_{k_i}) \le \lim_{\tau\uparrow\tau_{k_i}} \bar{V}(\tau),\ i = 0, 1, \ldots, \infty. \quad (9)$$

Furthermore, assume that function (8) satisfies

$$\dot{\bar{V}}(\tau) < -\bar{x}(\tau)^T Q\bar{x}(\tau) - \bar{u}(\tau)^T R\bar{u}(\tau) \quad (10)$$

for $\tau \in [\tau_{k_i}, \tau_{k_{i+1}}), i = 0, 1, \ldots, \infty$, and for any $\bar{S}(\cdot) \in \mathcal{P}$ and that $\bar{x}(\infty) = 0$. Then, from (9) and (10), one obtains the following upper bound on the robust performance objective:

$$\max_{\bar{S}(\cdot)\in\mathcal{P}} J(t_k) \le \bar{V}(\tau_{k_0}) = x(t_k)^T P(t_k)x(t_k). \quad (11)$$

## III. MPC WITH ON-LINE OPTIMIZATION

In this section a model predictive controller with piecewise constant control signal for uncertain continuous-time systems is proposed. This model predictive controller computes at each sampling instant $t_k$ the control input, i.e. it computes the feedback matrix $K(t_k)$, by solving on-line a convex optimization problem such that the upper bound of the robust performance objective is minimized and system (1) is stabilized. This is summarized in the next theorem.

*Theorem 1:* Consider system (1), the state and input constraints described by (2), a time sequence $t_k$ with time-varying sampling intervals $\delta_k = t_{k+1} - t_k$, $0 < \delta_k \le \delta_{\max}$, $\delta_{\max} > 0$, and a model predictive controller that computes the feedback matrix $K(t_k)$ of its control signal (6) for the sampling interval $\delta_k$ via the following optimization problem:

$$\min_{\epsilon, L, M, N} \quad \epsilon \quad (12)$$

subject to

$$\begin{bmatrix} -1 & -x(t_k)^T \\ \star & -M \end{bmatrix} \le 0 \quad (13a)$$

$$\begin{bmatrix} -1 & -c_j M - d_j L \\ \star & -M \end{bmatrix} \le 0 \quad (13b)$$

$$\begin{bmatrix} V_l + \delta_{\max} W_l & \delta_{\max} U_l^T & \begin{bmatrix} M \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ L^T \end{bmatrix} \\ \star & -\delta_{\max} M & 0 & 0 \\ \star & \star & -Q^{-1}\epsilon & 0 \\ \star & \star & \star & -R^{-1}\epsilon \end{bmatrix} < 0 \quad (13c)$$

$$\begin{bmatrix} V_l & \delta_{\max} U_l^T & \begin{bmatrix} M \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ L^T \end{bmatrix} & \delta_{\max} N \\ \star & -\delta_{\max} M & 0 & 0 & 0 \\ \star & \star & -Q^{-1}\epsilon & 0 & 0 \\ \star & \star & \star & -R^{-1}\epsilon & 0 \\ \star & \star & \star & \star & -\delta_{\max} M \end{bmatrix} < 0 \quad (13d)$$

$$l = 1, \ldots, n_s,\ j = 1, \ldots n_c$$

with

$$\begin{aligned} U_l &= \begin{bmatrix} A_l M & B_l L \end{bmatrix}, \\ V_l &= \begin{bmatrix} I \\ 0 \end{bmatrix} U_l + U_l^T \begin{bmatrix} I & 0 \end{bmatrix} - N \begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N^T \\ &\quad - \begin{bmatrix} I \\ -I \end{bmatrix} M \begin{bmatrix} I & -I \end{bmatrix}, \\ W_l &= U_l^T \begin{bmatrix} I & -I \end{bmatrix} + \begin{bmatrix} I \\ -I \end{bmatrix} U_l, \end{aligned}$$

where $M \in \mathbb{R}^{n_x \times n_x}$, $N = [N_1^T\ \ N_2^T]^T$ with $N_1, N_2 \in \mathbb{R}^{n_x \times n_x}$, $L \in \mathbb{R}^{n_u \times n_x}$, $\epsilon \in \mathbb{R}$. Furthermore, $x(t_k)$ is the measured system state at sampling instant $t_k$ and $c_j, d_j$, $j = 1, \ldots, n_c$ are vectors that describe the constraints (2). Then system (1) is robustly asymptotically stabilized without violating the constraints (2) via the piecewise constant control signal (6) with feedback matrix $K(t_k) = LM^{-1}$, if the optimization problem (12), (13) is feasible at time instant $t_0$. The upper bound of the robust performance objective is minimized, i.e. it is bounded by $x(t_k)^T P(t_k)x(t_k) \le \gamma(t_k)$ with $P(t_k) = \epsilon^{-1}M^{-1}$ and $\gamma(t_k) = \epsilon$.

*Proof.* The proof is divided into five parts. In the first part of the proof it is shown that the upper bound of the robust performance objective is minimized. In the second part it is shown that the predicted system (4a) is stabilized via the feedback control law (4b) with the feedback matrix $K(t_k)$ computed at sampling instant $t_k$. This can be interpreted in terms of invariant ellipsoids [1], that is used in the other parts of the proof. Then, in the third part of the proof, it is shown that the predicted system (4a) satisfies the state and input constraints. In the fourth part it is proven that feasibility of the optimization problem at time instant $t_k$, e.g. at the beginning, implies feasibility afterwards. Finally, in the fifth part, the asymptotic stability of the time-varying hybrid (impulsive) closed-loop system is proven.

*Minimization of the upper bound:* In the following it is shown that the upper bound of the robust performance index is minimized via the feedback control law (4a), that feedback matrix $K(t_k)$ is computed at time instant $t_k$. Suppose that there are no constraints, i.e. the LMIs (13b) are neglected. In a first step it is shown that the function (8) satisfies condition (10) if the LMIs (13c) and (13d) are feasible. To show this, consider the derivative of $\bar{V}(\tau)$ along the system (7) for $\tau_{k_i} \in [\tau_{k_i}, \tau_{k_{i+1}}), i = 0, \dots, \infty$, i.e.

$$\begin{aligned}\dot{\bar{V}}(\tau) &= 2\bar{x}(\tau)^T P(t_k)\dot{\bar{x}}(\tau)\\ &\quad + 2(\delta_{\max} - \mu(\tau))(\bar{x}(\tau) - \bar{\chi}(\tau))^T P(t_k)\dot{\bar{x}}(\tau)\\ &\quad - (\bar{x}(\tau) - \bar{\chi}(\tau))^T P(t_k)(\bar{x}(\tau) - \bar{\chi}(\tau))\\ &\quad + \delta_{\max}\dot{\bar{x}}(\tau)^T P(t_k)\dot{\bar{x}}(\tau)\\ &\quad - \int_{\tau-\mu(\tau)}^{\tau} \dot{\bar{x}}(s)^T P(t_k)\dot{\bar{x}}(s)ds.\end{aligned} \tag{14}$$

Using the fact that $\bar{x}(\tau) - \bar{\chi}(\tau)$ equals $\bar{x}(\tau) - \bar{x}(\tau - \mu(\tau))$ and that the last term in (14) can be overestimated by

$$\begin{aligned}-\int_{\tau-\mu(\tau)}^{\tau} \dot{\bar{x}}(s)^T P(t_k)\dot{\bar{x}}(s)ds &\le \mu(\tau)\bar{\xi}(\tau)^T Y P(t_k)^{-1} Y^T \bar{\xi}(\tau)\\ &\quad - 2\bar{\xi}(\tau)^T Y(\bar{x}(\tau) - \bar{\chi}(\tau)),\end{aligned}$$

where $Y = [Y_1^T \ Y_2^T]^T$ with $Y_1, Y_2 \in \mathbb{R}^{n_x \times n_x}$ and $\bar{\xi} = [\bar{x}^T \ \bar{\chi}^T]^T$, an upper bound of (14) is

$$\begin{aligned}\dot{\bar{V}}(\tau) \le \bar{\xi}(\tau)^T(X_1(\tau) + (\delta_{\max} - \mu(\tau))X_2(\tau))\bar{\xi}(\tau)\\ + \mu(\tau)\bar{\xi}(\tau)^T X_3\bar{\xi}(\tau)\end{aligned} \tag{15}$$

with

$$\begin{aligned}Z(\tau) &= [\bar{A}(\tau) \ \bar{B}(\tau)K(t_k)],\\ X_1(\tau) &= \begin{bmatrix} I \\ 0 \end{bmatrix} P(t_k)Z(\tau) + Z(\tau)^T P(t_k)\begin{bmatrix} I & 0 \end{bmatrix}\\ &- \begin{bmatrix} I \\ -I \end{bmatrix} P(t_k)\begin{bmatrix} I & -I \end{bmatrix} + \delta_{\max} Z(\tau)^T P(t_k) Z(\tau)\\ &- Y\begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} Y^T,\\ X_2(\tau) &= \begin{bmatrix} I \\ -I \end{bmatrix} P(t_k)Z(\tau) + Z(\tau)^T P(t_k)\begin{bmatrix} I & -I \end{bmatrix},\\ X_3 &= Y P(t_k)^{-1} Y^T.\end{aligned}$$

Hence, the function $\bar{V}(\tau)$ satisfies

$$\dot{\bar{V}}(\tau) + \bar{x}(\tau)^T Q\bar{x}(\tau) + \bar{\chi}(\tau)^T K(t_k)^T R K(t_k)\bar{\chi}(\tau) < 0 \tag{16}$$

for $\tau \in [\tau_{k_i}, \tau_{k_{i+1}}), i = 0, 1, \dots, \infty$, if

$$X_1(\tau) + (\delta_{\max} - \mu(\tau))X_2(\tau) + \mu(\tau)X_3 + X_4 < 0 \tag{17}$$

with $X_4 = -\mathrm{diag}([Q, K(t_k)^T R K(t_k)])$ holds for all $\mu(\tau) \in [0, \delta_{\max}]$. Inequality (17) is a convex combination of $X_1(\tau) + \delta_{\max} X_2(\tau) + X_4$ and $X_1(\tau) + X_4 + \delta_{\max} X_3(\tau) + X_4$ and thus it is satisfied if and only if the following inequalities hold:

$$X_1(\tau) + \delta_{\max} X_2(\tau) + X_4 < 0, \tag{18a}$$

$$X_1(\tau) + \delta_{\max} X_3 + X_4 < 0. \tag{18b}$$

By pre- and post-multiplying inequalities (18a) and (18b) with $\mathrm{diag}([P(t_k)^{-1}, P(t_k)^{-1}])$, defining the variables $N = \epsilon\mathrm{diag}([P(t_k)^{-1}, P(t_k)^{-1}])YP(t_k)^{-1}$, $M = \epsilon P(t_k)^{-1}$ $L = K(t_k)M$, using Schur complement operations, and taking into account that (18a) and (18b) are affine in $\bar{S}(\tau) \in \mathcal{P}$, one can conclude that inequalities (18a) and (18b) are satisfied if and only if inequalities (13c) and (13d) are feasible. Thus, inequality (16), i.e. condition (10), is satisfied if inequalities (13c) and (13d) are feasible for some $M, L$, and $\epsilon$.

In the second step it is shown that the upper bound of the robust performance index is minimized. Since condition (10) is satisfied, it is known that $x(t_k)^T P(t_k) x(t_k)$ is an upper bound of the robust performance objective at sampling instant $t_k$. From (13a), one obtains $x(t_k)^T M^{-1} x(t_k) \le 1$. Since $\epsilon = \gamma(t_k)$ and $M = \epsilon P(t_k)^{-1}$, this is equivalent to $x(t_k)^T P(t_k) x(t_k) \le \gamma(t_k)$. Hence, it follows from (12) and (13a) that the upper bound of the robust performance objective is minimized via the feedback control law (4a) with feedback matrix $K(t_k) = LM^{-1}$.

*Invariant ellipsoid:* In this part of the proof an invariant ellipsoid is introduced to incorporate the state and input constraints as LMI constraints in the next part of the proof. Suppose that there are no constraints, i.e. the LMIs (13b) are neglected, and that the optimization problem of Theorem 1 is feasible at sampling instant $t_k$. Then

$$\mathcal{E}(t_k) = \left\{\eta \in \mathbb{R}^n \mid \eta^T P(t_k)\eta \le \gamma(t_k)\right\} \tag{19}$$

is an invariant ellipsoid [1] for system (7), i.e. every trajectory $\bar{x}(\tau)$ of (7) with $x(t_k) \in \mathcal{E}(t_k)$ satisfies $\bar{x}(\tau) \in \mathcal{E}(t_k)\ \forall\tau \ge \tau_{k_0}$. Of course, $\mathcal{E}(t_k)$ is an invariant ellipsoid for the unconstrained system (7) if $\bar{V}(\tau)$, that satisfies $\bar{V}(\tau_{k_0}) = x(t_k)^T P(t_k) x(t_k) \le \gamma(t_k)$, is a Lyapunov function for system (7). As shown in [8], $\bar{V}(\tau)$ is a Lyapunov function for system (7) if its time derivative satisfies $\dot{\bar{V}}(\tau) \le -\theta||\xi(\tau)||^2$ for some $\theta > 0$ and for any $\bar{S}(\cdot) \in \mathcal{P}$. Here, the negative definiteness of $\dot{\bar{V}}(\tau)$ can be established from inequality (16) because $Q$ and $R$ are positive definite matrices. Hence, $\mathcal{E}(t_k)$ is an invariant ellipsoid for the predicted system (7) if the optimization problem of Theorem 1 is feasible at sampling instant $t_k$.

*Constraints:* Suppose that the optimization problem of Theorem 1 is feasible at sampling instant $t_k$. Hence, $\mathcal{E}(t_k)$ is an invariant ellipsoid for system (4a) under the feedback control law (4b), i.e. for system (7). Since the constraints (2) are described by a polytope, i.e. a set of linear inequalities, the ellipsoid $\mathcal{E}(t_k)$ is contained in the polytope $\mathcal{C}$ if and only if

$$z_j(P(t_k)^{-1}\gamma(t_k))z_j^T \leq 1, \ \ j = 1, \ldots, n_c, \tag{20}$$

where $z_j = c_j + d_j K(t_k)$, see e.g. [1]. Using the Schur complement operation, (20) becomes

$$\begin{bmatrix} -1 & -z_j\gamma(t_k)P(t_k)^{-1} - z_jK(t_k)\gamma(t_k)P(t_k)^{-1} \\ \star & -\gamma(t_k)P(t_k)^{-1} \end{bmatrix} \leq 0 \tag{21}$$

and with $\epsilon = \gamma(t_k)$, $M = \epsilon P(t_k)^{-1}$, $L = \epsilon K(t_k)P(t_k)^{-1}$ one obtains (13b). Hence, if the optimization of Theorem 1 problem is feasible, the invariant ellipsoid $\mathcal{E}(t_k)$ is contained in the polytope $\mathcal{C}$ which implies that system (7) fulfills the constraints (2) for all times, i.e. for $\tau \geq \tau_{k_0}$.

*Feasibility:* Suppose that the optimization problem of Theorem 1 is feasible at time instant $t_k$. Therefore, there exits a feedback matrix $K(t_k)$ of the feedback control law (4a) such that $\mathcal{E}(t_k)$ is an invariant ellipsoid for the predicted system (7). Thus, one has $\bar{x}(\tau_{k_{i+1}})^T P(t_k)\bar{x}(\tau_{k_{i+1}}) \leq \gamma(t_k)$, $i = 0, \ldots, \infty$. Since the sampling instant $t_{k+1}$ lies in the time interval $[\tau_{k_0}, \tau_{k_0} + \delta_{\max})$ and since the system state $x(\cdot)$, that is solution of system (1) for some $S(\cdot) \in \mathcal{P}$ driven by $u(t) = K(t_k)x(t_k)$ for $t \in [t_k, t_{k+1})$, equals some $\bar{x}(\cdot)$ of the predicted system (7), $x(t_{k+1})$ also satisfies $x(t_{k+1})^T P(t_k)x(t_{k+1}) \leq \gamma(t_k)$. Hence, if the optimization problem of Theorem 1 is feasible at time instant $t_k$ its feasibility is guaranteed, e.g. with matrices $K(t_k)$ and $P(t_k)$, at all subsequent sampling instants $t_{k+i}$, $i = 1, \ldots, \infty$.

*Asymptotic stability of the closed-loop system:* The dynamics of the closed-loop system can be described by the time-varying hybrid (impulsive) system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)K(t_k)\chi(t), \\ \dot{\chi}(t) &= 0, & t_k \leq t < t_{k+1}, \\ x(t_{k+1}) &= x(t_{k+1}^-), & t = t_{k+1}, \\ \chi(t_{k+1}) &= x(t_{k+1}^-), & k = 0, 1, \ldots, \infty, \end{aligned} \tag{22}$$

where $K(t_k)$, $K(t_{k+1})$, and so on are computed on-line. The stability of (22) is analyzed via the function

$$\begin{aligned} V(t) &= x^T(t)P(t_k)x(t) \\ &+ \int_{t-\nu(t)}^{t} (\delta_{\max} - t + s)\dot{x}(s)^T P(t_k)\dot{x}(s)\, ds \\ &+ (\delta_{\max} - \nu(t))(x(t) - \chi(t))^T P(t_k)(x(t) - \chi(t)), \end{aligned} \tag{23}$$

where $\nu(t) = t - t_k$, $t \in [t_k, t_{k+1})$, for $k = 0, 1, \ldots, \infty$. Note that, in contrast to function (8), the matrix $P(t_k)$ of function (23) changes at the sampling instants $t_k$. Furthermore, as shown in [8], function (23) satisfies $\alpha_1(t_k)||x(t)||^2 \leq V(t) \leq \alpha_2(t_k)||\xi(t)||^2$, where $\alpha_1(t_k) > 0$, $\alpha_2(t_k) > 0$, and $\xi = [x^T \ \chi^T]^T$. First, suppose that the optimization problem of Theorem 1 is feasible at sampling instant $t_0$. Therefore, the feasibility of the optimization problem is guaranteed for all future sampling instants $t_k$. Next, consider the time interval $[t_k, t_{k+1})$ with corresponding matrices $K(t_k)$ and $P(t_k)$. The time derivative of function (23) satisfies $\dot{V}(t) \leq -\alpha_3(t_k)||\xi(t)||^2$, $\alpha_3(t_k) > 0$. This can be established from equation (16) because it is satisfied for any $\bar{S}(\cdot) \in \mathcal{P}$ and for any sampling instant $\tau_{k_1} \in [\tau_{k_0}, \tau_{k_0} + \delta_{\max})$, i.e. it also satisfied for $S(\cdot) \in \mathcal{P}$ and for $t_{k+1}$. Hence, for $t \in [t_k, t_{k+1})$ the function (23) is bounded by

$$V(t) \leq e^{-\frac{\alpha_3(t_k)}{\alpha_2(t_k)}(t-t_k)} V(t_k). \tag{24}$$

Furthermore, at time instant $t_{k+1}$, the inequality

$$x(t_{k+1})^T P(t_{k+1})x(t_{k+1}) \leq x(t_{k+1})^T P(t_k)x(t_{k+1}) \tag{25}$$

is fulfilled because $P(t_{k+1})$ is the optimal solution of the optimization problem of Theorem 1 whereas $P(t_k)$ is only a feasible solution of this optimization problem (see fourth part of the proof) at sampling instant $t_{k+1}$. Hence, it follows that function (23) does not increase at sampling instant $t_{k+1}$. Now suppose that closed-loop system (22) encounters $s$ updates of the feedback matrix $K(t_k)$ at $t_0 < t_1 < \ldots < t_s < t$ on the time interval $[t_0, t]$. From (24) and (25), one obtains for $t \geq t_s$ the following upper bound:

$$\begin{aligned} V(t) &\leq e^{-\frac{\alpha_3(t_s)}{\alpha_2(t_s)}(t-t_s)} V(t_s) \\ &\vdots \\ &\leq e^{-\frac{\alpha_3(t_s)}{\alpha_2(t_s)}(t-t_s)} \times \ldots \times e^{-\frac{\alpha_3(t_0)}{\alpha_2(t_0)}\delta_0} V(t_0). \end{aligned} \tag{26}$$

Consequently, $||x(t)|| \leq (\alpha_4(t_0)/\alpha_1(t_s))^{\frac{1}{2}} e^{-\frac{\alpha_3(t_s)}{2\alpha_2(t_s)}(t-t_s)} \times \ldots \times e^{-\frac{\alpha_3(t_0)}{2\alpha_2(t_0)}\delta_0}||x(t_0)||$, where $\alpha_4(t_0) = \lambda_{\max}(P(t_0)) > 0$, and so the closed-loop system (22) is asymptotically stable, i.e the proposed model predictive controller asymptotically stabilizes system (1) if the optimization problem of Theorem 1 is feasible at time instant $t_0$. This completes the proof. ■

Summarizing, a robustly asymptotically stabilizing model predictive controller for linear continuous-time systems has been presented in this section. At each sampling instant $t_k$, the model predictive controller computes the feedback control law on-line via solving a convex optimization problem involving LMIs. Even though this convex optimization problem can be efficiently solved using semidefinite programming algorithms [2], the computation time of the optimizer may still take too much time in order to apply the MPC approach to processes with fast system dynamics. Hence, an off-line MPC approach, that reduces the on-line computations, is given in the next section.

## IV. MPC WITH OFF-LINE OPTIMIZATION

The off-line MPC approach for systems of the form (1) is described by Algorithm 1 and its stabilizing behavior is stated in Theorem 2. Note that Algorithm 1 coincides, except for the computations of the matrices, with the off-line MPC approach for discrete-time systems [10].

*Algorithm 1:*

1) Off-line: Compute, based on a given feasible system state $x_1 \in \mathbb{R}^{n_x}$, a sequence of $\epsilon_i, L_i, M_i, N_i$ for $i = 1, \dots, n_e$ as described below. Set $i := 1$.
   a) Compute $\epsilon_i, L_i, M_i, N_i$ for a given system state $x_i \in \mathbb{R}^{n_x}$ via the optimization problem of Theorem 1 with the additional constraint $M_{i-1} > M_i$, that is ignored for $i = 1$, and store $M_i^{-1}, K_i = L_i M_i^{-1}$ in a look-up table.
   b) If $i < n_e$, select a new system state $x_{i+1} \in \mathbb{R}^{n_x}$ such that $x_{i+1}^T M_i^{-1} x_{i+1} < 1$ holds. Set $i := i + 1$ and go to step a).
2) On-line: Suppose that the initial condition $x(t_0)$ of system (1) satisfies $x(t_0)^T M_1^{-1} x(t_0) \leq 1$ and let $x(t_k)$ be the system state at sampling instant $t_k$. Search over $M_i^{-1}$ in the look-up table to find the largest index $i$ such that $x(t_k)^T M_i^{-1} x(t_k) \leq 1$ holds. Set $K(t_k) = K_i$ and apply $u(t) = K(t_k)x(t_k)$ for $t \in [t_k, t_{k+1})$ to system (1) until the next sampling instant $t_{k+1}$.

*Theorem 2:* Consider the continuous-time system (1) with constraints (2). If the initial condition $x(t_0)$ of system (1) satisfies $x(t_0)^T M_1^{-1} x(t_0) \leq 1$, then the off-line MPC approach described in Algorithm 1 asymptotically stabilizes system (1).

*Proof:* The proof, that is based on the proof of [10], is omitted due to limited space. ■

Summarizing, in this section an off-line MPC approach to for continuous-time systems with piecewise constant control input has been presented. The advantage of this off-line MPC approach is that the on-line computations are reduced compared to the model predictive controller of Section III, see also Section III.C in [10] for a more detailed discussion. However, the drawback of the off-line MPC approach is that it is only suboptimal because the feedback matrix $K(t_k)$ is not necessarily recalculated at every sampling instant $t_k$, e.g. if $x(t_k)$ and $x(t_{k+1})$ are in the same ellipsoid.

## V. Examples

In this section the applicability of the proposed MPC approaches is demonstrated via two examples.

### *A. Spring-mass system*

In the following the model predictive controllers of Section III and IV with $Q = \mathrm{diag}([3, 3, 1, 1])$, $R = 0.1$, and $\delta_{\max} = 0.2$, are applied to control the uncertain spring-mass system [3], that is shown in Figure 1, with input constraint $|u| \leq 2$. Additionally, for the off-line MPC approach the system state $x_i$ of Algorithm 1 is discretized into $\{[1\ 0]^T, [0.8\ \ 0]^T, [0.7\ \ 0]^T, [0.6\ \ 0]^T, [0.5\ \ 0]^T, [0.4\ \ 0]^T, [0.3\ \ 0]^T, [0.2\ 0]^T, [0.15\ 0]^T, [0.1\ 0]^T\}$, see also Remark 5 in [10]. The simulation results in Figure 2 illustrate that the spring-mass system is asymptotically stabilized via the MPC approaches.

### *B. Continuous stirred tank reactor*

Consider the CSTR [10] shown in Figure 1. The dynamics of the CSTR is described by a polytopic system with matrices $A_1 = \begin{bmatrix} -1.6576 & -0.0094 \\ 6.5763 & -6.2465 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1.6576 & -0.0094 \\ 65.7625 & -5.4048 \end{bmatrix}$, $A_3 = \begin{bmatrix} -7.5763 & -0.0935 \\ 65.7625 & -5.4048 \end{bmatrix}, A_4 = \begin{bmatrix} -7.5763 & -0.0935 \\ 657.6253 & 3.0115 \end{bmatrix}$ and $B_i = \begin{bmatrix} 0.15 & 0 \\ 0 & -0.912 \end{bmatrix}, i = 1, \dots, 4$. Furthermore, the parameters of the MPC controllers are $\delta_{\max} = 0.1$, $|u_1| \leq 0.5$, $|u_2| \leq 1$, $Q = \mathrm{diag}([20, 20])$, $R = \mathrm{diag}([0.2, 0.2])$, and, only for the off-line MPC approach, $x_i$ in Algorithm 1 is given by the set $\{[0.14\ 0]^T, [0.09\ 0]^T, [0.07\ 0]^T, [0.06\ ]^T, [0.05\ 0]^T, [0.03\ 0]^T, [0.02\ 0]^T, [0.017\ 0]^T, [0.01\ 0]^T, [0.001\ 0]^T\}$. Figure 3 shows that the proposed MPC approaches robustly asymptotically stabilize the CSTR with similar performance.

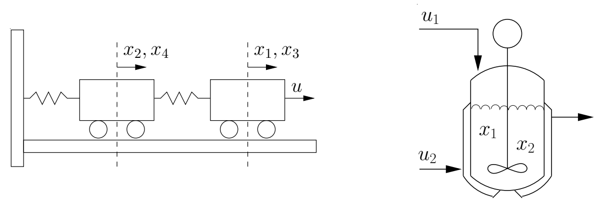


Fig. 1. Left: spring-mass system; $x_1, x_2$ are car positions, $x_3, x_4$ are car velocities. Right: continuous stirred tank reactor; $x_1$ represents reactor concentration and $x_2$ reactor temperature.

## VI. Summary

In this paper two new model predictive controllers with piecewise constant control signal for uncertain continuous-time system have been developed that can cope with state and input constraints as well as with time-varying sampling intervals. In particular, an on-line MPC approach and an off-line MPC approach, that has reduced on-line computations, have been presented. Finally, the proposed model predictive controllers have been successfully applied to control a spring-mass system and a CSTR.

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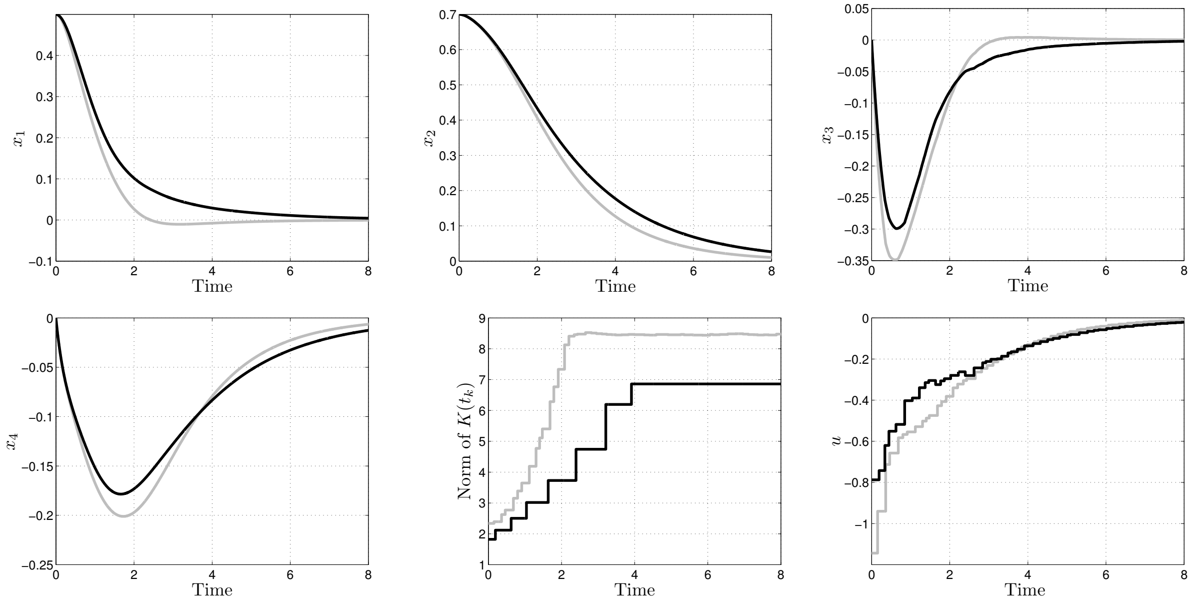


Fig. 2. Closed-loop response of the mass-spring system: gray lines, on-line MPC approach; black lines, off-line MPC approach. The system initial condition is $x_0 = [0.5\ 0.7\ 0\ 0]^T$, the input constraint is $|u| \leq 2$, and the time-varying sampling intervals $\delta_k$ are modeled by an uniform probability distribution between $\delta_{\min} = 0.1$ and $\delta_{\max} = 0.2$.

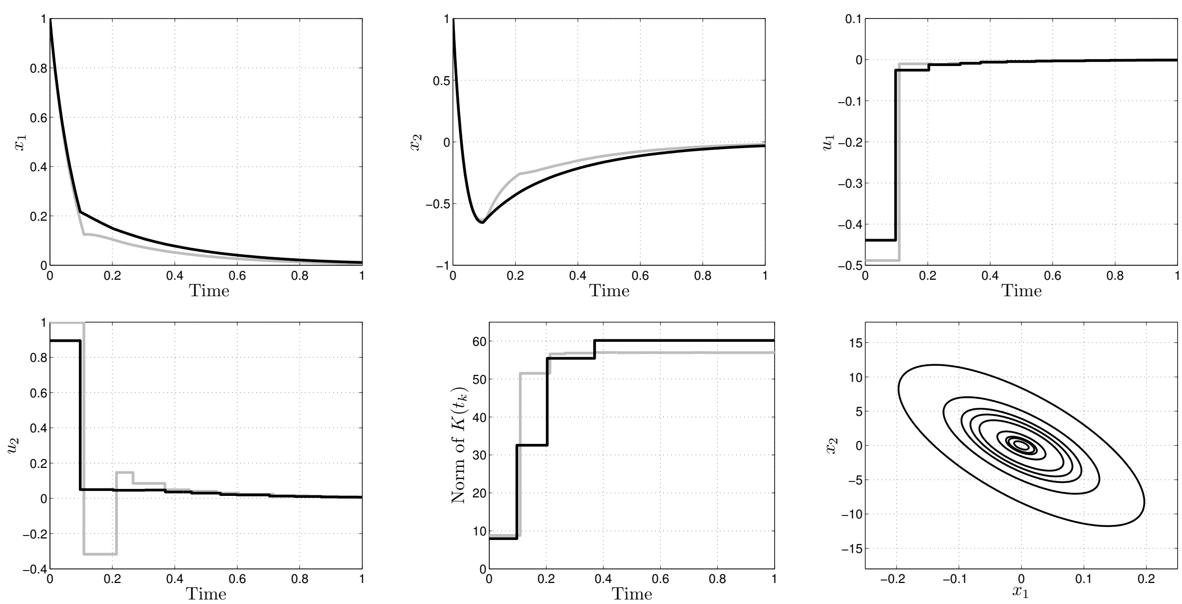


Fig. 3. Closed-loop response of the CSTR: gray lines, on-line MPC approach; black lines, off-line MPC approach. The system initial condition is $x_0 = [0.1\ 2]^T$, the input constraints are $|u_1| \leq 0.5$ and $|u_2| \leq 1$, and the time-varying sampling intervals $\delta_k$ are modeled by an uniform probability distribution between $\delta_{\min} = 0.05$ and $\delta_{\max} = 0.1$. Lower right figure shows the ellipsoids defined by $M_i^{-1}$ of the off-line MPC approach.