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Robust Output Feedback Model Predictive Control for Linear Systems via Moving Horizon Estimation

D. Sui, L. Feng, M. Hovd

***Abstract*— This paper provides a simple approach to the problem of robust output feedback model predictive control (MPC) for linear systems with state and input constraints, subject to bounded state disturbances and output measurement errors. The problem of estimating the state is addressed by using moving horizon estimation (MHE). For such an MHE estimator, it is shown that the state estimation error converges and stays in some set, which is taken into account in the design of the output feedback MPC controllers. In the MPC formulation where the nominal system is considered, the constraints are tightened in a monotonic sequence such that satisfaction of the input and state constraints is guaranteed. Robust stability of an invariant set for the closed-loop original system is ensured. The performance of the approach is assessed via a numerical example.**

***Index Terms*— Model predictive control; Moving horizon estimation; Constrained linear systems with bounded disturbances.**

I. INTRODUCTION

Model predictive control (MPC) is a feedback scheme in which an optimal control problem is solved at each time step and only the first step of the control sequence is applied [1]. Since MPC has the ability to handle hard constraints, it has received great attention in the literature. In most MPC formulations, the state feedback is assumed, which requires full knowledge of the state [2], [3]. In practice, the measurements contain noise, and often internal states are not measurable. Ignoring measurement errors may result in degradation in performance or even cause the instability.

This paper considers the problem of robust output feedback MPC for linear systems with state and input constraints, subject to bounded state disturbances and output measurement errors. The motivation of this paper is to provide an approach for computing output feedback MPC controllers that ensure the satisfaction of state and input constraints and the robust stability of the closed-loop original system.

For robust output feedback MPC, a common approach is to combine an observer with a standard predictive scheme, where the state estimate substitutes for the true system state. When the system dynamics is linear, a fixed linear observer (*e.g.* Luenberger observer, see [4], [5], [6]) is often employed. The equation for the Luenberger observer contains a term that corrects the current state estimates by an amount proportional to the prediction error: the estimation of the current output minus the actual measurement. Inclusion of this correction ensures stability and convergence of the observer. The design of the Luenberger observer is an important part in the output feedback controller design, which determines not only the performance of the observer, but also the size of the estimation error bound. However, few discussions about it are mentioned in the recent contributions [4], [5], [6]. The Kalman filter (optimal prediction) is the other most popular method of output feedback in practical implementations. The Kalman filter is the optimum estimator when the corrupting noise has a Gaussian probability distribution. Like the Luenberger observer, the Kalman filter also includes a correction factor to insure stability and convergence. However, the Kalman filter is known to lack robustness to modeling errors [7].

A promising approach is using moving horizon estimator (MHE). The ideas of MHE date bake to the early 1990s, see [8]. In MHE, the estimates of the states are obtained by solving an optimization problem, which penalizes the deviation between measurements and predicted outputs of a system. The basic strategy is to estimate the state using a moving and fixed-size window of data. When a new measurement becomes available, the oldest measurement is removed from the data window and the newest measurement is added. The problem size of the estimation is bounded, therefore, by looking at only a subset of the available information [9]. MHE approach is based on a batch of the most recent information, which results in a higher degree of robustness and so makes MHE well-suited in the presence of modeling uncertainties and/or numerical errors, see [10].

The design of output feedback MPC can be tackled by two approaches. One approach is to pursue the "certainty equivalence" principle and try to separate the estimation error from the state feedback by time scale separation and therefore make the observer dynamics sufficiently faster than controller dynamics. This may be achieved using high-gain [11] or deadbeat observers [12]. However, such approaches are not expected to be useful in the presence of noise, and therefore of little practical value in low-level control. Another approach is based on accounting for the errors in the state estimate by robust MPC controller design. With such an approach, a state estimator that provides estimation error bounds is typically required [4], [13], [14], [15]. In [5], state estimates with bounded error within an invariant set are provided by a simple Luenberger observer, and a tube-based robust predictive controller design is used, while the control paradigm is shifted from control of true

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process states to control of nominal estimator states.

In this paper, the proposed output feedback controller consists of an MHE estimator and a robustly stabilizing, tube-based, MPC controller. Under some conditions the MHE estimation error dynamics is stable and errors converge to the minimal disturbance invariant set, $\mathbb{E}$, of such a system. The errors are taken into account by introducing the set $\mathbb{E}$ in the controller design. Like the approach proposed in [5], the controller uses a tube, the center of which is obtained by solving a nominal MPC problem and within which the estimated state is guaranteed to remain. The problem is addressed by steering the tube to the origin. Due to considering the nominal system, the constraints in the optimization are tightened such that the satisfaction of the input and state constraints for the original system is guaranteed. Unlike the work in [5], in our approach the constraints are tightened in a monotonic sequence hence relaxed. Robust stability of an invariant set for the closed-loop original system is guaranteed. The computational complexity of the resulting controller is similar to that of the standard, nominal MPC controller. Furthermore, the proposed output feedback MPC problem can be solved off-line by using multi-parametric programming technique.

The paper is organized as follows. Section II discusses the class of systems considered, states several assumptions and reviews some definitions. Section III introduces the idea of MHE and shows the estimation error is contained in a disturbance invariant set. In Section IV, the framework of the robust output feedback MPC is introduced and its properties are stated. The effectiveness of the proposed output feedback controller is illustrated in Section V. Conclusions are given in Section VI.

Notation and Basic Definitions: Positive definite (semi-definite) square matrix A is denoted by $A \succ 0$ ($A \succeq 0$) and $A \succ (\succeq) B$ means $A - B \succ (\succeq) 0$. $\|x\|_A^2 = x^T A x$ with $A \succ 0$. $\|\cdot\|$ is the Euclidean norm. Let $\rho(A)$ denote spectral radius of a square matrix A. A set $X \subset \mathbb{R}^n$ is a C set if it is a compact, convex set that contains the origin in its (non-empty) interior. Suppose $X, Y \subset \mathbb{R}^n$, the interior of X is $int(X)$; $|X|$ is its cardinality; the P-difference of X and Y is $X \ominus Y = \{z \in \mathbb{R}^n : z + y \in X, \forall y \in Y\}$ and the Minkowski sum is $X \oplus Y = \{z \in \mathbb{R}^n : z = x + y, x \in X, y \in Y\}$. A polyhedron is the (convex) intersection of a finite number of open and/or closed half-spaces and a polytope is the closed and bounded polyhedron.

II. PRELIMINARIES

The following discrete-time, linear time-invariant system is considered,

$$x(t+1) = Ax(t) + Bu(t) + Dw(t), \quad \forall t \geq 0 \qquad \text{(1a)}$$

$$y(t) = Cx(t) + Ev(t), \quad \forall t \geq 0, \qquad \text{(1b)}$$

where t is the discrete time index, $x(\cdot)$, $u(\cdot)$ and $y(\cdot)$ are the state, input and measured output respectively and $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$. $w \in \mathbb{R}^{n_w}$ is an unknown state disturbance, $v \in \mathbb{R}^{n_v}$ is a measurement noise and disturbances w, v are known only to the extent that they lie, respectively, in the C sets $\mathbb{W}$ and $\mathbb{V}$.

System (1) is subject to the following sets of hard state and input constraints:

$$x(t) \in \mathbb{X}, \quad u(t) \in \mathbb{U}, \quad \forall t \geq 0. \qquad (2)$$

It is assumed in this paper that:

(A1) the couple (A,B) is controllable and (A,C) is observable;

(A2) $\mathbb{X}$ and $\mathbb{U}$ are polyheral and polytopic sets respectively, and both contain the origin as an interior point.

To make the results in the subsequent sections explicit, one definition is reviewed below.

Definition 1: (d-invariant set) A set $T \subset \mathbb{R}^{n_x}$ is disturbance invariant (d-invariant) for the system $x(t+1) = Ax(t) + Dw(t)$ and the constraint set $(\mathbb{X}, \mathbb{W})$ if $T \subseteq \mathbb{X}$ and $x(t+1) \in T$ for all $w(t) \in \mathbb{W}$ and $x(t) \in T$.

In most control problems, state feedback is assumed. In practice, the measurements contain noise, and perfect knowledge of the state is not realistic. A common approach is therefore to employ an observer and substitute the resulting state estimate for the true system state in the controller design. Denote the state estimate at time t as $\hat{x}(t)$ where $\hat{x}(t) \in \mathbb{R}^{n_x}$ and let the state estimation error be

$$e(t) = x(t) - \hat{x}(t), \quad \forall t \geq 0. \qquad (3)$$

Using a stable observer (*e.g.* Luenberger observer), the error $e(t)$ can be bounded by an invariant set, $\mathbb{E}$, see [5], or if $e(0) \in \mathbb{E}$, $e(t) \in \mathbb{E}$, $\forall t \geq 0$.

Proposition 1: [5] If the initial system and observer states, $x(0)$ and $\hat{x}(0)$, respectively, satisfy $e(0) \in \mathbb{E}$, then $x(t) \in \hat{x}(t) \oplus \mathbb{E}$ for all $t \geq 0$ and all admissible disturbances $w(t), v(t)$, $\forall t \geq 0$.

III. MOVING HORIZON ESTIMATION

MHE estimates the state by considering only a fixed amount of information data. It is assumed that the data is measured in the recent past. The basic strategy of MHE is to estimate the state using a moving and fixed-size window of data. The information vector is

$$I^{N_e}(t) = \mathrm{col}\big(y(t-N_e), \cdots, y(t),\ u(t-N_e), \cdots, u(t-1)\big), \quad (4)$$
$$t = N_e, N_e+1, \ldots,$$

where $N_e + 1$ is the number of measurements made at sliding-window stages from $t - N_e$ to t ($t \geq N_e$). When a new measurement becomes available, the oldest measurement is removed from the data window and the newest measurement is added. The problem size of MHE is bounded by considering a subset of the available information.

In [10], the objective of MHE is to compute the estimated state at time t on the basis of the information vector $I^{N_e}(t)$ and of a prediction $\overline{x}(t-N_e)$. The MHE estimator is derived by minimizing a quadratic function, the first term of which is the usual prediction error computed on the basis of the last measures, the second of which is a weighted term penalizing the distance of the current estimated state from its prediction. At any stage $t = N_e, N_e+1, \ldots$, the MHE problem is formulated, *i.e.*

$$\min_{\hat{x}(t-N_e|t)} V(\hat{x}(t-N_e|t); I^{N_e}(t), \bar{x}(t-N_e)) = \sum_{k=t-N_e}^{t} \|y(k) - C\hat{x}(k|t)\|^2 + \mu\|\hat{x}(t-N_e|t) - \bar{x}(t-N_e)\|^2 \quad (5a)$$

$$\text{s.t.}\, \hat{x}(k+1|t) = A\hat{x}(k|t) + Bu(k), \quad k = t-N_e, \ldots, t-1, \quad (5b)$$

where $\hat{x}(k|t)$ denotes the estimated value of x at time point k derived using (5b); μ is a nonnegative scalar to tune a trade-off between the two components of the cost. Problem (5) is just a simple quadratic programming problem. Without the inequality constraints, a single explicit solution can be found. The solution to problem (5) is defined as $\hat{x}^*(t-N_e|t)$ and it yields the sequence of the state estimates $\{\hat{x}^*(k|t)\}_{k=t-N_e}^{t}$. At the current time t, the state estimated by MHE is denoted as $\hat{x}(t) = \hat{x}^*(t|t)$. The optimal predictions are determined as

$$\bar{x}(t-N_e) = A\hat{x}^*(t-N_e-1|t-1) + Bu(t-N_e-1), \quad (6)$$
$$t = N_e+1, N_e+2, \ldots,$$

where $\bar{x}(0)$ is given. The estimation error defined in [10] differs from the one defined in (3), *i.e.*

$$\tilde{e}(t) = x(t) - \hat{x}^*(t|t+N_e), \quad (7)$$

and it is shown to satisfy the following dynamics

$$\tilde{e}(t+1) = \tilde{A}\tilde{e}(t) + \tilde{G}\mathbf{w}(t) + \tilde{H}\mathbf{v}(t), \quad (8)$$

where $\mathbf{w}(t) = [w^T(t), \cdots, w^T(t+N_e)]^T$, $\mathbf{v}(t) = [v^T(t+1), \cdots, v^T(t+N_e+1)]^T$ and $\mathbf{w}(t) \in \mathbb{W}_e := \underbrace{\mathbb{W} \times \cdots \times \mathbb{W}}_{N_e+1}$, $\mathbf{v}(t) \in \mathbb{V}_e := \underbrace{\mathbb{V} \times \cdots \times \mathbb{V}}_{N_e+1}$; the matrixes $\tilde{A} = (\mu I + O)^{-1}\mu A$, $\tilde{G} = (\mu I + O)^{-1}[\mu D| - \tilde{S}_{N_e}^T T_{N_e}\tilde{D}_{N_e}]$ and $\tilde{H} = (\mu I + O)^{-1}[-S_{N_e}^T \tilde{E}_{N_e+1}]$, with

$$S_{N_e} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N_e} \end{bmatrix}, \quad \tilde{S}_{N_e} = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N_e} \end{bmatrix},$$

$$\tilde{T}_{N_e} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ C & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ CA^{N_e-2} & CA^{N_e-3} & \cdots & CA & C \end{bmatrix}, T_{N_e} = [S_{N_e-1}|\tilde{T}_{N_e}],$$

$$O = S_{N_e}^T S_{N_e}, \tilde{D}_{N_e} = diag(\underbrace{D, \ldots, D}_{N_e}), \tilde{E}_{N_e} = diag(\underbrace{E, \ldots, E}_{N_e}).$$

If suitable conditions on the scalar parameter μ are satisfied, we have $\rho(\tilde{A}) < 1$ (more details are referred to [16]). In this paper, following the definition of the estimation error made in (3) and the definition of $\tilde{e}(t)$, it is easy to derive the following relation

$$e(t) = A^{N_e}\tilde{e}(t-N_e) + A^{N_e-1}Dw(t-N_e) + \cdots + Dw(t-1). \quad (9)$$

From (8) and (9), the explicit error dynamics of MHE can be easily generated,

$$e(t+1) = \bar{A}e(t) + \bar{G}\mathbf{w}(t-N_e) + \bar{H}\mathbf{v}(t-N_e), \quad (10)$$

where $\bar{A} = A^{N_e}\tilde{A}A^{-N_e}$, $\bar{G} = A^{N_e}\tilde{G} - \bar{A}\mathbf{A}_1 + \mathbf{A}_2$ and $\bar{H} = A^{N_e}\tilde{H}$ with $\mathbf{A}_1 = [A^{N_e-1}D, \cdots, D, 0]$ and $\mathbf{A}_2 = [0, A^{N_e-1}D, \cdots, D]$. Let $\alpha(t) = \bar{G}\mathbf{w}(t-N_e) + \bar{H}\mathbf{v}(t-N_e)$. Thus $\alpha(t)$ lies in a C set $\mathbb{Q}$ defined by

$$\mathbb{Q} = \bar{G}\mathbb{W}_e \oplus \bar{H}\mathbb{V}_e. \quad (11)$$

Equation (10) can be rewritten, *i.e.*

$$e(t+1) = \bar{A}e(t) + \alpha(t). \quad (12)$$

The fact $\rho(\tilde{A}) < 1$ implies $\rho(\bar{A}) < 1$. Thus there exists a C set $\mathbb{E}$ such that it is d-invariant for system (12). It follows

$$\bar{A}\mathbb{E} \oplus \mathbb{Q} \subseteq \mathbb{E}, \quad (13)$$

which implies that if $e(0) \in \mathbb{E}$, $e(t) \in \mathbb{E}$, $\forall t \geq 0$. Since the set $\mathbb{E}$ is the upper set of the error, it is desired to be as small as possible. In this paper the set $\mathbb{E}$ is chosen as the outer bound of the minimal d-invariant set of system (12). Efforts to compute such a set for linear systems have appeared in the literature, see for example [17], [18].

It is easy to show that the estimated state yielded by the MHE estimator satisfies the following uncertain equation, *i.e.*

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + \hat{L}e(t) - \hat{G}\mathbf{w}(t-N_e) - \hat{H}\mathbf{v}(t-N_e), \quad (14)$$

where $\hat{L} = A - \bar{A}$, $\hat{G} = \bar{G} - \mathbf{G}_1$ with $\mathbf{G}_1 = diag(0, \ldots, 0, D)$ and $\hat{H} = \bar{H}$. Let $\beta(t) = \hat{L}e(t) - \hat{G}\mathbf{w}(t-N_e) - \hat{H}\mathbf{v}(t-N_e)$. Suppose $e(0) \in \mathbb{E}$, then $\beta(t)$ lies in a C set $\mathbb{T}$ defined by

$$\mathbb{T} = \hat{L}\mathbb{E} \oplus (-\hat{G}\mathbb{W}_e) \oplus (-\hat{H}\mathbb{V}_e). \quad (15)$$

Equation (14) can be also rewritten as a linear system dynamics with a uncertainty β, *i.e.* $\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + \beta(t)$. Hence, if the initial error can be chosen appropriately such that $e(0) \in \mathbb{E}$, the estimated state dynamics (14) can be regard as a nominal system of (1a) with an additional, unknown but bounded uncertainty.

Remark 1: In the literature of output feedback linear MPC, a common approach, see [5], [4], [15] is to employ a fixed linear observer, e.g. the Luenberger observer to estimate the state, *i.e.* $\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t))$, where the observer matrix $L \in \mathbb{R}^{n_x \times n_y}$ is designed to satisfy $\rho(A_L) < 1$ ($A_L = A - LC$). Then the estimated state satisfies the following equation

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + LCe(t) + LEv(t). \quad (16)$$

If $e(t) \in \Upsilon$ (like $\mathbb{E}$, Υ is a d-invariant set of error dynamics), dynamics (16) follows that $\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + \delta(t)$, where the uncertainty $\delta \in \Theta := LC\Upsilon \oplus LE\mathbb{V}$ is bounded. The result about state estimation dynamics obtained by using Luenberger observer is similar as that obtained by MHE. The design of the observer matrix L is a key aspect when employing the Luenberger observer in the output feedback controller design, which affects the performance of the observer, the error bound Υ and the size of the set Θ. However, many papers, *e.g.* [5], [4], [15], have a lack of the consideration of the design of L.

IV. OUTPUT FEEDBACK MPC

A. Problem formulation

The output feedback MPC controller $u(\cdot)$ takes the form following [19], [20], which is parameterized by $c(\cdot) \in \mathbb{R}^{n_u}$ as

$$u(t) = K\hat{x}(t) + c(t) \tag{17}$$

for some given $K \in \mathbb{R}^{n_u \times n_x}$ such that $\Phi := A + BK$ is asymptotically stable ($\rho(\Phi) < 1$). The motivation of the proposed approach is to find $c(t)$ such that robust constraint satisfaction can be guaranteed for all $t \geq 0$ and robust closed-loop stability can be ensured. To achieve it, the state estimation error should be taken into account by introducing its associated estimation error set $\mathbb{E}$. Given an initial state estimate and bounded on the estimation error such that $e(0) \in \mathbb{E}$, the estimated state satisfies

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + \beta(t), \tag{18}$$

where $\beta(t) \in \mathbb{T}$. To design the robust output feedback MPC controller, the constraint tightening approach is used, which is introduced by [21] and extended by [20]. Recently, its most modifications are developed, see [22], [23], [24], [25]. This approach avoids huge complexity by using only a nominal prediction model and modifying the constraints to achieve robustness. The key idea is that the effect of the persistent disturbance β is taken into account by using the strengthened input/output constraints. Moreover, like the approach proposed in [5], the controller uses a tube, and within which the estimated state is guaranteed to remain.

The disturbance-free system is

$$\widetilde{x}(t+1) = A\widetilde{x}(t) + B\widetilde{u}(t), \ \forall t \geq 0. \tag{19}$$

At time t with the given $\hat{x}(t)$, the finite horizon MPC problem over $\mathbf{c}(t) = [c^T(0|t), c^T(1|t), \cdots, c^T(N-1|t)]^T$ is:

$$\min_{\mathbf{c}(t),\widetilde{x}(0|t)} J(\mathbf{c}(t),\widetilde{x}(0|t);\hat{x}(t)) = \sum_{k=0}^{N-1} \|c(k|t)\|_{\Psi}^2 \tag{20a}$$

$$s.t. \quad \hat{x}(t) \in \widetilde{x}(0|t) \oplus \mathbb{T}, \tag{20b}$$

$$\widetilde{x}(k+1|t) = A\widetilde{x}(k|t) + B\widetilde{u}(k|t), \forall k \geq 0, \tag{20c}$$

$$\widetilde{u}(k|t) = K\widetilde{x}(k|t) + c(k|t), k = 0,1,\ldots,N-1, \tag{20d}$$

$$\widetilde{u}(k|t) = K\widetilde{x}(k|t), \forall k \geq N, \tag{20e}$$

$$\widetilde{x}(k|t) \in \mathbb{X}_k, \ \widetilde{u}(k|t) \in \mathbb{U}_k, k = 0,1,\ldots,N-1, \tag{20f}$$

$$\widetilde{x}(N|t) \in \widetilde{\mathbb{X}}_f, \tag{20g}$$

where $\Psi \succ 0$; N is the prediction horizon; the notations $\widetilde{x}(k|t)$ and $\widetilde{u}(k|t)$ denote the state and input at time $t+k$ derived by using (20c)-(20e) based on the estimated state $\hat{x}(t)$. The sets $\mathbb{X}_k$, $\mathbb{U}_k$ and $\mathbb{X}_f$ are appropriately strengthened, given by

$$\mathbb{X}_k = \mathbb{X}_t \ominus F_k, \quad \mathbb{U}_k = \mathbb{U} \ominus KF_k, \quad \widetilde{\mathbb{X}}_f = \mathbb{X}_f \ominus F_N \tag{21}$$

where

$$F_k := \mathbb{T} \oplus \Phi\mathbb{T} \oplus \cdots \oplus \Phi^k\mathbb{T} \tag{22}$$

and $\mathbb{X}_t = \mathbb{X} \ominus \mathbb{E}$. The terminal set $\mathbb{X}_f$ is chosen to be the maximal d-invariant set of system

$$x(t+1) = \Phi x(t) + \beta(t), \tag{23a}$$

$$s.t. \quad x(t) \in \mathbb{X}_t, \ Kx(t) \in \mathbb{U}, \ \beta(t) \in \mathbb{T}, \tag{23b}$$

in the sense that $\Phi\mathbb{X}_f \oplus \mathbb{T} \subseteq \mathbb{X}_f$. In problem (20), the 'tube' $\mathbb{T}$ is computed by (15) and the center of the tube at the initial time is treated as a decision variable. The MPC controller applied to system (1) at time t is

$$u^*(t) := K\hat{x}(t) + c^*(0|t) \tag{24}$$

where $c^*(0|t)$ is the first control of the optimal solution of problem (20). Let

$$X_N := \{\hat{x}(t) \ \exists \mathbf{c}(t), \widetilde{x}(0|t) \text{ such that } (20b)\text{–}(20g) \text{ are feasible}\}$$

be the domain of attraction of system (18) under (24).

Remark 2: The proposed MPC problem (20) is also suitable for the system combined with a fixed linear observer, *e.g.* Luenberger observer. In that case, compared with the work of [5], in our approach the constraints are tightened in a monotonic sequence and relaxedly.

B. Feasibility and stability

To show the feasibility and robust stability of the proposed output feedback MPC, we first define the following set sequence

$$\mathbf{c}(t+1) = [c^{*T}(1|t), \cdots, c^{*T}(N-1|t), 0]^T \tag{25}$$

which is obtained by the concatenation of the optimal "tail" at time t, with a terminal zero element.

Lemma 1: Suppose Assumptions (A1)-(A2) hold and $e(t) \in \mathbb{E}$. For system (1) under the output feedback MPC controller (24), if there exists a feasible solution of problem (20) for $\hat{x}(t)$, then there also exists a feasible solution for $\hat{x}(t+1)$.

Proof: At time $t+1$, the estimated state $\hat{x}(t+1)$ is

$$\hat{x}(t+1) = \Phi\hat{x}(t) + Bc^*(0|t) + \beta(t). \tag{26}$$

Since $\hat{x}(t) \in \widetilde{x}^*(0|t) \oplus \mathbb{T}$ and $\beta(t) \in \mathbb{T}$, we have

$$\hat{x}(t+1) \in \Phi\widetilde{x}^*(0|t) + Bc^*(0|t) \oplus \Phi\mathbb{T} \oplus \mathbb{T}, \tag{27}$$

or $\hat{x}(t+1) \in \widetilde{x}(1|t) \oplus \Phi\mathbb{T} \oplus \mathbb{T}$. Hence,

$$\hat{x}(t+1) \in \widetilde{x}(0|t+1) \oplus \mathbb{T}, \tag{28}$$

where $\widetilde{x}(0|t+1) \in \widetilde{x}(1|t) \oplus \Phi\mathbb{T}$. Employing the control sequence $\mathbf{c}(t+1)$, we have

$$\widetilde{x}(k|t+1) \in \widetilde{x}(k+1|t) \oplus \Phi^{k+1}\mathbb{T}, \quad k = 0,1,\ldots,N,$$

$$\widetilde{u}(k|t+1) \in \widetilde{u}(k+1|t) \oplus K\Phi^{k+1}\mathbb{T}, k = 0,1,\ldots,N.$$

Due to the fact that $\widetilde{x}(N|t) \in \widetilde{\mathbb{X}}_f$, we have $\widetilde{x}(N|t) + y \in \mathbb{X}_f$ for all $y \in F_N$. Thus $K\widetilde{x}(N|t) + Ky \in \mathbb{U}$, which implies $\widetilde{u}(N|t) = K\widetilde{x}(N|t) \in \mathbb{U} \ominus KF_N = \mathbb{U}_N$. The fact $\mathbb{X}_f \subseteq \mathbb{X}_t$ means $\widetilde{\mathbb{X}}_f \subseteq \mathbb{X}_N$. It implies $\widetilde{x}(N|t) \in \mathbb{X}_N$. From equation (20f) and the above discussion, we know that $\widetilde{x}(k|t) \in \mathbb{X}_k$, $\widetilde{u}(k|t) \in \mathbb{U}_k$, $k = 0, 1, \ldots, N$, which implies that

$$\begin{aligned} \widetilde{x}(k|t+1) &\in \mathbb{X}_{k+1} \oplus \Phi^{k+1}\mathbb{T}, \quad k = 0, 1, \ldots, N-1, \\ \widetilde{u}(k|t+1) &\in \mathbb{U}_{k+1} \oplus K\Phi^{k+1}\mathbb{T}, k = 0, 1, \ldots, N-1. \end{aligned}$$

This, together with (22), show that

$$\widetilde{x}(k|t+1) \in \mathbb{X}_k,\ \widetilde{u}(k+1|t) \in \mathbb{U}_k, k = 0, 1, \ldots, N-1.$$

Since $\mathbb{X}_f$ is a d-invariant set, $\widetilde{x}(N|t) + y \in \mathbb{X}_f,\ \forall y \in F_N \Rightarrow \Phi\widetilde{x}(N|t) + \Phi y \oplus \mathbb{T} \in \mathbb{X}_f \Rightarrow \widetilde{x}(N+1|t) \in \mathbb{X}_f \ominus \mathbb{T} \ominus \Phi F_N = \mathbb{X}_f \ominus F_{N+1}$. Hence,

$$\begin{aligned} \widetilde{x}(N|t+1) &\in \widetilde{x}(N+1|t) \oplus \Phi^{N+1}\mathbb{T} \subseteq \mathbb{X}_f \ominus F_{N+1} \oplus \Phi^{N+1}\mathbb{T} \\ &\subseteq \mathbb{X}_f \ominus F_N = \widetilde{\mathbb{X}}_f. \end{aligned}$$

From the above, the set sequence $[\mathbf{c}(t+1), \widetilde{x}(0|t+1)]$ is feasible for $x(t+1)$. ■

We can now establish the main result:

Theorem 1: Suppose Assumptions (A1)-(A2) hold, system (1) with the proposed output feedback MPC controller (24) has the following properties, for any $\hat{x}(0) \in X_N$ and $e(0) \in \mathbb{E}$: (i) $x(t) \in X$ and $u^*(t) \in U$ for all $t \geqslant 0$; (ii) $\lim_{t\to\infty} c(t) = 0$, where $c(t) = c^*(0|t)$; (iii) $\hat{x}(t) \to F_\infty$, as $t \to \infty$, where $F_\infty = \lim_{k\to\infty} F_k$; (iv) $x(t) \to F_\infty \oplus \mathbb{E}$, as $t \to \infty$.

Proof: Since, by assumption, $\hat{x}(0)$ is feasible, $\hat{x}(t)$ is feasible for all $t \geqslant 0$ following Lemma 1. Due to the fact that $\widetilde{x}(0|t) \in \mathbb{X}_t \ominus \mathbb{T}$, $\widetilde{u}(0|t) \in \mathbb{U} \ominus K\mathbb{T}$ and $\hat{x}(t) \in \widetilde{x}(0|t) \oplus \mathbb{T}$, we have $\hat{x}(t) \in \mathbb{X}_t$, $u^*(t) \in \mathbb{U}$, $\forall t \geq 0$. Following proposition 1, $x(t) \in \hat{x}(t) \oplus \mathbb{E}$ for all $t \geq 0$ and all admissible disturbances $w(t), v(t)$, which implies $x(t) \in \mathbb{X}$, $\forall t \geq 0$. Thus property (i) holds. Suppose the optimal cost is defined by $J^*(t) = \sum_{k=0}^{N-1} \|c^*(k|t)\|^2_\Psi$. At time $t+1$, there exists a feasible cost $J^f(t+1) = \sum_{k=1}^{N-1} \|c^*(k|t)\|^2_\Psi$. Hence,

$$J^*(t+1) - J^*(t) \leqslant -\|c^*(0|t)\|^2_\Psi. \tag{29}$$

It is easy to see that $\{J^*(t)\}$ is non-increasing and bounded by 0. As $t \to \infty$, it converges to $J^*(\infty) < +\infty$. Summing (29), we have $+\infty > J^*(0) - J^*(\infty) \geqslant \sum_{t=0}^{\infty} \|c^*(0|t)\|^2_\Psi \geqslant 0 \Rightarrow \lim_{t\to\infty} c(t) = 0$. Therefore, property (ii) is proven. Thanks to Assumptions (A1)-(A2) and $\rho(\Phi) < 1$,

$$\begin{aligned} \lim_{t\to\infty} \hat{x}(t) &= \lim_{t\to\infty} \big[\Phi^t \hat{x}(0) + \sum_{k=1}^{t} \Phi^{k-1} Bc(t-k) + \sum_{k=1}^{t} \Phi^{k-1} \beta(t-k)\big] \\ &= \lim_{t\to\infty} \big[\sum_{k=1}^{t} \Phi^{k-1} \beta(t-k)\big], \end{aligned}$$

which, in turn, proves (iii). Property (iv) of the theorem follows from the fact that $x(t) \in \hat{x}(t) \oplus \mathbb{E}$ for all $t \geq 0$. ■

C. Multi-parametric programming in output feedback MPC

For problem (20), constraints (20b)-(20g) can be expressed collectively as a matrix inequality

$$\tilde{G}\tilde{\mathbf{z}}(t) \leq \tilde{V} + \tilde{W}\hat{x}(t),$$

where matrixes $\tilde{G}, \tilde{V}, \tilde{W}$ can be easily obtained and $\tilde{\mathbf{z}}(t) = [\mathbf{c}^T(t), \widetilde{x}(0|t)^T]^T$. From the above, the optimization problem (20) falls into a class of multi-parametric quadratic programming (mp-QP) problems, see [26], [27]. Using the algorithm described in [28], one can compute the explicit solution of problem (20) off-line for all $\hat{x}(t) \in X_N$, *i.e.* $\tilde{\mathbf{z}}^*(\hat{x}(t)) = \mathscr{L}_i\hat{x}(t) + \mathscr{G}_i$, if $\hat{x}(t) \in \mathscr{R}_i$, $\forall i \in \mathscr{I}$, where each $\mathscr{L}_i$ and $\mathscr{G}_i$ are associated with a convex polytope $\mathscr{R}_i$ in $\mathbb{R}^{n_x}$ that forms a partition of X_N in the sense that $X_N = \cup_{i\in\mathscr{I}} \mathscr{R}_i$ and $int(\mathscr{R}_i) \cap int(\mathscr{R}_j) = \emptyset$ for all $i \neq j, i, j \in \mathscr{I}$. Thereby, the explicit output feedback MPC control law can be expressed as

$$u^*(\hat{x}(t)) = L_i\hat{x}(t) + g_i, \quad \text{if} \quad \hat{x}(t) \in \mathscr{R}_i,\ \forall i \in \mathscr{I}, \tag{30}$$

where $L_i \in \mathbb{R}^{n_u \times n_x}$ and $g_i \in \mathbb{R}^{n_u}$. Clearly, the availability of (30) means that $(L_i, g_i), i \in \mathscr{I}$ can be computed off-line leaving the on-line computational effort to the identification of $\mathscr{R}_i$ when $\hat{x} \in X_N$ and the evaluation of u^*.

V. EXAMPLE

The example is taken from [5]. The system is a double integrator:

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t), \\ y(t) &= [1\ \ 1]x(t) + v(t) \end{aligned}$$

with additive disturbances $(w, v) \in \mathbb{W} \times \mathbb{V}$ where $\mathbb{W} = \{w \in \mathbb{R}^2 : \| w \|_\infty \leqslant 0.1\}$ and $\mathbb{V} = \{v \in \mathbb{R} : \| v \| \leqslant 0.05\}$. The state and control constraints are $(x, u) \in \mathbb{X} \times \mathbb{U}$ where $\mathbb{X} = \{x \in \mathbb{R}^2 : x_1 \in [-50,\ 3],\ x_2 \in [-50,\ 3]\}$ and $\mathbb{U} = \{u \in \mathbb{R} : \| u \| \leqslant 3\}$ (x_i is the ith coordinate of a vector of x). $K = [-1\ \ -1]$. The d-invariant sets $\mathbb{E}$ is obtained using results in [17]. The horizon is $N = 13$. Figure 1 shows the responses of the proposed controller starting from the initial state $\hat{x}(0) = [-5, -10.89]$. The domain of attraction X_{13} is shown as dash line. The domain of attraction for the true system is $X_{13} \oplus \mathbb{E}$, shown as dash-dot line. From Figure 1, it is shown that the estimate state $\hat{x}(t)$ finally converges to the set F_∞.

VI. CONCLUSION

The main contribution of this paper is to provide a simple approach to the problem of robust output feedback MPC for linear systems, subject to bounded state disturbances and output measurement errors, which employs a combination of an MHE observer with a tube-based robust MPC controller. The satisfaction of state and input constraints are guaranteed and the closed-loop stability is ensured. The proposed approach can be easily written as an multi-parametric programming problem, thus be solved off-line to relax the on-line computational burden.

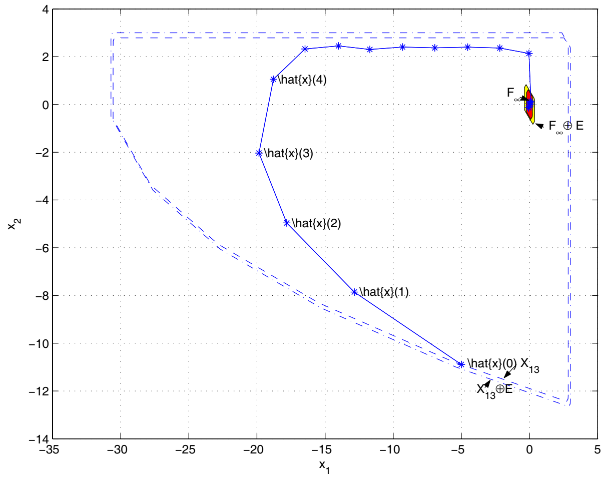

Fig. 1. Closed-loop responses of robust output-feedback MPC.

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