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Non-fragile H_∞ Filter Design for Discrete-Time Systems via LMI Approach

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***Abstract*— This paper presents new non-fragile H_∞ filter design methods for linear discrete-time systems. The filter to be designed is assumed to be with additive gain variations, which reflect the FWL effects in filter digital implementations. A notion of structured vertex separator is proposed to approach the problem, and exploited to develop sufficient conditions for the non-fragile H_∞ filter design in terms of solutions to a set of linear matrix inequalities (LMIs). Moreover, to reduce the design conservativeness, the slack variable method is adopted to realize the decoupling between the Lyapunov matrix and the system dynamic matrix. The designs render the augmented system asymptotically stable and guarantee the H_∞ attenuation level less than a prescribed level. A numerical example is given to illustrate the design methods and the design benefits.**

I. Introduction

In the course of filter implementation based on different design algorithms, it turns out that the filters can be sensitive with respect to errors in the filter coefficients ([3], [16]). The sources for this include, but not limited to, imprecision in analogue-digital conversion, fixed word length, finite resolution instrumentation and numerical roundoff errors. By means of several examples, it is demonstrated in the control design formalism [8] that relatively small perturbations in controller parameters could even destabilize the closed-loop system. So a significant issue is how to design a filter or controller for a given plant such that the filter or controller is insensitive to some amount of error with respect to its gain, i.e., the designed filter or controller is resilient or non-fragile. This issue has received some attention from the control systems community, and some relevant results have appeared in the last decade to tackle the problem of designing controllers that are capable of tolerating some level of controller gain variations ([3],[5],[7], [15]). The problem of resilient Kalman filtering with respect to estimator gain perturbations is considered in [16]. In [10], the problem of designing robust resilient linear filtering for a class of continuous-time systems with norm-bounded uncertainty is investigated. Recently, [2] develops an approach of designing the optimal filter transfer function and its realization.

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Noting that the above works deal with the non-fragile problem with the consideration of norm-bounded type of gain uncertainty. However, this type of uncertainty cannot reflect the uncertain information due to the FWL effects exactly. Correspondingly, the interval type of uncertainty [9] is more exact than the former type to describe the uncertain information, but till now, there is no work on the non-fragile filter design problem with taking account of interval gain uncertainty. On the other hand, when considering the the interval type of uncertainty [9], the vertices of the set of uncertain parameters grow exponentially with the number of uncertain parameters, which may result in numerical problem for systems with high dimensions. These problems motivate our work in this paper.

This paper is concerned with the problem of non-fragile H_∞ filter design for linear discrete-time systems. The filter to be designed is assumed to be with additive gain variations of the interval type, which are due to the FWL effects when the filter is implemented. Firstly, an LMI-based sufficient condition is given for the solvability of the non-fragile H_∞ filtering problem, which requires checking all of the vertices of the set of uncertain parameters that grows exponentially with the number of uncertain parameters. It will be very difficult to apply the result to the systems with high orders. To overcome the difficulty, a notion of structured vertex separator is proposed to approach the problem, and exploited to develop sufficient conditions for the non-fragile H_∞ filter design in terms of solutions to a set of LMIs. The structured vertex separator-based method can significantly reduce the number of the LMI constraints involved in the design condition. It should be mentioned that the similar method has been used in [17]. Moreover, we adopt the slack variable method [1] to realize the decoupling between the Lyapunov matrix and the system dynamic matrix, which reduces the design conservativeness. The designs guarantee the asymptotic stability of the estimation errors, and the H_∞ performance of the system from the exogenous signals to the estimation errors less than a prescribed level. It should be mentioned that the existing method given in [16] and [11], for the non-fragile problem with norm-bounded gain variations, is also applicable for the non-fragile H_∞ filtering problem considered here. But this method is more conservative than our new proposed one, which will be shown in Section III.

Notation: For a column-rank deficient matrix H, N_H denotes a matrix whose columns form a basis for the null space of H. I denotes the identity matrix with an appropriate dimension. $\mathbf{0}_{i\times j}$ represents zero matrix of i rows and j columns. The symbol $*$ within a matrix represents the the symmetric entries.

II. Problem Statement and Preliminaries

A. Problem statement

Consider an LTI discrete-time model described by

$$\begin{aligned} x(k+1) &= Ax(k) + B_1\omega(k), \\ z(k) &= C_1x(k), \\ y(k) &= C_2x(k) + D_{21}\omega(k). \end{aligned} \tag{1}$$

where $x(k) \in R^n$ is the state, $y(k) \in R^p$ is the measured output, $\omega(k) \in R^r$ is the disturbance input and $z(k) \in R^q$ is the regulated output, respectively. A, B_1, C_1, C_2 and D_{21} are known constant matrices of appropriate dimensions.

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Consider a filter with gain variations of the following form:

$$\begin{array}{ll} \xi(k+1) & = (A_F + \Delta A_F)\xi(k) + (B_F + \Delta B_F)y(k), \\ z_F(k) & = (C_F + \Delta C_F)\xi(k). \end{array} \quad (2)$$

where $\xi(k) \in R^n$ is the filter state, $z_F(k)$ is the estimation of $z(k)$, and the constant matrices A_F, B_F and C_F are filter matrices to be designed, $\Delta A_F, \Delta B_F$ and ΔC_F represent the interval type of additive gain variations with the following form:

$$\begin{array}{ll} \Delta A_F & = [\delta_{aij}]_{n\times n}, |\delta_{aij}| \le \delta_a, i,j = 1,\cdots,n, \\ \Delta B_F & = [\delta_{bij}]_{n\times p}, |\delta_{bij}| \le \delta_a, i = 1,\cdots,n, j = 1,\cdots,p, \\ \Delta C_F & = [\delta_{cij}]_{q\times n}, |\delta_{cij}| \le \delta_a, i = 1,\cdots,q, j = 1,\cdots,n. \end{array} \quad (3)$$

Remark 1. The additive gain variation model of form (3) is from [9], which has been extensively used to describe the FWL effects. Let $e_k \in R^n, h_k \in R^p$ and $g_k \in R^q$ denote the column vectors in which the kth element equals 1 and the others equal 0. Then the gain variations of the form (3) can be described as :

$$\Delta A_F = \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{aij} e_i e_j^T, \ \Delta B_F = \sum_{i=1}^{n}\sum_{j=1}^{p} \delta_{bij} e_i h_j^T,$$

$$\Delta C_F = \sum_{i=1}^{q}\sum_{j=1}^{n} \delta_{cij} g_i e_j^T.$$

Combining filter (2) with system (1), we obtain the filter error system as:

$$\begin{array}{ll} x_e(k+1) & = A_e x_e(k) + B_e\omega(k), \\ z_e(k) & = C_e x_e(k). \end{array} \quad (4)$$

where $x_e(k) = [x(k)^T, \xi(k)^T]^T$, $z_e(k) = z(k) - z_F(k)$ is the estimation error, and

$$A_e = \begin{bmatrix} A & 0 \\ (B_F + \Delta B_F)C_2 & A_F + \Delta A_F \end{bmatrix},$$

$$B_e = \begin{bmatrix} B_1 \\ (B_F + \Delta B_F)D_{21} \end{bmatrix}, C_e = \begin{bmatrix} C_1 & -C_F - \Delta C_F \end{bmatrix}.$$

The transfer function matrix of the augmented system (4) from ω to z_e is given by

$$G_{z_e\omega}(z) = C_e(zI - A_e)^{-1}B_e.$$

Then the problem under consideration in this paper is as follows:
Non-fragile H_∞ filtering problem with additive filter gain variations: Given a positive constant γ, find a filter described by (2) with the gain variations of the form (3) such that the resulting system (4) is asymptotically stable and $\|G_{z_e\omega}(z)\| < \gamma$.

B. Useful lemmas

In this part, some useful lemmas are presented firstly.

Lemma 1: [14] Let matrices $Q = Q^T$, G, and a compact subset of real matrices $\mathbf{H}$ be given. Then the following statements are equivalent:
(i) for each $H \in \mathbf{H}$

$$\xi^T Q \xi < 0 \text{ for all } \xi \neq 0 \text{ such that } HG\xi = 0;$$

(ii) there exists $\Theta = \Theta^T$ such that

$$Q + G^T\Theta G < 0, \mathbf{N}_H^T\Theta\mathbf{N}_H \ge 0 \text{ for all } H \in \mathbf{H}.$$

Lemma 2: [4] Let $G_{az\omega}(z) = C_a(zI - A_a)^{-1}B_a$, then A_a is Shur stable and $\|G_{az\omega}(z)\| < \gamma$ for some constant $\gamma > 0$ if and only if there exists a symmetric matrix $X > 0$, such that

$$\begin{bmatrix} -X & 0 & XA_a & XB_a \\ * & -I & C_a & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0. \quad (5)$$

Now, to facilitate the presentation of Lemma 3, we denote

$$G_{0z_e\omega}(z) = C_{e0}(zI - A_{e0})^{-1}B_{e0}, \quad (6)$$

where

$$A_{e0} = \begin{bmatrix} A & 0 \\ B_F C_2 & A_F \end{bmatrix}, \ B_{e0} = \begin{bmatrix} B_1 \\ B_F D_{21} \end{bmatrix}, \quad C_{e0} = \begin{bmatrix} C_1 & -C_F \end{bmatrix}. \quad (7)$$

Then, we have

Lemma 3: Let $\gamma > 0$ be a given constant. Then the following statements are equivalent:
(i) A_{e0} is Shur stable, and $\|G_{0z_e\omega}(z)\| < \gamma$;
(ii) there exists a symmetric positive matrix $X > 0$ such that

$$\begin{bmatrix} -X & 0 & XA_{e0} & XB_{e0} \\ * & -I & C_{e0} & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0; \quad (8)$$

(iii) there exist a symmetric positive matrix $X > 0$ and a matrix G such that

$$\begin{bmatrix} X - G - G^T & 0 & G^T A_{e0} & G^T B_{e0} \\ * & -I & C_{e0} & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0; \quad (9)$$

(iv) there exist a nonsingular matrix T, and a symmetric matrix $P > 0$ with

$$P = \begin{bmatrix} Y & N \\ N & -N \end{bmatrix}, \quad (10)$$

such that

$$\begin{bmatrix} -P & 0 & PA_{ea} & PB_{ea} \\ * & -I & C_{ea} & 0 \\ * & * & -P & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \quad (11)$$

where

$$A_{ea} = \begin{bmatrix} A & 0 \\ B_{Fa}C_2 & A_{Fa} \end{bmatrix}, \ B_{ea} = \begin{bmatrix} B_1 \\ B_{Fa}D_{21} \end{bmatrix}, \quad C_{ea} = \begin{bmatrix} C_1 & -C_{Fa} \end{bmatrix}. \quad (12)$$

and

$$A_{Fa} = T^{-1}A_F T, \quad B_{Fa} = T^{-1}B_F, \quad C_{Fa} = C_F T. \quad (13)$$

(v) there exist a nonsingular matrix T, a symmetric matrix $X > 0$ and a matrix G with structure

$$G = \begin{bmatrix} Y & N \\ N & -N \end{bmatrix}, \quad (14)$$

such that

$$\begin{bmatrix} X - G - G^T & 0 & G^T A_{ea} & G^T B_{ea} \\ * & -I & C_{ea} & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \quad (15)$$

holds, where A_{ea}, B_{ea} and C_{ea} are defined by (12).

Proof: Due to the limit of the space, it is omitted. ■

Lemma 4: Let matrices Q, F_1 and F_2 be constant matrices with appropriate dimensions. Then the following statements are equivalent:
(i)

$$Q + F_1\Omega F_2 + (F_1\Omega F_2)^T < 0,$$

where $\Omega = diag[\delta_1, \ \cdots, \delta_s]$, for $|\delta_i| \le \delta_a, \ i = 1,\cdots,s$.
(ii)

$$Q + F_1\Omega F_2 + (F_1\Omega F_2)^T < 0, \text{ for } \Omega \in \Omega_v,$$

where $\Omega_v = \{\Omega : \ \delta_i \in \{-\delta_a, \ \delta_a\}, i = 1, \cdots, s\}$.
(iii) there exists a symmetric matrix $\Theta \in R^{2s\times 2s}$ such that

$$\begin{bmatrix} Q & F_1 \\ F_1^T & 0 \end{bmatrix} + \begin{bmatrix} F_2 & 0 \\ 0 & I \end{bmatrix}^T \Theta \begin{bmatrix} F_2 & 0 \\ 0 & I \end{bmatrix} < 0, \qquad (16)$$

$$\begin{bmatrix} I \\ \Omega \end{bmatrix}^T \Theta \begin{bmatrix} I \\ \Omega \end{bmatrix} \geq 0, \text{ for all } \ \Omega \in \Omega_v. \qquad (17)$$

Proof: Due to the limit of the space, it is omitted. ■

III. Non-fragile H_∞ filter design with additive gain variations

In this section, an LMI-based method for designing H_∞ filters with respect to additive gain uncertainties is presented, and further, a comparison between the new proposed method and the existing method is given.

A. Non-fragile H_∞ filter design methods

Let G with structure (14), i.e., $G = \begin{bmatrix} Y & N \\ N & -N \end{bmatrix}$. To facilitate the presentation, we denote $S = Y + N$ and

$$M_0(\Delta A_F, \Delta B_F, \Delta C_F) = \begin{bmatrix} \Xi_1 & \Xi_2 & 0 & S^TA & S^TA & S^TB_1 \\ * & \Xi_3 & 0 & \Xi_5 & \Xi_6 & \Xi_7 \\ * & * & -I & \Xi_4 & C_1 & 0 \\ * & * & * & -\bar{P}_{11} & -\bar{P}_{12} & 0 \\ * & * & * & * & -\bar{P}_{22} & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix}, \qquad (18)$$

where

$$\begin{aligned} \Xi_1 &= \bar{P}_{11} - S - S^T, \Xi_2 = \bar{P}_{12} - S - S^T, \\ \Xi_3 &= \bar{P}_{22} - S - S^T + N + N^T, \Xi_4 = C_1 - C_F - \Delta C_F, \\ \Xi_5 &= (S-N)^T A + F_B C_2 + N^T(\Delta B_F C_2 + \Delta A_F) + F_A, \\ \Xi_6 &= (S-N)^T A + F_B C_2 + N^T \Delta B_F C_2, \\ \Xi_7 &= (S-N)^T B_1 + F_B D_{21} + N^T \Delta B_F D_{21}. \end{aligned} \qquad (19)$$

Then the following theorem presents a sufficient condition for the solvability of the non-fragile H_∞ filtering problem with additive gain uncertainties.

Theorem 1. Consider system (1). Let $\gamma > 0$ and $\delta_a > 0$ be given constants. If there exist matrices $F_A, F_B, C_F, S, N, \bar{P}_{12}$ and $\bar{P}_{11} > 0, \bar{P}_{22} > 0$, such that the following LMIs hold:

$$\begin{aligned} &M_0(\Delta A_F, \Delta B_F, \Delta C_F) < 0, \delta_{aij}, \ \delta_{bik}, \ \delta_{clj} \in \{-\delta_a, \delta_a\}, \\ &i, j = 1, \cdots, n; \ k = 1, \cdots, p; \ l = 1, \cdots, q, \end{aligned} \qquad (20)$$

then filter (2) with additive uncertainty described by (3) and

$$A_F = (N^T)^{-1} F_A, \quad B_F = (N^T)^{-1} F_B, \quad C_F = C_F \qquad (21)$$

solves the non-fragile H_∞ filtering problem for system (1).

Proof: By Lemma 3, it is sufficient to show that there exist a matrix G with structure (14) and a symmetric positive matrix $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0$ such that

$$M_1 = \begin{bmatrix} P - G - G^T & 0 & G^T A_e & G^T B_e \\ * & -I & C_e & 0 \\ * & * & -P & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \qquad (22)$$

holds for all δ_{aij}, δ_{bik} and δ_{clj} satisfying (3).
Denote

$$\Gamma_1 = \begin{bmatrix} I & I \\ I & 0 \end{bmatrix}, \bar{\Gamma}_1 = diag\{\Gamma_1, I, \Gamma_1, I\},$$

$$\bar{P}_{11} = P_{11} + P_{12} + P_{12}^T + P_{22},$$

$$\bar{P}_{12} = P_{11} + P_{12}^T, \quad \bar{P}_{22} = P_{11}.$$

Then (22) is equivalent to

$$M_2 = \bar{\Gamma}_1 M_1 \bar{\Gamma}_1^T = \begin{bmatrix} \Xi_1 & \Xi_2 & 0 & S^TA & S^TA & S^TB_1 \\ * & \Xi_3 & 0 & \Pi_1 & \Pi_2 & \Pi_3 \\ * & * & -I & \Xi_4 & C_1 & 0 \\ * & * & * & -\bar{P}_{11} & -\bar{P}_{12} & 0 \\ * & * & * & * & -\bar{P}_{22} & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \qquad (23)$$

holds for all δ_{aij}, δ_{bik} and δ_{clj} satisfying (3), where $\Xi_1, \Xi_2, \Xi_3, \Xi_4$ are defined by (19), and

$$\begin{aligned} \Pi_1 &= (S-N)^T A + N^T (B_F C_2 + \Delta B_F C_2 + A_F + \Delta A_F), \\ \Pi_2 &= (S-N)^T A + N^T B_F C_2 + N^T \Delta B_F C_2, \\ \Pi_3 &= (S-N)^T B_1 + N^T B_F D_{21} + N^T \Delta B_F D_{21}. \end{aligned}$$

Obviously, M_2 is convex for each δ_i, for all $\delta_i \in \{\delta_{aij}, \delta_{bik}, \delta_{clj}$ satisfying (3)$\}$, so by using (21), (23) is equivalent to (20). ■

Remark 2. Theorem 1 presents a sufficient condition in terms of solutions to a set of LMIs for the solvability of the non-fragile H_∞ filtering problem. By the proofs of Theorem 1 and Lemma 3, Theorem 1 also shows that the non-fragile H_∞ filtering problem becomes a convex one when the state-space realizations of the designed filters with gain variations admit the slack variable matrix G with the structure of (14). For the case that the designed filter contains no gain variations, from Lemma 3, it follows that the design condition given in Theorem 1 reduces to a necessary and sufficient condition for the standard H_∞ filtering problem, which means that the structure constraint (14) on the slack matrix G does not result in any conservativeness for the standard H_∞ filter design. For the non-fragile filter design method given in Theorem 1, it should be noted that the number of LMIs involved in (20) is $2^{n^2+np+nq}$, which results in the difficulty of implementing the LMI constraints in computation. For example, when $n = 5$ and $p = q = 1$, the number of LMIs involved in (20) is 2^{35}, which already exceeds the capacity of the current LMI solver in Matlab. To overcome the difficulty raising from implementing the design condition given in Theorem 1, the following method is developed. To facilitate to formulate Theorem 2, denote

$$F_{a1} = [f_{a11} \ f_{a12} \ \cdots \ f_{a1l_a}], F_{a2} = [f_{a21}^T \ f_{a22}^T \ \cdots \ f_{a2l_a}^T]^T, \qquad (24)$$

where $l_a = n^2 + np + nq$, and

$$f_{ak1} = \begin{bmatrix} \mathbf{0}_{1\times n} & (N^T e_i)^T & \mathbf{0}_{1\times q} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times r} \end{bmatrix}^T,$$
$$f_{ak2} = \begin{bmatrix} \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times q} & e_j^T & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times r} \end{bmatrix},$$
for $k = (i-1)n + j, \ i, j = 1, \cdots, n$.

$$f_{a1k} = \begin{bmatrix} \mathbf{0}_{1\times n} & (N^T e_i)^T & \mathbf{0}_{1\times q} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times r} \end{bmatrix}^T,$$
$$f_{a2k} = \begin{bmatrix} \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times q} & h_j^T C_2 & h_j^T C_2 & h_j^T D_{21} \end{bmatrix},$$
for $k = n^2 + (i-1)p + j, \ i = 1, \cdots, n, j = 1, \cdots, p$.

$$f_{a1k} = \begin{bmatrix} \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & -g_i^T & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times r} \end{bmatrix}^T,$$
$$f_{a2k} = \begin{bmatrix} \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times q} & e_j^T & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times r} \end{bmatrix},$$
for $k = n^2 + np + (i-1)n + j, \ i = 1, \cdots, q, j = 1, \cdots, n$.

Let $k_0, k_1, \cdots, k_{s_a}$ be integers satisfying $k_0 = 0 < k_1 < \cdots < k_{s_a} = l_a$, and matrix Θ have the following structure

$$\Theta = \begin{bmatrix} diag[\theta_{11}^1 \cdots \theta_{11}^{s_a}] & diag[\theta_{12}^1 \cdots \theta_{12}^{s_a}] \\ diag[\theta_{12}^1 \cdots \theta_{12}^{s_a}]^T & diag[\theta_{22}^1 \cdots \theta_{22}^{s_a}] \end{bmatrix}, \qquad (25)$$

where $\theta_{11}^i, \theta_{12}^i$ and $\theta_{22}^i \in R^{(k_i - k_{i-1})\times(k_i - k_{i-1})}, i = 1, \cdots, s_a$. Then, we have

Theorem 2. Consider system (1). Let $\gamma > 0$ and $\delta_a > 0$ be given constants. If there exist matrices $F_A, F_B, C_F, S, N, \bar{P}_{12}, \bar{P}_{11} >$

$0, \bar{P}_{22} > 0$ and symmetric matrix Θ with the structure described by (25) such that the following LMIs hold:

$$\begin{bmatrix} Q & F_{a1} \\ F_{a1}^T & 0 \end{bmatrix} + \begin{bmatrix} F_{a2} & 0 \\ 0 & I \end{bmatrix}^T \Theta \begin{bmatrix} F_{a2} & 0 \\ 0 & I \end{bmatrix} < 0, \quad (26)$$

$$\begin{aligned} &\begin{bmatrix} I \\ diag[\delta_{k_{i-1}+j} \cdots \delta_{k_i}] \end{bmatrix}^T \begin{bmatrix} \theta_{11}^i & \theta_{12}^i \\ (\theta_{12}^i)^T & \theta_{22}^i \end{bmatrix} \\ &\times \begin{bmatrix} I \\ diag[\delta_{k_{i-1}+j} \cdots \delta_{k_i}] \end{bmatrix} \geq 0, \\ &\text{for all } \delta_{k_{i-1}+j} \in \{-\delta_a, \delta_a\}, \\ &j = 1, \cdots, k_i - k_{i-1}, i = 1, \cdots, s_a, \end{aligned} \quad (27)$$

where

$$Q = \begin{bmatrix} \Xi_1 & \Xi_2 & 0 & S^T A & S^T A & S^T B_1 \\ * & \Xi_3 & 0 & (S-N)^T A + F_B C_2 + F_A & (S-N)^T A + F_B C_2 & (S-N)^T B_1 + F_B D_{21} \\ * & * & -I & C_1 - C_F & C_1 & 0 \\ * & * & * & -\bar{P}_{11} & -\bar{P}_{12} & 0 \\ * & * & * & * & -\bar{P}_{22} & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix}. \quad (28)$$

with Ξ_1, Ξ_2, Ξ_3 defined by (19). Then filter (2) with additive gain uncertainties described by (3) and the filter gain parameters given by (21) solves the non-fragile H_∞ filtering problem for system (1).

Proof. By (20), we have

$$M_0 = Q + \Delta Q + \Delta Q^T < 0, \quad (29)$$

where

$$\Delta Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta Q_1 & \Delta Q_2 & \Delta Q_3 \\ 0 & 0 & 0 & \Delta Q_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with

$$\begin{aligned} \Delta Q_1 &= \sum_{i,j=1}^{n} \delta_{aij} N^T e_i e_j^T + \sum_{i=1}^{n}\sum_{j=1}^{p} \delta_{bij} N^T e_i h_j^T C_2, \\ \Delta Q_2 &= \sum_{i=1}^{n}\sum_{j=1}^{p} \delta_{bij} N^T e_i h_j^T C_2, \\ \Delta Q_3 &= \sum_{i=1}^{n}\sum_{j=1}^{p} \delta_{bij} N^T e_i h_j^T D_{21}, \\ \Delta Q_4 &= -\sum_{i=1}^{q}\sum_{j=1}^{n} \delta_{cij} g_j e_j^T. \end{aligned}$$

By using (24), it follows that (29) is equivalent to

$$\begin{aligned} M_0 &= Q + \textstyle\sum_{i=1}^{l_a} \delta_i f_{a1i} f_{a2i} + (\sum_{i=1}^{l_a} \delta_i f_{a1i} f_{a2i})^T \\ &= Q + F_{a1} \tilde{\Delta}_a F_{a2} + (F_{a1} \tilde{\Delta}_a F_{a2})^T < 0, \end{aligned} \quad (30)$$

where $\tilde{\Delta}_a = diag[\delta_1, \cdots, \delta_{l_a}]$, for all $\delta_i \in \{-\delta_a, \delta_a\}$. By Lemma 4, it follows that (30) is further equivalent to that there exists a symmetric matrix $\Theta \in R^{l_a \times l_a}$ such that (26) and

$$\begin{bmatrix} I \\ \tilde{\Delta}_a \end{bmatrix}^T \Theta \begin{bmatrix} I \\ \tilde{\Delta}_a \end{bmatrix} \geq 0 \quad (31)$$

hold for all $\delta_i \in \{-\delta_a, \delta_a\}, \ i = 1, \cdots, l_a$. Notice that the set of Θ satisfying (25) is a subset of the set of Θ satisfying (31), hence the conclusion follows.

Remark 3. From the proof of Theorem 2, it follows that when $s_a = 1$, the set of Θ satisfying (25) is equal to the set of Θ satisfying (31), and the design conditions given in Theorem 2 and Theorem 1 are equivalent to checking all the parameter vertices. Θ satisfying (26) and (31) (or (27) with $s_a = 1$) is said to be a *vertex separator* [6]. Notice that the number of LMIs involved in (31) or (27) with $s_a = 1$ still is $2^{n^2+np+nq}$, so that the difficulty of implementing the LMI constraints remains. To overcome the difficulty, Theorem 2 presents a sufficient condition for the non-fragile H_∞ filter design in terms of the separator Θ with the structure described by (25), where the number of LMIs involved in (27) is $\sum_{i=1}^{s_a} 2^{k_i - k_{i-1}}$, which can be controlled not to grow exponentially by reducing the value of max $k_i - k_{i-1}: \ i = 1, \cdots, s_a$. Compared with the Θ being of full block in (26) and (31), Θ with the structure described by (25) satisfying (26) and (27) is said to be a *structured vertex separator*. However, it should be noted that the design condition given in Theorem 2 may be more conservative than that given in Theorem 1 because of the structure constraint on Θ. But the design condition proposed in Theorem 2 solves the numerical computation problem, which cannot be solved by the design condition given in Theorem 1. On the other hand, in Theorem 2, the smaller value of s_a is, the less conservativeness is introduced.

B. Comparison with the existing design method

In the following, we will introduce the result of non-fragile H_∞ filter design with norm-bounded gain variations. And at the same time, the relationship with our result is discussed.

Similar to [10] and [17], the norm-bounded type of gain variations $\Delta A_F, \Delta B_F$ and ΔC_F can be overbounded [13] by the following norm-bounded uncertainty:

$$\begin{aligned} \Delta A_F = \ & M_a F_1(t) E_a, \Delta B_F = M_b F_2(t) E_b, \\ & \Delta C_F = M_c F_3(t) E_c, \end{aligned} \quad (32)$$

where

$$\begin{aligned} M_a &= [M_{a1} \cdots M_{an^2}], E_a = [E_{a1}^T \cdots E_{an^2}^T]^T, \\ M_b &= [M_{b1} \cdots M_{bnp}], E_b = [E_{b1}^T \cdots E_{bnp}^T]^T, \\ M_c &= [M_{c1} \cdots M_{cnq}], E_c = [E_{c1}^T \cdots E_{cnq}^T]^T, \end{aligned}$$

with

$$\begin{aligned} & M_{ak} = e_i, E_{ak} = e_j^T \\ & \text{for } k = (i-1)n + j, \ i, j = 1, \cdots, n, \\ & M_{bk} = e_i, E_{bk} = h_j^T \\ & \text{for } k = n^2 + (i-1)p + j, \ i = 1, \cdots, n, \ j = 1, \cdots, p, \\ & M_{ck} = g_i, E_{ck} = e_j^T \\ & \text{for } k = n^2 + np + (i-1)n + j, \ i = 1, \cdots, q, \ j = 1, \cdots, n. \end{aligned}$$

and $F_i^T(t) F_i(t) \leq \delta_a^2 I$ for $i = 1, 2, 3$, represent the uncertain parameters, here δ_a is the same as before. Then, the following lemma is presented to design the non-fragile H_∞ filter with gain variations (32) by using the method [10] and [17].

Lemma 5: Consider system (1). Let $\gamma > 0, \delta_a > 0$ be given constants. If there exist matrices $\bar{F}_A, \bar{F}_B, C_F, \bar{S} > 0, \bar{N} < 0$ and scalar $\varepsilon > 0$, such that the following LMI holds:

$$\begin{bmatrix} \bar{Q} & M_{a1} & \delta_a \varepsilon M_{a2}^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0, \quad (33)$$

where

$$M_{a1} = \begin{bmatrix} 0 & 0 & 0 \\ \bar{N} M_a & \bar{N} M_b & 0 \\ 0 & 0 & -M_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M_{a2} = \begin{bmatrix} 0 & 0 & 0 & E_a & 0 & 0 \\ 0 & 0 & 0 & E_bC_2 & E_bC_2 & E_bD_{21} \\ 0 & 0 & 0 & E_c & 0 & 0 \end{bmatrix}.$$

$$\bar{Q} = \begin{bmatrix} -\bar{S} & -\bar{S} & 0 & \bar{S}A & \bar{S}A & \bar{S}B_1 \\ * & -\bar{S}+\bar{N} & 0 & Q_1 & Q_2 & Q_3 \\ * & * & -I & C_1 - C_F & C_1 & 0 \\ * & * & * & -\bar{S} & -\bar{S} & 0 \\ * & * & * & * & -\bar{S}+\bar{N} & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix},$$

with $Q_1 = (\bar{S} - \bar{N})A + \bar{F}_A + \bar{F}_B C_2,\ Q_2 = (\bar{S} - \bar{N})A + \bar{F}_B C_2,\ Q_3 = (\bar{S} - \bar{N})B_1 + \bar{F}_B D_{21}$, then filter (2) with additive norm-bounded gain uncertainty described by (32) and $A_F = (\bar{N}^T)^{-1}\bar{F}_A,\ \ B_F = (\bar{N}^T)^{-1}\bar{F}_B,\ \ C_F = C_F$ solves the non-fragile H_∞ filtering problem for system (1).

Lemma 5 presents a method for designing non-fragile H_∞ filters with norm-bounded gain variations via the existing technique. To show the relationship between the conditions for designing non-fragile H_∞ filters given in Lemma 5 and Theorem 2, the following Lemma is presented.

Lemma 6: Consider system (1), if Lemma 5 is feasible, then Theorem 2 is feasible.

Proof: Let $S = \bar{P}_{11} = \bar{P}_{12} = \bar{S}, N = \bar{N}, \bar{P}_{22} = \bar{S} - \bar{N} > 0$, then it is easy to see that $Q = \bar{Q}, F_{a1} = M_{a1}$ and $F_{a2} = M_{a2}$, i.e., condition (33) becomes

$$\begin{bmatrix} Q & F_{a1} & \delta_a \varepsilon F_{a2}^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0. \tag{34}$$

In Theorem 2, when $s_a = l_a$, according to (34) and $F_i^T(t)F_i(t) \le \delta_a^2 I,\ i = 1, 2, 3$, and by the Schur complement, there exists a matrix Θ with the structure

$$\Theta = \begin{bmatrix} \varepsilon\delta_a^2 I & \mathbf{0}_{l_a \times l_a} \\ \mathbf{0}_{l_a \times l_a} & -\varepsilon I \end{bmatrix}, \tag{35}$$

such that the following LMIs hold:

$$\begin{bmatrix} Q & F_{a1} \\ F_{a1}^T & 0 \end{bmatrix} + \begin{bmatrix} F_{a2} & 0 \\ 0 & I \end{bmatrix}^T \Theta \begin{bmatrix} F_{a2} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} Q + \varepsilon\delta_a^2 F_{a2}^T F_{a2} & F_{a1} \\ F_{a1}^T & -\varepsilon I \end{bmatrix} < 0, \tag{36}$$

$$\begin{bmatrix} I \\ \delta_i \end{bmatrix}^T \begin{bmatrix} \theta_{11}^i & \theta_{12}^i \\ (\theta_{12}^i)^T & \theta_{22}^i \end{bmatrix} \begin{bmatrix} I \\ \delta_i \end{bmatrix} = \varepsilon\delta_a^2 - \varepsilon\delta_i^2 \ge 0, \quad \text{for all } i = 1, \cdots, l_a. \tag{37}$$

Thus, the proof is complete. ■

Remark 4. From the proof of Lemma 6, it follows that Lemma 5 is more conservative than Theorem 2 with $s_a = l_a$. However, as indicated in Remark 3, the case of $s_a = l_a$ is the worst case of the method. So the existing non-fragile H_∞ filter design method with the norm-bounded gain variations is more conservative than the non-fragile H_∞ filter design method proposed in this paper.

C. Evaluation of H_∞ performance index

In Theorem 2, for obtaining the convex design condition, we restrict the slack variable matrix G with structure (14), which may result in more conservative evaluation of the H_∞ performance index bound. So, in this subsection, for a designed filter, the matrix variable G without the restriction is exploited for obtaining less conservative evaluation of the H_∞ performance index bound.

When the filter parameter matrices A_F, B_F and C_F are known, the problem of minimizing γ subject to (3) for a given $\delta_a > 0$ and $\| G_{z_e\omega}(z) \| < \gamma$ can be converted into the one: minimize γ^2 subject to the following LMIs:

$$\begin{bmatrix} P - G - G^T & 0 & G^T A_e & G^T B_e \\ * & -I & C_e & 0 \\ * & * & -P & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \quad \text{for all } \delta_{aij},\ \delta_{bik},\ \delta_{clj} \in \{-\delta_a, \delta_a\},\ i, j = 1, \cdots, n;\ k = 1, \cdots, p;\ l = 1, \cdots, q, \tag{38}$$

where A_e, B_e and C_e are defined as in (5).

Similar to the design condition given in Theorem 1, the above method is also with the numerical computation problem. To solve the problem, the following lemma provides a solution using the structured vertex separator approach.

Denote

$$G_{a1} = [g_{a11}\ g_{a12}\ \cdots\ g_{a1l_a}], G_{a2} = [g_{a21}^T\ g_{a22}^T\ \cdots\ g_{a2l_a}^T]^T. \tag{39}$$

where

$$g_{a1k} = \left[\begin{pmatrix} \mathbf{0}_{1\times n} & e_i^T \end{pmatrix} G \quad \mathbf{0}_{1\times q} \quad \mathbf{0}_{1\times 2n} \quad \mathbf{0}_{1\times r} \right]^T,$$
$$g_{a2k} = \left[\mathbf{0}_{1\times 2n} \quad \mathbf{0}_{1\times q} \quad \mathbf{0}_{1\times n} \quad e_j^T \quad \mathbf{0}_{1\times n} \right],$$
for $k = (i-1)n + j,\ i, j = 1, \cdots, n.$

$$g_{a1k} = \left[\begin{pmatrix} \mathbf{0}_{1\times n} & e_i^T \end{pmatrix} G \quad \mathbf{0}_{1\times q} \quad \mathbf{0}_{1\times 2n} \quad \mathbf{0}_{1\times r} \right]^T,$$
$$g_{a2k} = \left[\mathbf{0}_{1\times 2n} \quad \mathbf{0}_{1\times q} \quad h_j^T C_2 \quad \mathbf{0}_{1\times n} \quad h_j^T D_{21} \right],$$
for $k = n^2 + (i-1)p + j,\ i = 1, \cdots, n, j = 1, \cdots, p.$

$$g_{a1k} = \left[\mathbf{0}_{1\times 2n} \quad -g_i^T \quad \mathbf{0}_{1\times 2n} \quad \mathbf{0}_{1\times r} \right]^T,$$
$$g_{a2k} = \left[\mathbf{0}_{1\times 2n} \quad \mathbf{0}_{1\times q} \quad \mathbf{0}_{1\times n} \quad e_j^T \quad \mathbf{0}_{1\times n} \right],$$
for $k = n^2 + np + (i-1)n + j,\ i = 1, \cdots, q, j = 1, \cdots, n.$

Then we have

Lemma 7: Consider the system (1). Let $\gamma > 0, \delta_a > 0$ be constants and filter parameter matrices A_F, B_F, C_F be given. Then $\| G_{z_e\omega}(z) \| < \gamma$ holds for all δ_{aij}, δ_{bik} and δ_{clj} satisfying (3), if there exist a matrix G, a positive definite matrix $P > 0$ and a symmetric matrix Θ with the structure described by (25) such that (27) and the following LMI hold:

$$\begin{bmatrix} Q_s & G_{a1} \\ G_{a1}^T & 0 \end{bmatrix} + \begin{bmatrix} G_{a2} & 0 \\ 0 & I \end{bmatrix}^T \Theta \begin{bmatrix} G_{a2} & 0 \\ 0 & I \end{bmatrix} < 0 \tag{40}$$

where

$$Q_s = \begin{bmatrix} P - G - G^T & 0 & G^T A_{e0} & G^T B_{e0} \\ * & -I & C_{e0} & 0 \\ * & * & -P & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix}$$

with A_{e0}, B_{e0} and C_{e0} are defined by (7).

Proof: By using (38) and (39), it is similar to the proof of Theorem 2, and omitted here. ■

Remark 5. For evaluating the H_∞ performance bound of the transfer function from ω to z_e, the condition given in Lemma 7 usually is less conservative than that given in Theorem 2 because no structure constraint on the slack variable matrix G in Lemma 7 is imposed.

IV. Example

An example is given to illustrate the effectiveness of the proposed method. Consider a linear system of form (1) with

$$A = \begin{bmatrix} 0 & 1 & -0.5 \\ -1 & -0.5 & 1 \\ -1 & 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} -1 & 0 \\ 0.5 & 0 \\ -1 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} -1 & 0.5 & 2 \end{bmatrix}, D_{21} = \begin{bmatrix} 0 & 0.9 \end{bmatrix}.$$

For the case that the designed filter contains no gain variations, by the standard H_∞ filtering method [12] for discrete-time systems, the optimal H_∞ performance index of the standard closed-loop system is achieved as $\gamma_{opt} = 3.7282$.
In the following, tables and a figure are given to compare the proposed method with the existing method.

A. Comparison

In this subsection, tables are given to provide a comparison between the proposed method given by Theorem 2 and the existing method given by Lemma 5.
Firstly, the H_∞ performance indexes achieved by the designs are showed in Table 1.

TABLE I
PERFORMANCE INDEXES BY DESIGN WITH $\delta_a = 0.05$

	Lem. 5	Th.2($s_a = 15$)	Th.2 ($s_a = 5$)
γ	4.7319	4.1791	4.1790

From Table 1, we can see that compared with the optimal H_∞ performance index bound $\gamma_{opt} = 3.7282$, the performance index of the filter designed by Lemma 5 is degraded 26.92%. The performance indexes of the filters designed by Theorem 2 are degraded the same as 12.09% (for $s_a = 15$ or $s_a = 5$), which are much more improved than 26.92%.
For convenience, denote the filter designed by Lemma 5 as F_{norm}, denote the filters designed by Theorem 2 as F_{in15} for $s_a = 15$ and F_{in5} for $s_a = 5$, respectively. For these designed filters, Lemma 7 gives better performance indexes shown in Table 2.

TABLE II
PERFORMANCE INDEXES EVALUATION BY LEMMA 7 WITH $\delta_a = 0.05$

	F_{norm}	F_{in15}	F_{in5}
$\gamma(s_a = 15)$	4.3539	4.1069	$--$
$\gamma(s_a = 5)$	4.3439	$--$	4.1060

Obviously, compared with $\gamma_{opt} = 3.7282$, by Lemma 7, the H_∞ performance indexes of filter F_{norm} are degraded 16.78% for $s_a = 15$ and 16.51% for $s_a = 5$. Correspondingly, the performance indexes of filters F_{in15} and F_{in5} are degraded 10.16% for $s_a = 15$ and 10.13% for $s_a = 5$, respectively.

V. CONCLUSION

In this paper, the problem of non-fragile H_∞ filter design for linear discrete-time systems has been addressed, where the filter to be designed is assumed to be with additive gain variations of interval type due to the FWL effects. A notion of structured vertex separator is proposed to approach the problem, and exploited to develop sufficient conditions for the non-fragile H_∞ filter design in terms of solutions to a set of LMIs. The designs guarantee the asymptotic stability of the estimation errors, and the H_∞ performance of the system from the exogenous signals to the estimation errors less than a prescribed level. A comparison between our method and the existing method for non-fragile H_∞ filter design is presented to indicate the superiority of our proposed method. A numerical example has shown the effectiveness of the proposed approach.

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