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Dynamic Output Feedback Covariance Control of Linear Stochastic Dissipative Partial Differential Equations

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Abstract— **In this work, we develop a method for dynamic output feedback covariance control of the state covariance of linear dissipative stochastic partial differential equations (PDEs) using spatially distributed control actuation and sensing with noise. Such stochastic PDEs arise naturally in the modeling of surface height profile evolution in thin film growth and sputtering processes. We begin with the formulation of the stochastic PDE into a system of infinite stochastic ordinary differential equations (ODEs) by using modal decomposition. A finite-dimensional approximation is then obtained to capture the dominant mode contribution to the surface roughness profile (i.e., the covariance of the surface height profile). Subsequently, a state feedback controller and a Kalman-Bucy filter are designed on the basis of the finite-dimensional approximation. The dynamic output feedback covariance controller is subsequently obtained by combining the state feedback controller and the state estimator. The steady-state expected surface covariance under the dynamic output feedback controller is then estimated on the basis of the closed-loop finite-dimensional system. An analysis is performed to obtain a theoretical estimate of the expected surface covariance of the closed-loop infinite-dimensional system. Applications of the linear dynamic output feedback controller to the linearized stochastic Kuramoto-Sivashinsky equation are presented.**

I. Introduction

The recent efforts on feedback control and optimization of thin film growth processes to achieve desired material microstructure (see, for example, [4], [6], [7] and the references therein) have been motivated by the fact that the electrical and mechanical properties of thin films strongly depend on microstructural features such as interface width, island density and size distributions [2], [17], which significantly affect device performance. To fabricate thin film devices with high and consistent performance, it is desirable that the operation of thin film growth processes is tightly controlled.

Stochastic PDEs arise naturally in the modeling of surface morphology of ultra thin films in a variety of material preparation processes [11], [24], [25], [10]. Methods for state feedback covariance control for linear [19], [18], [21] and nonlinear [20] stochastic PDEs have been developed. However, in the design of a state feedback controller, it is assumed that the full state of the PDE can be measured in real-time at all positions and times. This assumption is not practical in many applications, where process output measurements are typically available from a finite (usually small) number of measurement sensors. Therefore, there is a strong motivation to develop dynamic output feedback covariance control methods for stochastic PDEs, which couple a state feedback control law to a dynamic state-observer that utilizes information from a few measurement sensors. The observer-based covariance control structure for linear stochastic ODE systems was proposed in [12], [15], in which a Kalman filter is used as a state estimator and the estimated state is used by the feedback controller. However, the problem of output feedback covariance control for nonlinear systems and infinite-dimensional systems has not been studied.

In this work, a method is developed for dynamic output feedback covariance control of the state covariance of linear dissipative stochastic PDEs. Spatially distributed control actuation and sensor measurements with noise are considered when designing the dynamic output feedback controller. We initially formulate the stochastic PDE into a system of infinite stochastic ODEs by using modal decomposition and construct a finite-dimensional approximation to capture the dominant mode contribution to the surface covariance of the height profile. Subsequently, a state feedback controller and a Kalman-Bucy filter are designed on the basis of the finite-dimensional approximation. The dynamic output feedback controller is obtained by combining the state feedback controller and the state estimator. Analysis of the closed-loop stability and the steady-state surface covariance under the dynamic output feedback controller are provided for the finite-dimensional approximation and the infinite-dimensional system. Applications of the linear dynamic output feedback controller to the linearized stochastic Kuramoto-Sivashinsky equation (KSE) are presented.

II. Preliminaries

A. Stochastic PDEs with Distributed Control

We focus on linear dissipative stochastic PDEs with distributed control of the following form:

$$\frac{\partial h}{\partial t} = \mathscr{A} h + \sum_{i=1}^{p} b_i(x) u_i(t) + \xi(x,t) \qquad (1)$$

subject to homogeneous boundary conditions and the initial condition $h(x,0) = h_0(x)$, where $x \in [-\pi, \pi]$ is the spatial coordinate, t is the time, $h(x,t)$ is the state of the PDE which corresponds to the height of the surface in a thin film growth process at position x and time t, $\mathscr{A}$ is a dissipative spatial differential operator, $u_i(t)$ is the i^{th} manipulated input,

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p is the number of manipulated inputs and $b_i(x)$ is the i^{th} actuator distribution function (i.e., $b_i(x)$ determines how the control action computed by the i^{th} control actuator, $u_i(t)$, is distributed (e.g., point or distributed actuation) in the spatial interval $[-\pi,\pi]$). $\xi(x,t)$ is a Gaussian noise with the following expressions for its mean and covariance:

$$\begin{array}{rcl}\langle\xi(x,t)\rangle & = & 0 \\ \langle\xi(x,t)\xi(x',t')\rangle & = & \sigma^2\delta(x-x')\delta(t-t')\end{array} \tag{2}$$

where σ is a real number, $\delta(\cdot)$ is the Dirac function, and $\langle\cdot\rangle$ denotes the expected value.

Our objective is to control the surface covariance of the process, Cov_h, which is represented by the expected value of the standard deviation of the surface height from the desired height and is given as follows:

$$Cov_h(t) = \left\langle \int_{-\pi}^{\pi}[h(x,t)-h_d]^2dx \right\rangle \tag{3}$$

where $h_d(t)$ is the desired surface height.

To study the dynamics of Eq.1, we initially consider the eigenvalue problem of the linear spatial differential operator of Eq.1 subject to the operator homogenous boundary conditions, which takes the form:

$$\mathscr{A}\bar{\phi}_n(x) = \lambda_n\bar{\phi}_n(x), \quad n=1,2,\cdots \tag{4}$$

where λ_n and $\bar{\phi}_n$ denote the n^{th} eigenvalue and eigenfunction, respectively. To simplify our development and motivated by most practical applications, we consider stochastic PDEs for which $\mathscr{A}$ is a highly dissipative operator (i.e., a second-order or fourth-order linear self-adjoint operator) and has eigenvalues which are real and satisfy $\lambda_1 \geq \lambda_2 \cdots$ and the sum $\sum_{i=1}^{\infty}\left|\frac{1}{\lambda_i}\right|$ converges to a finite positive number. Furthermore, the eigenfunctions $\{\bar{\phi}_1(x),\bar{\phi}_2(x),\cdots\}$ form a complete orthonormal set.

To present the method for feedback controller design, we initially formulate Eq.1 into an infinite-dimensional stochastic ODE system using modal decomposition. To this end, we first expand the solution of Eq.1 into an infinite series in terms of the eigenfunctions of the operator of Eq.4 as follows:

$$h(x,t) = \sum_{n=1}^{\infty}\alpha_n(t)\bar{\phi}_n(x) \tag{5}$$

where $\alpha_n(t)$ $(n=1,2,\cdots,\infty)$ are time-varying coefficients. Substituting the above expansion for the solution, $h(x,t)$, into Eq.1 and taking the inner product with the adjoint eigenfunction, $\bar{\phi}_n^*(x)$, the following system of infinite stochastic ODEs is obtained:

$$\frac{d\alpha_n}{dt} = \lambda_n\alpha_n + \sum_{i=1}^{p}b_{i\alpha_n}u_i(t) + \xi_\alpha^n(t), n=1,\ldots,\infty \tag{6}$$

where

$$b_{i\alpha_n} \quad = \quad \int_{-\pi}^{\pi}\bar{\phi}_n^*(x)b_i(x)dx \tag{7}$$

and

$$\xi_\alpha^n(t) = \int_{-\pi}^{\pi}\xi(x,t)\bar{\phi}_n^*(x)dx \tag{8}$$

The covariance of $\xi_\alpha^n(t)$ can be computed by using the following result:

Result 1: If (1) $f(x)$ is a deterministic function, (2) $\eta(x)$ is a random variable with $\langle\eta(x)\rangle = 0$ and covariance $\langle\eta(x)\eta(x')\rangle = \sigma^2\delta(x-x')$, and (3) $\varepsilon = \int_a^b f(x)\eta(x)dx$, then ε is a real random number with $\langle\varepsilon\rangle = 0$ and covariance $\langle\varepsilon^2\rangle = \sigma^2\int_a^b f^2(x)dx$ [1].

Using Result 1, we obtain $\langle\xi_\alpha^n(t)\xi_\alpha^n(t')\rangle = \sigma^2\delta(t-t')$.

In this work, the controlled variable is the surface covariance defined in Eq.3. Without loss of generality, we pick $h_d(t)=0$. Therefore, $Cov_h(t)$ can be rewritten in terms of $\alpha_n(t)$ as follows [19]:

$$\begin{aligned}&Cov_h(t) = \langle\int_{-\pi}^{\pi}[h(x,t)-0]^2dx\rangle \\ &= \langle\int_{-\pi}^{\pi}[\sum_{i=1}^{\infty}\alpha_i(t)\phi_i(x)]^2dx\rangle = \langle\sum_{i=1}^{\infty}\alpha_i(t)^2\rangle = \sum_{i=1}^{\infty}\langle\alpha_i(t)^2\rangle\end{aligned} \tag{9}$$

Eq.9 provides a direct link between the surface covariance and the state covariance of the system of infinite stochastic ODEs of Eq.6.

B. Model Reduction

Owing to its infinite-dimensional nature, the system of Eq.6 cannot be directly used as a basis for feedback controller design that can be implemented in practice (i.e., the practical implementation of such a controller will require the computation of infinite sums which cannot be done by a computer). Instead, we will use finite-dimensional approximations of the system of Eq.6 for the purpose of model-based output feedback controller design. Specifically, we rewrite the system of Eq.6 as follows:

$$\begin{aligned}\frac{dx_s}{dt} &= \Lambda_s x_s + B_s u + \xi_s \\ \frac{dx_f}{dt} &= \Lambda_f x_f + B_f u + \xi_f\end{aligned} \tag{10}$$

where

$$\begin{aligned}&x_s = [\alpha_1 \quad \cdots \quad \alpha_m]^T \quad x_f = [\alpha_{m+1} \quad \alpha_{m+2} \quad \cdots]^T \\ &\xi_s = [\xi_\alpha^1 \quad \cdots \quad \xi_\alpha^m]^T \quad \xi_f = [\xi_\alpha^{m+1}\xi_\alpha^{m+2}\cdots]^T \\ &\Lambda_s = diag[\lambda_1 \quad \cdots \quad \lambda_m] \\ &\Lambda_f = diag[\lambda_{m+1} \quad \lambda_{m+2} \quad \cdots] \\ &B_s = \begin{bmatrix} b_{1\alpha_1} & \cdots & b_{p\alpha_1} \\ \vdots & \ddots & \vdots \\ b_{1\alpha_m} & \cdots & b_{p\alpha_m}\end{bmatrix} \\ &B_f = \begin{bmatrix} b_{1\alpha_{m+1}} & \cdots & b_{p\alpha_{m+1}} \\ b_{1\alpha_{m+2}} & \cdots & b_{p\alpha_{m+2}} \\ \vdots & \vdots & \vdots\end{bmatrix}\end{aligned} \tag{11}$$

Note that the x_s subsystem is m^{th}-order and the x_f subsystem is infinite-dimensional.

The expression of Cov_h in Eq.9 can be re-written in the following form:

$$\begin{aligned}Cov_h(t) &= \sum_{i=1}^{\infty}\langle\alpha_i(t)^2\rangle = \sum_{i=1}^{m}\langle\alpha_i(t)^2\rangle + \sum_{i=m+1}^{\infty}\langle\alpha_i(t)^2\rangle \\ &= \langle x_s(t)^T x_s(t)\rangle + \langle x_f(t)^T x_f(t)\rangle = Tr[P_s(t)] + Tr[P_f(t)]\end{aligned} \tag{12}$$

where P_s and P_f are covariance matrices of the x_s and x_f which are defined as $P_s = \langle x_s x_s^T\rangle$ and $P_f = \langle x_f x_f^T\rangle$, respectively. $Tr[\cdot]$ denotes the trace of a matrix.

Neglecting the x_f subsystem, the following finite-dimensional approximation is obtained:

$$\frac{d\tilde{x}_s}{dt} = \Lambda_s\tilde{x}_s + B_s u + \xi_s \tag{13}$$

and the surface covariance of the infinite-dimensional stochastic system, Cov_h, can be approximated by $\widetilde{Cov}_h$, which is computed from the state of the finite-dimensional approximation of Eq.13 as follows:

$$\widetilde{Cov}_h(t) = Tr[\tilde{P}_s(t)] \tag{14}$$

where the tilde symbol denotes that the variable is associated with the finite-dimensional system. The reader may refer to [9], [23], [5] for further results on model reduction of dissipative PDEs.

C. State Feedback Control

When the state of the finite-dimensional system of Eq.13 is available, a linear state feedback controller can be designed to regulate the surface covariance:

$$u = G\tilde{x}_s \tag{15}$$

where G is the gain matrix, which should be carefully designed so as to stabilize the closed-loop finite-dimensional system and obtain the desired closed-loop surface covariance. Note that the linear state feedback controller of Eq.15 has been used, in our previous work, to control the surface covariance in both thin film growth and ion-sputtering processes [19], [18].

Since the above state feedback control assumes a full knowledge of the states of the process at all positions and times, which may be a restrictive requirement for certain practical applications, we proceed to design output feedback controllers by combining the state feedback control law and a state observer.

III. OUTPUT FEEDBACK CONTROL

In this section, we design linear output feedback controllers by combining the state feedback control law of Eq.15 and a dynamic state observer which estimates the state of the finite-dimensional system of Eq.13 using the measured process output with sensor noise. First, a dynamic state observer is developed using a Kalman-Bucy filter approach, which yields an optimal estimate of the state of the finite-dimensional system by minimizing the mean square estimation error. The dynamic state observer is then coupled to the state feedback controller of Eq.15 to construct a dynamic output feedback controller. For the special case where the number of measurement sensors is equal to the order of the finite-dimensional system, a static output feedback controller may be designed by following a static state estimation approach proposed in [3], [8].

A. Measured Output with Sensor Noise

The state feedback controller of Eq.15 requires the availability of the state $\tilde{x}_s$, which implies that the value of the surface height profile, $h(x,t)$, is available at any location and time. However, from a practical point of view, measurements of the surface height profile are only available at a finite number of locations. Motivated by this, we design an output feedback controller that uses measurements of the surface height at distinct locations to enforce a desired closed-loop surface covariance. The sensor noise is modeled as a Gaussian white noise and is added to the surface height measurements. Specifically, the measured process output is expressed as follows:

$$y(t) = \begin{bmatrix} h(x_1,t)+\xi_y^1(t) & h(x_2,t)+\xi_y^2(t) & \cdots & h(x_q,t)+\xi_y^q(t)\end{bmatrix}^T \tag{16}$$

where x_i $(i = 1,2,\cdots,q)$ denotes a location of a point measurement sensor and q is the number of measurement sensors. $\xi_y^1(t)$, $\xi_y^2(t)$, $\cdots$, $\xi_y^q(t)$ are independent Gaussian white noises with the following expressions for their means and covariances:

$$\begin{aligned}&\langle\xi_y^i(t)\rangle = 0\\ &\langle\xi_y^i(t)\xi_y^j(t')\rangle = \sigma^2\delta_{ij}\delta(t-t')\\ &i = 1,2,\cdots q \quad j = 1,2,\cdots q\end{aligned} \tag{17}$$

where σ is a constant and δ_{ij} is the Kronecker delta function. Note that the sensor noises are independent of the system noises, ξ_s and ξ_f.

Using Eq.5, the vector of measured outputs, $y(t)$, can be written in terms of x_s and x_f as follows:

$$\begin{aligned} y(t) &= \begin{bmatrix} \sum_{n=1}^{\infty}\alpha_n(t)\phi_n(x_1)+\xi_y^1(t)\\ \sum_{n=1}^{\infty}\alpha_n(t)\phi_n(x_2)+\xi_y^2(t)\\ \vdots\\ \sum_{n=1}^{\infty}\alpha_n(t)\phi_n(x_q)+\xi_y^q(t)\end{bmatrix} \\ &= C_s x_s(t) + C_f x_f(t) + \xi_y(t)\end{aligned} \tag{18}$$

where

$$\begin{aligned} C_s &= \begin{bmatrix}\phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_m(x_1)\\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_m(x_2)\\ \vdots & \vdots & \ddots & \vdots\\ \phi_1(x_q) & \phi_2(x_q) & \cdots & \phi_m(x_q)\end{bmatrix}\\ C_f &= \begin{bmatrix}\phi_{m+1}(x_1) & \phi_{m+2}(x_1) & \cdots\\ \phi_{m+1}(x_2) & \phi_{m+2}(x_2) & \cdots\\ \vdots & \vdots & \ddots\\ \phi_{m+1}(x_q) & \phi_{m+2}(x_q) & \cdots\end{bmatrix}\end{aligned} \tag{19}$$

and

$$\xi_y(t) = \begin{bmatrix} \xi_y^1(t) & \xi_y^2(t) & \cdots & \xi_y^q(t) \end{bmatrix}^T \tag{20}$$

Consequently, the system of Eq.10 with the measured process output vector can be written as follows:

$$\begin{aligned} \frac{dx_s}{dt} &= \Lambda_s x_s + B_s u + \xi_s \\ \frac{dx_f}{dt} &= \Lambda_f x_f + B_f u + \xi_f \\ y &= C_s x_s + C_f x_f + \xi_y \end{aligned} \tag{21}$$

Neglecting the x_f subsystem, the following finite-dimensional stochastic ODE system can be obtained:

$$\begin{aligned} \frac{d\tilde{x}_s}{dt} &= \Lambda_s \tilde{x}_s + B_s u + \xi_s \\ \tilde{y} &= C_s \tilde{x}_s + \xi_y \end{aligned} \tag{22}$$

where the tilde symbols in $\tilde{x}_s$ and $\tilde{y}$ denote the correspondence to a reduced-order system. The system of Eq.22 is used as the basis for output feedback controller design.

B. Dynamic Output Feedback Control

To design a dynamic output feedback controller, we first construct a dynamic state estimator using information from the measured output vector. Specifically, a Kalman-Bucy filter is designed for the optimal estimation of the state of the finite-dimensional system of Eq.22 as follows [12]:

$$\frac{d\hat{x}_s}{dt} = \Lambda_s \hat{x}_s + B_s u + K(y - C_s \hat{x}_s), \quad \hat{x}_s(0) = \hat{x}_{s0} \tag{23}$$

where $\hat{x}_s$ is the estimate of the state and K is a gain matrix, which is computed as follows [12]:

$$K = Q C_s^T V_y^{-1} \tag{24}$$

where V_y is the sensor noise intensity matrix defined by

$$\langle \xi_y(t) \xi_y(t')^T \rangle = V_y \delta(t - t') \tag{25}$$

and Q is the covariance matrix for the state estimation error and is defined as

$$Q = \lim_{t \to \infty} \langle \tilde{e}(t) \tilde{e}(t)^T \rangle \tag{26}$$

where $\tilde{e}(t)$ is the estimation error:

$$\tilde{e} = \tilde{x}_s - \hat{x}_s \tag{27}$$

The covariance matrix for the state estimation error, Q, is the unique nonnegative-definite solution of the following algebraic Riccati equation [12]:

$$\Lambda_s Q + Q \Lambda_s - Q C_s^T V_y^{-1} C_s Q + V_s = 0 \tag{28}$$

where V_s is the noise intensity matrix of the ξ_s defined by

$$\langle \xi_s(t) \xi_s(t')^T \rangle = V_s \delta(t - t') \tag{29}$$

The dynamic output feedback controller is designed by combining the state feedback controller of Eq.15 and the state estimator of Eq.23 and takes the form:

$$\begin{aligned} \frac{d\hat{x}_s}{dt} &= \Lambda_s \hat{x}_s + B_s u + K(y - C_s \hat{x}_s), \quad \hat{x}_s(0) = \hat{x}_{s0} \\ u &= G \hat{x}_s \end{aligned} \tag{30}$$

By applying the dynamic output feedback controller of Eq.30 to the finite-dimensional system of Eq.22, the following closed-loop finite dimensional system can be obtained:

$$\begin{aligned} \frac{d\tilde{x}_s}{dt} &= \Lambda_s \tilde{x}_s + B_s u + \xi_s \\ \tilde{y} &= C_s \tilde{x}_s + \xi_y \\ \frac{d\hat{x}_s}{dt} &= \Lambda_s \hat{x}_s + B_s u + K(\tilde{y} - C_s \hat{x}) \\ u &= G \hat{x}_s \end{aligned} \tag{31}$$

The closed-loop finite dimensional system of Eq.31 can be written in terms of $\tilde{x}_s$ and e using Eq.27 as follows:

$$\begin{aligned} \frac{d\tilde{x}_s}{dt} &= (\Lambda_s + B_s G)\tilde{x}_s - B_s G \tilde{e} + \xi_s \\ \frac{d\tilde{e}}{dt} &= (\Lambda_s - K C_s)\tilde{e} + \xi_s - K \xi_y \end{aligned} \tag{32}$$

The stability of the closed-loop finite-dimensional system of Eq.32 depends on the stability properties of the matrices $(\Lambda_s + B_s G)$ and $(\Lambda_s - K C_s)$. Specifically, the stability of $(\Lambda_s + B_s G)$ depends on the appropriate design of the state feedback controller and the stability of $(\Lambda_s - K C_s)$ depends on the appropriate design of the Kalman-Bucy filter. Owing to its cascaded structure, the system of Eq.32 is asymptotically stable if both $(\Lambda_s + B_s G)$ and $(\Lambda_s - K C_s)$ are stable matrices. This results in the existence of a steady-state covariance matrix (e.g., a covariance matrix as $t \to \infty$) of the closed-loop stochastic system [12]. To investigate the steady-state covariance matrix of the closed-loop system of Eq.32, we rewrite Eq.32 as follows:

$$\frac{d}{dt}\begin{bmatrix} \tilde{x}_s \\ e \end{bmatrix} = \begin{bmatrix} \Lambda_s + B_s G & -B_s G \\ \mathbf{0} & \Lambda_s - K C_s \end{bmatrix} \begin{bmatrix} \tilde{x}_s \\ e \end{bmatrix} + \begin{bmatrix} I_s & \mathbf{0} \\ I_s & -K \end{bmatrix} \begin{bmatrix} \xi_s \\ \xi_y \end{bmatrix} \tag{33}$$

where I_s is a m^{th}-order elementary matrix and $\mathbf{0}$ denotes a zero matrix with an appropriate size.

The steady-state covariance matrix of the system of Eq.33 is defined as follows:

$$\tilde{P} = \lim_{t \to \infty} \left\langle \begin{bmatrix} \tilde{x}_s(t) \\ \tilde{e}(t) \end{bmatrix} \begin{bmatrix} \tilde{x}_s(t)^T & \tilde{e}(t)^T \end{bmatrix} \right\rangle = \begin{bmatrix} \tilde{P}_s & \tilde{P}_{se} \\ \tilde{P}_{es} & \tilde{P}_e \end{bmatrix} \tag{34}$$

where $\tilde{P}_s$, $\tilde{P}_e$, $\tilde{P}_{se}$ and $\tilde{P}_{es}$ are covariance matrices of the form:

$$\tilde{P} = \lim_{t \to \infty} \left\langle \begin{bmatrix} \tilde{x}_s(t) \\ e(t) \end{bmatrix} \begin{bmatrix} \tilde{x}_s(t)^T & e(t)^T \end{bmatrix} \right\rangle = \begin{bmatrix} \tilde{P}_s & P_{se} \\ P_{es} & P_e \end{bmatrix} \tag{35}$$

$\tilde{P}$ is the unique positive-definite solution of the following Lyapunov equation [12]:

$$\begin{bmatrix} \Lambda_s + B_s G & -B_s G \\ \mathbf{0} & \Lambda_s - K C_s \end{bmatrix} \tilde{P} + \tilde{P} \begin{bmatrix} \Lambda_s + B_s G & -B_s G \\ \mathbf{0} & \Lambda_s - K C_s \end{bmatrix}^T + \begin{bmatrix} I_s & \mathbf{0} \\ I_s & -K \end{bmatrix} \begin{bmatrix} V_s & \mathbf{0} \\ \mathbf{0} & V_y \end{bmatrix} \begin{bmatrix} I_s & \mathbf{0} \\ I_s & -K \end{bmatrix}^T = \mathbf{0} \tag{36}$$

When the solution of $\tilde{P}$ is available, the surface covariance of the finite-dimensional system, $\widetilde{Cov_h}$, can be obtained by using only $\tilde{P}_s$.

C. Analysis of Closed-loop Infinite-dimensional System

We now proceed to characterize the accuracy with which the surface covariance in the closed-loop infinite-dimensional system is controlled by the finite-dimensional linear dynamic output feedback controller. By applying the controller of Eq.30 to the infinite-dimensional system of Eq.10 and substituting the estimation error in Eq.27, the infinite-dimensional closed-loop system takes the following form:

$$\begin{aligned} \frac{dx_s}{dt} &= (\Lambda_s + B_s G)x_s - B_s G e + \xi_s \\ \frac{de}{dt} &= (\Lambda_s - KC_s)e - KC_f x_f + \xi_s - K\xi_y \\ \varepsilon\frac{dx_f}{dt} &= \Lambda_{f\varepsilon} x_f + \varepsilon(B_f G x_s - B_f G e) + \varepsilon\xi_f \end{aligned} \quad (37)$$

where $\varepsilon = \frac{|\lambda_1|}{|\lambda_{m+1}|}$, and $\Lambda_{f\varepsilon} = \varepsilon\Lambda_f$ is an infinite-dimensional stable matrix.

The infinite-dimensional system of Eq.37 is then a singularly-perturbed system driven by white noise. We now proceed to characterize the accuracy with which the surface covariance is controlled in the closed-loop infinite-dimensional system. Theorem 1 provides a characterization of the surface covariance enforced by dynamic output feedback controller in the closed-loop infinite dimensional system. The proof of Theorem 1 can be found in the journal version of this paper [13] and is omitted here due to space limitation.

Theorem 1: Consider the surface covariance of the finite-dimensional system of Eq.32, $\widetilde{Cov}_h$

$$\tilde{P}_s = \lim_{t\to\infty}\langle \tilde{x}_s(t)\tilde{x}_s(t)^T\rangle, \quad \widetilde{Cov}_h = Tr\{\tilde{P}_s\} \quad (38)$$

and the surface covariance of the infinite-dimensional system of Eq.37, Cov_h

$$x = [\ x_s^T \quad x_f^T\ ]^T, \quad P = \lim_{t\to\infty}\langle x(t)x(t)^T\rangle, \quad Cov_h = Tr\{P\} \quad (39)$$

where $\langle\cdot\rangle$ denotes the expected value. Then, there exists $\varepsilon^* > 0$ such that if $\varepsilon \in (0, \varepsilon^*]$, $\widetilde{Cov}_h$ and Cov_h satisfy:

$$Cov_h = \widetilde{Cov}_h + O(\sqrt{\varepsilon}) \quad (40)$$

IV. Application to the Linearized Stochastic KSE

In this section, we present applications of the proposed linear output feedback covariance controller to the linearized stochastic KSE to demonstrate the effectiveness of the proposed output feedback covariance controllers. The stochastic KSE is a fourth-order nonlinear stochastic partial differential equation that describes the evolution of the height fluctuation for surfaces in a variety of material preparation processes including surface erosion by ion sputtering [10], [16], surface smoothing by energetic clusters [14] and ZrO_2 thin film growth by reactive ion beam sputtering [22]. The linearized stochastic KSE around the zero solution ($h(x,t) = 0$) takes the following form:

$$\begin{aligned} &\frac{\partial h}{\partial t} = -\frac{\partial^2 h}{\partial x^2} - \kappa\frac{\partial^4 h}{\partial x^4} + \sum_{i=1}^{p} b_i(x)u_i(t) + \xi(x,t) \\ &y(t) = \\ &\left[h(x_1,t) + \xi_y^1(t) \quad h(x_2,t) + \xi_y^2(t) \quad \cdots \quad h(x_q,t) + \xi_y^q(t)\right]^T \end{aligned} \quad (41)$$

subject to periodic boundary conditions:

$$\frac{\partial^j h}{\partial x^j}(-\pi,t) = \frac{\partial^j h}{\partial x^j}(\pi,t), \quad j = 0,\cdots,3 \quad (42)$$

and the initial condition $h(x,0) = h_0(x)$, where $x \in [-\pi,\pi]$ is the spatial coordinate and $\kappa > 0$ is the instability parameter of the stochastic KSE.

The eigenvalue problem of the linear operator of Eq.41 takes the form:

$$\begin{aligned} \mathscr{A}\bar{\phi}_n(x) &= -\frac{d^2\bar{\phi}_n(x)}{dx^2} - \kappa\frac{d^4\bar{\phi}_n(x)}{dx^4} = \lambda_n\bar{\phi}_n(x) \\ \frac{d^j\bar{\phi}_n}{dx^j}(-\pi) &= \frac{d^j\bar{\phi}_n}{dx^j}(+\pi); \quad j = 0,\cdots,3; \quad n = 1,\cdots,\infty \end{aligned} \quad (43)$$

A direct computation of the solution of the above eigenvalue problem yields $\lambda_0 = 0$ with $\psi_0 = 1/\sqrt{2\pi}$, and $\lambda_n = n^2 - \kappa n^4$ (λ_n is an eigenvalue of multiplicity two) with eigenfunctions $\phi_n = (1/\sqrt{\pi})\sin(nx)$ and $\psi_n = (1/\sqrt{\pi})\cos(nx)$ for $n = 1,\cdots,\infty$. Note that the $\bar{\phi}_n$ in the general eigenvalue problem formulation of Eq.4 denotes either ϕ_n or ψ_n. From the expression of the eigenvalues, it follows that for a fixed value of $\kappa > 0$, the number of unstable eigenvalues of the operator $\mathscr{A}$ in Eq.43 is finite and the distance between two consecutive eigenvalues (i.e. λ_n and λ_{n+1}) increases as n increases.

For $0 < \kappa < 1$, the operator of Eq.4 possesses unstable eigenvalues. Thus, the zero solution of the open-loop system of Eq.41 is unstable, which implies that the surface covariance increases with time due to the open-loop instability of the zero solution. An appropriately designed feedback controller is necessary to regulate the surface covariance to a desired value.

Using modal decomposition, the linearized stochastic KSE is formulated into an infinite-dimensional stochastic ODE system as follows:

$$\begin{aligned} \frac{d\alpha_n}{dt} &= (n^2 - \kappa n^4)\alpha_n + \sum_{i=1}^{p} b_{i\alpha_n}u_i(t) + \xi_\alpha^n(t) \quad & n = 1,\dots,\infty \\ \frac{d\beta_n}{dt} &= (n^2 - \kappa n^4)\beta_n + \sum_{i=1}^{p} b_{i\beta_n}u_i(t) + \xi_\beta^n(t) \quad & n = 0,1,\dots,\infty \end{aligned} \quad (44)$$

A finite-dimensional approximation of Eq.44 can be then derived by neglecting the fast modes (i.e., modes of order $m+1$ and higher) and a system of the form of Eq.13 is obtained for covariance controller design.

A linear state feedback controller is initially designed on the basis of the finite-dimensional approximation by following the method proposed in [18], which takes the following form:

$$u = B_s^{-1}(\Lambda_{cs} - \Lambda_s)\tilde{x}_s \quad (45)$$

where the matrix Λ_{cs} contains the desired poles of the closed-loop system; $\Lambda_{cs} = diag[\lambda_{c\beta 0}\ \lambda_{c\alpha 1}\ \cdots\ \lambda_{c\alpha m}\ \lambda_{c\beta 1}\ \cdots\ \lambda_{c\beta m}]$. $\lambda_{c\beta 0}$, $\lambda_{c\alpha i}$ and $\lambda_{c\beta i}$ $(i = 1, \cdots, m)$ are desired poles of the closed-loop finite-dimensional system, which satisfy $Re\{\lambda_{c\alpha i}\} < 0$ for $i = 1, \cdots, m$ and $Re\{\lambda_{c\beta i}\} < 0$ for $i = 0, 1, \cdots, m$.

To simplify the development, we assume that $p = 2m+1$ (i.e., the number of control actuators is equal to the dimension of the finite dimensional system) and pick the actuator distribution functions, $b_i(x)$, in the following form:

$$b_i(x) = \begin{cases} 1/\sqrt{2\pi}; & i = 1 \\ (1/\sqrt{\pi})\sin[(i-1)x]; & i = 2, \cdots, m+1 \\ (1/\sqrt{\pi})\cos[(i-m-1)x]; & i = m+2, \cdots, 2m+1 \end{cases} \quad (46)$$

Note that the actuator distribution functions are selected such that B_s^{-1} exists. The following parameters are used in the simulation:

$$\kappa = 0.1 \quad \sigma = 0.1 \quad m = 5 \quad (47)$$

We design the linear state feedback controller such that all the desired poles in Λ_{cs} are equal to -10.0. The surface covariance of the infinite-dimensional system under the state feedback controller is 0.55. The method to determine the values of the closed-loop poles to regulate the surface covariance to a set-point value can be found in [18] and is omitted here for brevity.

Eleven measurement sensors are used and are evenly placed on the spatial domain $[-\pi, \pi]$. A perfect initial surface is assumed and zero initial state estimates are used for all simulations.

$$h_0(x) = 0 \quad x_s(0) = \hat{x}_s(0) = \mathbf{0} \quad x_f(0) = \mathbf{0} \quad (48)$$

A 50^{th} order stochastic ODE approximation of Eq.41 is used to simulate the process (a higher-order approximation leads to identical numerical results). The Dirac delta function involved in the covariances of ξ_α^n and ξ_β^n is approximated by $\frac{1}{\Delta t}$, where Δt is the integration time step. Since it is a stochastic process, the surface covariance profile is obtained by averaging the results of 1000 independent simulation runs using the same parameters.

In the closed-loop simulation under linear dynamic output feedback control, a Kalman-Bucy filter is designed to estimate the state of the finite-dimensional system. The gain matrix K is obtained from the solution of the algebraic Riccati equation of Eqs.24 and 28. $\widetilde{Cov}_h$ is the surface covariance of the closed-loop finite-dimensional system under the finite-dimensional output feedback covariance controller and is the solution of the Lyapunov equation of Eq.36. According to Theorem 1, $\widetilde{Cov}_h$ is an $O(\sqrt{\varepsilon})$ approximation of the closed-loop surface covariance of the infinite-dimensional system, Cov_h, i.e., the closed-loop surface covariance of the infinite-dimensional system is an $O(\sqrt{\varepsilon})$ approximation of the desired value. To regulate the surface covariance to a desired value, the ε should be sufficiently small, which can be achieved by appropriately selecting the size of the finite-dimensional approximation used for covariance controller design. In this design, when $m = 5$, $\varepsilon = 0.01$, which is a sufficiently small number compared to the desired closed-loop surface covariance.

Since we use 11 measurement sensors, $q = 2m+1$ and the observer gain matrix is a square matrix. The desired surface covariance is 1.1347. The gain matrices for both the state observer, K, and the state feedback control law, G, are determined based on this desired surface covariance Note that because of the existence of the sensor noise, the surface covariance under the output feedback covariance controller is higher than the one under state feedback control where the same gain matrix, G is used and the full state of the surface is accessible. The closed-loop simulation result under the dynamic output feedback controller with 11 measurement sensors is shown in Fig.1. The controller successfully drives the surface covariance of the closed-loop infinite-dimensional system to a level which is within the range of the theoretical estimate of Theorem 1, i.e., $\sqrt{\varepsilon} \simeq 0.1$ and $Cov_h = \widetilde{Cov}_h + O(0.1)$. The result shown in Fig.1 also confirms that the surface covariance contribution from the x_f subsystem is negligible and that the contribution from the x_s subsystem is dominant. Therefore, the design of the output feedback covariance controller based on the x_s subsystem can regulate the surface covariance of the infinite-dimensional closed-loop system to the desired level.

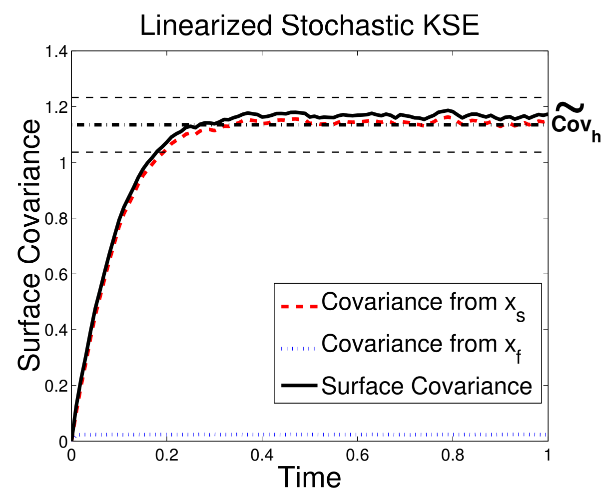

Fig. 1. The closed-loop surface covariance under linear dynamic output feedback control using 11 measurement sensors. The horizontal dashed lines represent the range in which the surface covariance Cov_h is expected to be based on the theoretical estimates of Theorem 1.

For dynamic output feedback control design, the number of the measurements is not needed to be equal to the dimension of the finite-dimensional system. A number of measurement sensors that is larger than the dimension of the finite-dimensional system results in a more accurate state estimation from the Kalman-Bucy filter. Therefore, the closed-loop surface covariance can be closer to the set-point value compared to the one in which the number of measurement sensors is equal to the dimension of the finite-dimensional system. On the other hand, when the number of the measurement sensors is smaller than the dimension of the finite-dimensional system but is equal to or larger than the number of unstable modes of the system, it is still possible to design a stable Kalman-Bucy filter for state estimation. Fig.2 shows the comparison of closed-loop simulation results when different numbers of measurement sensors are used for state estimation. The feedback control law is the same for all simulations. Specifically, Fig.2 shows results from three closed-loop simulation runs with 7, 11 and 15 measurement sensors. It is clear that the control system which uses a larger number of measurement sensors is capable to control the surface covariance to a lower level. On the other hand, since the dimension of the finite-dimensional system is 11, it is possible to stabilize the surface covariance to a finite value when the number of measurement sensors is smaller than the dimension of the finite-dimensional system.

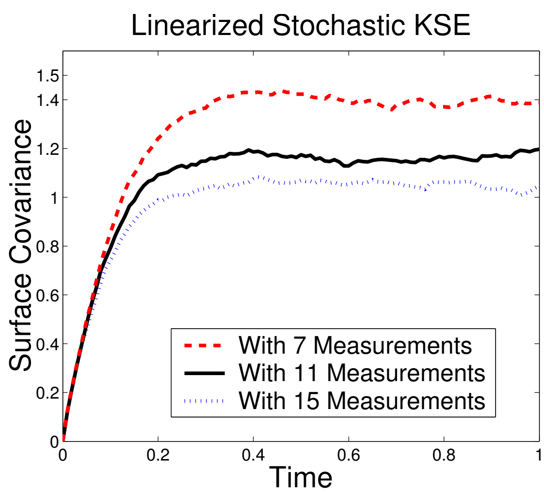

Fig. 2. Comparison of the surface covariance under linear dynamic output feedback controllers with 7, 11 and 15 measurement sensors.

However, there is a minimum number of measurement sensors required by the dynamic output feedback controller to stabilize the system. In this study, a minimum of 7 measurement sensors are required. When the number of measurement sensors is fewer than the minimum number, 7, the output feedback controller cannot stabilize the closed-loop system. The results for a number of measurement sensors less than the minimum can be found in the journal version of this paper and thus omitted here due to the page limit [13].

V. Conclusions

In this work, we developed a method for dynamic output feedback covariance control of the state covariance of linear dissipative stochastic PDEs using spatially distributed control actuation and sensing with measurement noise. Application of the linear dynamic output feedback controller to the linearized stochastic Kuramoto-Sivashinsky equations was presented.

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