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# On the Computation of an Upper Bound on the Gap Metric for a Class of Nonlinear Systems

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***Abstract*— This work deals with the computation of upper bounds on the gap metric and the corresponding stability margin. The suggested bounds can be computed for a class of a nonlinear systems which satisfy an inequality. Comparing to previous works, where results are highly dependent on the studied cases, our methods are applicable to a wider range of nonlinear systems. The results are based on two inequalities derived for the gap metric and the stability margin with respect to the gain of the relevant systems. An example is provided to illustrate the derived bounds for both our method and a previous method that is based on the direct computation.**

***Index Terms*— Nonlinear systems, the gap metric, the stability margin.**

## I. Introduction

Model uncertainty often has a significant effect on stability and performance of feedback control systems. For linear time-invariant (LTI) systems, much work has been done to study this effect. One important concept used to measure system uncertainty is the gap metric which was introduced to systems and control theory by Zames and El-Sakkary [10]. For LTI systems, it has been shown that a perturbed system can be stabilized by any controller which is designed for the nominal system if and only if the distance between the perturbed system and the nominal system is small in the gap metric. The computation of the gap metric for LTI system was developed by T. T. Georgiou [4].

The extension of the gap metric to larger classes of systems was initiated in [5], where the metric is extended to time-varying linear plants. Later, the parallel projection operator for nonlinear systems [2] and its relationship to the differential stabilizability of nonlinear feedback systems [3] paved the road to the extension of the gap metric to a pseudo-metric on nonlinear operators, [6].

Unlike the LTI system case, there is no generally applicable method of computing the gap metric for nonlinear systems. In fact, there are only a few examples in literature for the computation of the gap metric. Moreover, those methods are highly dependent upon the case of interest. This is also the case for the corresponding stability margin which can be used to determine the ball of uncertainty in the sense of the gap metric.

This paper deals with the computation of the gap metric and stability margin for nonlinear systems. We will consider the extension of the gap metric to nonlinear systems given in [6]. We derive upper bounds on the gap metric and the stability margin with respect to the operator norm (gain) of the plant , perturbed system and controller and based on our earlier work [9] on the upper bound of the gain of nonlinear systems, we find the upper bounds. The suggested methods are only applicable to a class of nonlinear systems which satisfy an inequality.

The paper is organized as follows: In Section II, first, we introduce the notation and present some preliminaries results. Then, we study a representation for unforced nonlinear systems, called the $\zeta_A$ representation [8]. This representation can be used to derive upper bounds on the gain of nonlinear systems [9]. In Section III, the gap metric for the nonlinear systems is introduced.

The main contribution of this paper is contained in Section IV where Theorems 4.1 and 4.2 are stated and proved. These theorems provide upper bounds on the gap metric and the stability margin, respectively. In Section IV, an example is also solved to illustrate the effectiveness of the results and comparison between the direct computation and the suggested methods. Since the literature suffers from the lack of widely-applicable computation methods and there are just a few examples which are highly dependent to the studied systems, it is indeed hard to construct example which both satisfies our required condition and is compatible by the previously suggested methods such as the method used in [6].

## II. Notation and Preliminaries

### A. Notation

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. For $x \in \mathbb{R}$, $\operatorname{sgn}(x)$ is 1 if $x > 0$, $-1$ if $x < 0$ and 0 otherwise. $\mathbb{R}^n$ denotes the space of $n \times 1$ real vectors. The Euclidean norm in $\mathbb{R}^n$ is denoted by $\|\cdot\|$. $I_{n\times n}$ denotes the $n \times n$ identity matrix. $\mathcal{L}_p^r$ denotes Lebesgue $p$-space of $r$-vector valued functions on $[0, \infty]$, with norm defined as $\|f\|_p := \left(\int_0^\infty \|f(t)\|^p \, dt\right)^{1/p}$ for $1 \le p \le \infty$ and $\|f\|_\infty := \operatorname{ess\,sup}_{t\in\mathbb{R}} \|f(t)\|$. Usually $r$ is a finite integer; we drop $r$ and write $\mathcal{L}_p$ instead of $\mathcal{L}_p^r$. Let $\mathcal{L}$ denote $\mathcal{L}_p$ for any $0 < p \le \infty$. Let $\mathbf{T}_\tau$ denote the Truncation operator: for $f(t)$, $0 \le t < \infty$, $\mathbf{T}_\tau f(t) = f(t)$ on $[0, \tau]$, and zero otherwise. We also denote the truncation of $f(t)$ by $f_\tau(t) := \mathbf{T}_\tau f(t)$. Let $\|f(t)\|_\tau$ denote $\|\mathbf{T}_\tau f\|$.

Let $\mathcal{U} := \mathcal{L}$ and $\mathcal{Y} := \mathcal{L}$ denote input and output signal spaces, respectively. A nonlinear time-varying system can be thought of as a possibly unbounded operator $H : \mathcal{D}_h \to \mathcal{Y}$ where $\mathcal{D}_h \subseteq \mathcal{U}$. The action of $H$ on any $u \in \mathcal{D}_h$ is denoted by $Hu$. A system $H$ is called *stable* if $\mathcal{D}_h = \mathcal{U}$. For an operator $H : \mathcal{U} \to \mathcal{Y}$, let $\gamma(H)$ stand for the induced norm (gain) of the operator defined as

$$\gamma(H) := \sup_{\substack{u \in \mathcal{U} \\ u \neq 0}} \frac{\|Hu\|_T}{\|u\|_T} \tag{1}$$

where the supremum is taken over all $u \in \mathcal{U}$ and all $T$ in $\mathbb{R}^+$ for which $u_T \neq 0$. Let $\gamma_p(H)$ stand for $\gamma(H)$ in $\mathcal{L}_p$. A system $H$ is called *finite gain stable (fg-stable)* if $H0 = 0$ and $\gamma(H) < \infty$.

### *B. $\zeta_A$ and $\zeta_{AB}$ Representations*

Our proposed method to compute the upper bounds requires the gain of the relevant nonlinear systems. We will use the $\zeta_A$ representation of nonlinear systems, which have recently been introduced in [8], to compute the required gains [9]. In this section, we briefly explain the $\zeta_A$ and $\zeta_{AB}$ representations; for more details, see [8] and [9].

Assume that the nonlinear system of interest, $N$, is

$$N: \ \dot{x}(t) = f(t, x(t)) \tag{2}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz [7]. We also assume that the initial condition of the system is finite. Let $A \in \mathbb{R}^{n \times n}$. Define

$$\begin{aligned} &\Phi(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n \\ &\Phi(t, x) := f(t, x) - Ax \end{aligned} \tag{3}$$

$$\Gamma : \mathcal{L}_p \to \mathcal{L}_p, \quad \Gamma(z(t)) := \int_0^t e^{A(t-\tau)} z(\tau)\, d\tau, \tag{4a}$$

and

$$\Omega : \mathcal{L}_p \to \mathcal{L}_p, \quad \Omega(x(t)) := e^{At} x(0) \tag{4b}$$

The nonlinear system is equivalent to the structure represented in Fig. 1. This representation of the nonlinear system is called the $\zeta_A$ representation with ordered operator set $[\Phi, \Gamma, \Omega]$ [8].

For forced nonlinear systems, suppose that the system of interest is

$$N: \ \dot{x}(t) = f(t, x(t), u(t)) \tag{5}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz. Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Define

$$\Phi(x, u, t) := f(t, x, u) - Ax - Bu \tag{6}$$

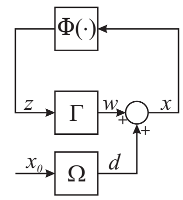


Fig. 1: $\zeta_A$ representation

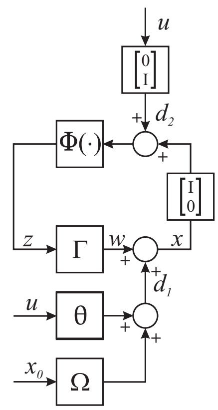


Fig. 2: $\zeta_{AB}$ representation

Let

$$\Theta : \mathcal{L}_p \to \mathcal{L}_p, \quad \Theta(u(t)) := \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau, \tag{7}$$

and $\Gamma$ and $\Omega$ be defined in the same formulas as in (4). The nonlinear system is equivalent to the structure represented in Fig. 2. This representation of the nonlinear system is called the $\zeta_{AB}$ representation with ordered operator set $[\Phi, \Gamma, \Omega]$ [8].

It is important to note that $\left[\begin{array}{c|c} A & I \\ \hline I & 0 \end{array}\right]$ and $\left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right]$ are state-space realizations for $\Gamma$ and $\Theta$, respectively. Since $A$ and $B$ are chosen arbitrary, $\zeta_A$ and $\zeta_{AB}$ representations are not unique. A useful choice for the $\zeta_{AB}$ representation is $B = 0$, which implies $\theta = 0$ and simplifies the $\zeta_{AB}$ structure as the structure shown in Fig. 3. For forced systems, this representation is also called $\zeta_A$ representation.

The following theorems taken from [9] provide upper bounds on the gain of nonlinear systems based on $\zeta_A$ representation.

*Theorem 2.1:* Let $[\Phi, \Gamma, \Omega]$ be a $\zeta_A$ representation for a forced system, $N$. If

$$\gamma_p(\Gamma)\gamma_p(\Phi) < 1 \tag{8}$$

then

$$\gamma_p(N) \le \frac{\gamma_p(\Gamma)\gamma_p(\Phi)}{1 - \gamma_p(\Gamma)\gamma_p(\Phi)} \tag{9}$$

*Proof:* See [9]. ■

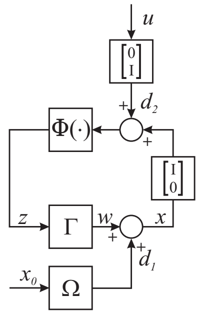


Fig. 3: The $\zeta_A$ representation for forced systems

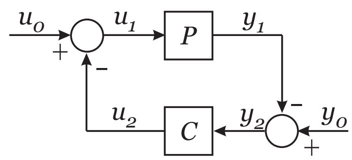


Fig. 4: The standard feedback configuration, $[P, C]$

*Theorem 2.2:* Let $[\Phi, \Gamma, \Omega]$ be a $\zeta_A$ representation for a forced system, $N$. If $\gamma_2(\Gamma)\gamma_2(\Phi) < 1$ then

$$\gamma_2(N) \leq \frac{\gamma_2(\Gamma)\gamma_2(\Phi)}{\sqrt{1 - \gamma_2(\Gamma)^2\gamma_2(\Phi)^2}} \tag{10}$$

*Proof:* See [9]. ■

## III. The Gap Metric

Let $[P, C]$ denote the feedback configuration shown in Figure 4. This configuration is standard in literature, e.g. [6] and can be described by the following equations.

$$\begin{aligned} y_1 &= Pu_1 \\ u_2 &= Cy_2 \\ u_0 &= u_1 + u_2 \\ y_0 &= y_1 + y_2 \end{aligned} \tag{11}$$

where $P$ and $C$ denote the nominal plant and the controller and $u_0$ and $y_0$ are the input and measurement disturbances, respectively. Let $u_i \in \mathcal{U}$, $y_i \in \mathcal{Y}$ and $w_i := \left[\begin{smallmatrix} u_i \\ y_i \end{smallmatrix}\right]$ for $i \in \{0, 1, 2\}$ and $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$. We assume that the product of the instantaneous gains of $P$ and $C$ is less than one. This assumption guarantees the well-posedness of the feedback configuration, e.g. [6] [1]. Similar to [6], we assume that the feedback configuration is always well-posed. The closed-loop operator is defined as

$$H_{P,C} : \mathcal{W} \to \mathcal{W} \times \mathcal{W}\,,\ H_{P,C} : w_0 \mapsto (w_1, w_2). \tag{12}$$

The graph of the plant is

$$\mathcal{G}_P = \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \,:\, u \in \mathcal{U},\, Pu \in \mathcal{Y} \right\} \subset \mathcal{W}. \tag{13}$$

If the domain of $P$ is $U$, the condition $Pu \in \mathcal{Y}$ is unnecessary. To have compatible notation with [6], we define the graph of $C$ as follows

$$\mathcal{G}_C = \left\{ \begin{pmatrix} Cy \\ y \end{pmatrix} \,:\, Cy \in \mathcal{U},\, y \in \mathcal{Y} \right\} \subset \mathcal{W}. \tag{14}$$

In some literature, e.g [2], this graph is also called inverse graph. Let

$$\mathcal{M} := \mathcal{G}_P\,,\ \mathcal{N} := \mathcal{G}_C. \tag{15}$$

The following operators are useful in the study of the closed-loop system stability.

$$\Pi_{\mathcal{M}||\mathcal{N}} := \Pi_1 H_{P,C}\,,\ \Pi_{\mathcal{N}||\mathcal{M}} := \Pi_2 H_{P,C} \tag{16}$$

where $\Pi_i : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ denote the natural projection onto the $i$th component ($i \in \{1, 2\}$) of $\mathcal{W} \times \mathcal{W}$. Therefore

$$\begin{aligned} \Pi_{\mathcal{M}||\mathcal{N}} &: w_0 \mapsto w_1 \\ \Pi_{\mathcal{N}||\mathcal{M}} &: w_0 \mapsto w_2 \end{aligned} \tag{17}$$

*Definition 3.1: Parallel Projection* [2]
A stable operator $\Pi : \mathcal{L} \to \mathcal{L}$ (with $\Pi 0 = 0$) is called a parallel projection if for any $x_1, x_2 \in \mathcal{L}$

$$\Pi(\Pi x_1 + (I - \Pi)x_2) = \Pi x_1 \tag{18}$$

where $I$ denotes the identity on $\mathcal{L}$.
Thus, $\Pi_{\mathcal{M}||\mathcal{N}}$ and $\Pi_{\mathcal{N}||\mathcal{M}}$ are parallel projections considering that for any $w_1, w_2 \in \mathcal{W}$

$$\Pi(\Pi w_1 + (I - \Pi)w_2) = \Pi w_1, \tag{19}$$

for $\Pi \in \left\{\Pi_{\mathcal{M}||\mathcal{N}}, \Pi_{\mathcal{N}||\mathcal{M}}\right\}$.

Consider the *summation operator*

$$\Sigma_{\mathcal{M},\mathcal{N}} : \mathcal{M} \times \mathcal{N} \to \mathcal{W} : (m, n) \mapsto m + n. \tag{20}$$

The stability of the standard feedback interconnection, Fig. 4, is equivalent to $\Sigma_{\mathcal{M},\mathcal{N}}$ having an inverse defined on the whole of $\mathcal{W}$ which is bounded. In fact, if $\Sigma_{\mathcal{M},\mathcal{N}}$ has a bounded inverse, then $\Sigma^{-1}_{\mathcal{M},\mathcal{N}} = H_{P,C}$. It can be shown that a necessary condition for $[P, C]$ to be stable is that $\mathcal{M}$ and $\mathcal{N}$ are closed subsets of $\mathcal{W}$ [2]. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be closed subsets of a Banach space $\mathcal{W}$. We define

$$\vec{\delta}(\mathcal{W}_1, \mathcal{W}_2) := \begin{cases} \inf\{\|(\mathcal{T} - I)|_{\mathcal{W}_1}\|\}, & \mathcal{T} \text{ is a causal} \\ & \text{bijective map from } \mathcal{W}_1 \text{ to } \mathcal{W}_2 \\ & \text{with } \mathcal{T}0 = 0, \\ \infty, & \text{if no such operator } \mathcal{T} \text{ exists,} \end{cases}$$

$$\delta(\mathcal{W}_1, \mathcal{W}_2) = \max\left\{\vec{\delta}(\mathcal{W}_1, \mathcal{W}_2), \vec{\delta}(\mathcal{W}_2, \mathcal{W}_1)\right\}. \tag{21}$$

*Theorem 3.1:* Consider the feedback system shown in Fig. 4. Let $\mathcal{M} := \mathcal{G}_P$ and $\mathcal{N} := \mathcal{G}_C$. Assume that $[P, C]$ is fg-stable. Suppose that $P$ is perturbed to $P_1$ and $\mathcal{M}_1 := \mathcal{G}_{P1}$. If

$$\vec{\delta}(\mathcal{M}, \mathcal{M}_1) < \left\|\Pi_{\mathcal{M}||\mathcal{N}}\right\|^{-1} \tag{22}$$

then $[P_1, C]$ is fg-stable. Furthermore

$$\|\Pi_{\mathcal{M}_1||\mathcal{N}}\| < \|\Pi_{\mathcal{M}||\mathcal{N}}\| \frac{1+\vec{\delta}(\mathcal{M},\mathcal{M}_1)}{1-\|\Pi_{\mathcal{M}||\mathcal{N}}\|\vec{\delta}(\mathcal{M},\mathcal{M}_1)} \quad (23)$$

*Proof:* See [6]. ■

## IV. Upper bounds on the Gap Metric and the stability margin

In this section, we suggest a method to find an upper bound on the gap metric between two nonlinear systems as well as a method to compute an upper bound on $\Pi_{\mathcal{M}||\mathcal{N}}$.

*Theorem 4.1:* Consider nonlinear dynamical systems given by

$$\begin{aligned} N:\ \dot{x} &= f(x,u),\ x_0 = 0; \\ \hat{N}:\ \dot{\hat{x}} &= \hat{f}(\hat{x},u),\ \hat{x}_0 = 0. \end{aligned} \quad (24)$$

Let $\gamma(N)$ and $\gamma(\hat{N})$ denote their gain respectively. Then

$$\delta(N,\hat{N}) \leq \gamma(N) + \gamma(\hat{N}). \quad (25)$$

*Proof:* We have

$$\begin{aligned} \|x-\hat{x}\| &\leq \|x\| + \|\hat{x}\| \\ &\leq \gamma(N)\|u\| + \gamma(\hat{N})\|u\| \\ &\leq \big(\gamma(N)+\gamma(\hat{N})\big)\|u\| \\ &\leq \big(\gamma(N)+\gamma(\hat{N})\big)\left\|\begin{bmatrix}u\\x\end{bmatrix}\right\| \end{aligned} \quad (26)$$

Define $\mathcal{T}$ as

$$\mathcal{T}\begin{bmatrix}u\\x\end{bmatrix} := \begin{bmatrix}u\\\hat{x}\end{bmatrix}. \quad (27)$$

■

It is trivial that $\mathcal{T}$ is bijective. We have

$$\begin{aligned} \vec{\delta}(N,\hat{N}) &= \|I-\mathcal{T}\| \\ &= \sup \frac{\|(I-\mathcal{T})\left[\begin{smallmatrix}u\\x\end{smallmatrix}\right]\|}{\|\left[\begin{smallmatrix}u\\x\end{smallmatrix}\right]\|} \\ &= \sup \frac{\left\|\left[\begin{smallmatrix}u-u\\x-\hat{x}\end{smallmatrix}\right]\right\|}{\|\left[\begin{smallmatrix}u\\x\end{smallmatrix}\right]\|} \\ &= \sup \frac{\|x-\hat{x}\|}{\|\left[\begin{smallmatrix}u\\x\end{smallmatrix}\right]\|} \\ &\leq \gamma(N)+\gamma(\hat{N}) \qquad \text{using (26)} \end{aligned} \quad (28)$$

Similarly

$$\vec{\delta}(\hat{N},N) \leq \gamma(N)+\gamma(\hat{N}) \quad (29)$$

Consequently,

$$\begin{aligned} \delta(N,\hat{N}) &= \max\{\vec{\gamma}(N,\hat{N}),\vec{\gamma}(\hat{N},N)\} \\ &\leq \delta(N)+\delta(\hat{N}). \end{aligned} \quad (30)$$

*Theorem 4.2:* Consider the standard feedback configuration depicted in Fig. 4. Suppose that $\gamma(P)\gamma(C) < 1$. Let $\Pi_{\mathcal{M}||\mathcal{N}}$ be defined as 15 and (16). Then

$$\|\Pi_{\mathcal{M}||\mathcal{N}}\| \leq \frac{\big(1+\gamma(P)\big)\big(1+\gamma(C)\big)}{1-\gamma(P)\gamma(C)} \quad (31)$$

*Proof:* From the feedback configuration, we have

$$\begin{aligned} \|u_1\| &\leq \|u_0\| + \gamma(C)\|y_0-y_1\| \\ &\leq \|u_0\| + \gamma(C)\|y_0\| + \gamma(C)\gamma(P)\|u_1\| \end{aligned} \quad (32)$$

Consequently

$$\|u_1\| \leq \frac{1}{1-\gamma(C)\gamma(P)}\|u_0\| + \frac{\gamma(C)}{1-\gamma(C)\gamma(P)}\|y_0\| \quad (33)$$

Therefore

$$\begin{aligned} \left\|\begin{bmatrix}u_1\\y_1\end{bmatrix}\right\| &\leq \|u_1\| + \|y_1\| \\ &\leq \|u_1\| + \gamma(P)\|u_1\| \\ &\leq \frac{1+\gamma(P)}{1-\gamma(C)\gamma(P)}\|u_0\| + \frac{\gamma(C)\big(1+\gamma(P)\big)}{1-\gamma(C)\gamma(P)}\|y_0\| \end{aligned} \quad (34)$$

Since $\|a\| \leq \left\|\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\right\|$,

$$\begin{aligned} \left\|\begin{bmatrix}u_1\\y_1\end{bmatrix}\right\| &\leq \frac{1+\gamma(P)+\gamma(C)\big(1+\gamma(P)\big)}{1-\gamma(C)\gamma(P)}\left\|\begin{bmatrix}u_0\\y_0\end{bmatrix}\right\| \\ &= \frac{\big(1+\gamma(P)\big)\big(1+\gamma(C)\big)}{1-\gamma(C)\gamma(P)}\left\|\begin{bmatrix}u_0\\y_0\end{bmatrix}\right\|. \end{aligned} \quad (35)$$

On the other hand, Equation (17) implies

$$\Pi_{\mathcal{M}||\mathcal{N}}\begin{bmatrix}u_0\\y_0\end{bmatrix} = \begin{bmatrix}u_1\\y_1\end{bmatrix}. \quad (36)$$

Thus

$$\|\Pi_{\mathcal{M}||\mathcal{N}}\| = \sup_{\left\|\left[\begin{smallmatrix}u_0\\y_0\end{smallmatrix}\right]\right\|\neq 0} \frac{\left\|\left[\begin{smallmatrix}u_1\\y_1\end{smallmatrix}\right]\right\|}{\left\|\left[\begin{smallmatrix}u_0\\y_0\end{smallmatrix}\right]\right\|}. \quad (37)$$

Using (35)

$$\|\Pi_{\mathcal{M}||\mathcal{N}}\| \leq \frac{\big(1+\gamma(P)\big)\big(1+\gamma(C)\big)}{1-\gamma(C)\gamma(P)}. \quad (38)$$

■

*Example 4.1:* Consider the feedback configuration of Fig. 4. Assume that the plant is the circuit shown in Fig. 5, where the inductance of the SSR is nonlinear and $L(\cdot)$ is defined as Fig. 6 and $R = 10$. The state equation of the system is

$$\begin{aligned} \dot{x}(t) &= L^{-1}(u_1(t) - Rx(t)),\ x(0) = 0 \\ y_1(t) &= x(t) \end{aligned} \quad (39)$$

where $x(t) := i_L(t)$ and $u_1(t) := V_s(t)$. Let $C = -c$ where $c$ is a positive non-zero constant. Let $\mathcal{U} = \mathcal{Y} = \mathcal{L}_\infty$. Since the instantaneous gains of $P$ and $C$ are zero and one, respectively, the loop is well-posed. First, we will find the

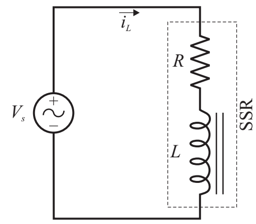


Fig. 5: $P$ in Example 4.1

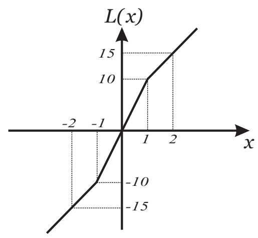


Fig. 6: Inductance of SSR

$\left\|\Pi_{\mathcal{M}||\mathcal{N}}\right\|$ by a direct method similar to the solution of Example 1 in [6]. Then, we will compute the upper bound on $\left\|\Pi_{\mathcal{M}||\mathcal{N}}\right\|$ by the suggested method.

*I. Direct computation:*

The feedback equation is

$$\dot{x} = L^{-1}(u_0 + cy_0 - (10+c)x), \; x(0) = 0. \tag{40}$$

We have

$$\Pi_{\mathcal{M}||\mathcal{N}} : \begin{bmatrix} u_0 \\ y_0 \end{bmatrix} \mapsto \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} u_0 + cy_0 - cx \\ x \end{bmatrix} \tag{41}$$

Let $v_0 := u_0 + cy_0$. For any $v_0$, $u_0 = y_0$ gives the mapping with the smallest input norm. Therefore, $v_0 = (1+c)u_0$ and

$$\begin{aligned} \left\|\Pi_{\mathcal{M}||\mathcal{N}}\right\| &= \left\| \begin{bmatrix} u_0 \\ y_0 \end{bmatrix} \mapsto \begin{bmatrix} u_0 + cy_0 - cx \\ x \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} u_0 \\ y_0 \end{bmatrix} \mapsto \begin{bmatrix} v_0 - cx \\ x \end{bmatrix} \right\| \\ &= (1+c) \left\| v_0 \mapsto \begin{bmatrix} v_0 - cx \\ x \end{bmatrix} \right\| \\ &= (1+c) \times \\ &\quad \max\{\left\|v_0 \mapsto \left(v_0 - cx\right)\right\|, \left\|v_0 \mapsto x\right\|\} \end{aligned} \tag{42}$$

We now show that $\|v_0 \mapsto x\| = {}^1/_{10+c}$. Suppose that for any arbitrary chosen interval $[0, T]$, the maximum of $x(t)$, which is positive, occurs at $t_0 \in [0, T]$. Then, for any $\epsilon > 0$, there exists $t_1$ such that $0 < t_1 < t_0$, $x(t_1) > x(t_0) - \epsilon$ and $\dot{x}(t_1) > 0$. Consequently, $L^{-1}\big(v_0(t_1) - (10 + c)x(t_1)\big) > 0$. Since $\operatorname{sgn}\big(L^{-1}(x)\big) = \operatorname{sgn}(x)$, $v_0(t_1) > (10+c)x(t_1)$. Thus, $v_0(t_1) > (10+c)x(t_0) - (1+c)\epsilon$ for any $\epsilon$. Similarly, if the minimum of $x(t)$ in $[0, T]$, which is negative, occurs at $\acute{t}_0$, for any $\acute{\epsilon} > 0$, there exists $\acute{t}_1$ such that $v_0(\acute{t}_1) < (10+c)x(\acute{t}_0) - (1+c)\acute{\epsilon}$. Consequently, $\|v_0\|_T \geq (10+c)\|x\|_T$. To show that this upper bound on $\|v_0 \mapsto x\|$ can be approached arbitrary closely, let $v_0 = 1$ for all $t$. It is trivial that $x(t) = \big(1 - e^{-(1+0.1c)t}\big)/(10+c)$. So $\|v_0\| = 1$ and $\|x\| = {}^1/_{10+c}$. Consequently, $\|v_0 \mapsto x\| = {}^1/_{10+c}$. Next, we compute $\left\|v_0 \mapsto \big(v_0 - cx\big)\right\|$. Trivially, $\left\|v_0 \mapsto \big(v_0 - cx\big)\right\| \leq 1 + \|v_0 \mapsto (cx)\| = 1 + \frac{c}{10+c}$. This upper bound can be approached arbitrarily closely by the input $v_0 = 1$ for $0 \leq t < T$ and $v_0 = -1$ for $t \geq T$. We have $x(t) = \big(1 - e^{-(1+0.1c)t}\big)/(10+c)$ for $0 \leq t < T$. Thus, $\big(v_0 - cx\big)(T) = -(1 + \frac{c}{10+c}) + e^{-(1+0.1c)T}$. Therefore, $\|v_0\| = 1$ and $\|v_0 - cx\| = 1 + \frac{c}{10+c}$ which implies that $\left\|v_0 \mapsto \big(v_0 - cx\big)\right\| = 1 + \frac{c}{10+c}$. Consequently, $\left\|\Pi_{\mathcal{M}||\mathcal{N}}\right\| = 1 + \frac{c}{10+c}$.

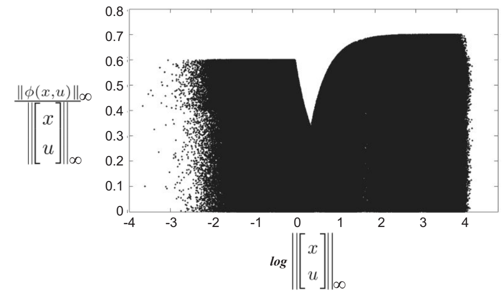


Fig. 7: $\frac{\|\Phi(x,u)\|}{\left\|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\right\|}$ versus $\log \left\|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\right\|$

*II. The suggested method:*

To find $\gamma(P)$, let $\Phi(x, u) = L^{-1}(u - 10x) + 3x/2$ and $\Gamma := \left[\begin{array}{c|c} -3/2 & 1 \\ \hline 1 & 0 \end{array}\right]$. We use the computational methods introduced in [8]. Fig. 7 shows the plot of $\frac{\|\Phi(x,u)\|}{\left\|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\right\|}$ versus $\left\|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\right\|$ for $2 \times 10^6$ randomly chosen input vector. Therefore, $\gamma(\Phi) = 0.7$. We have $\gamma(\Gamma) = 2/3$. Theorem 2.1 implies that

$$\gamma(P) \leq 0.639. \tag{43}$$

Since $C = -c$ is a constant, $\gamma(C) = c$. Theorem 4.2 implies that $\left\|\Pi_{\mathcal{M}||\mathcal{N}}\right\| \leq \frac{1.639(1+c)}{1-0.639\,c}$ if $c < 1.56$. Apparently, the obtained upper bound is closer to the actual value when $c$ approaches zero.

*Example 4.2:* Consider the plant introduced in the previous example. Suppose that the system is perturbed by time delay $h$. That is

$$P_1 : \begin{cases} \dot{x}(t) = L^{-1}(u_1(t) - Rx(t)), \; x(0) = 0 \\ y_1(t) = x(t-h) \end{cases} \tag{44}$$

First, we will compute an upper bound on the gap between the plant $P$ and the perturbation $P_1$ by a direct method similar to the solution of Example 1 in [6]. Then, we will compute the upper bound on the gap by the suggested method.

*I. Direct computation:*

Let $\mathcal{M}_1 := \mathcal{G}_{P_1}$ and define a mapping $\mathfrak{T} : \mathcal{M} \to \mathcal{M}_1$ as

$$\mathfrak{T}\begin{bmatrix} u_1(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ x(t-h) \end{bmatrix} \tag{45}$$

Thus

$$\begin{aligned} |x(t) - x(t-h)| &\le \sup_{\hat{t} \in [t-h,t]} |\dot{x}(\hat{t})| \cdot h \\ &\le \sup_{\hat{t} \in [t-h,t]} |L^{-1}\big(u(\hat{t}) - 10x(\hat{t})\big)| \cdot h \end{aligned} \tag{46}$$

Since $L^{-1}(\cdot)$ is an strictly increasing function,

$$\begin{aligned} &|x(t) - x(t-h)| \\ &\quad \le L^{-1}\big(\sup_{\hat{t} \in [t-h,t]} |u(\hat{t}) - 10x(\hat{t})|\big) \cdot h \\ &\quad \le L^{-1}\big(\sup_{\hat{t} \in [t-h,t]} |u(\hat{t})| + \sup_{\hat{t} \in [t-h,t]} |10x(\hat{t})|\big) \cdot h \\ &\quad \le L^{-1}\big(\sup_{\hat{t} \in [0,t]} |u(\hat{t})| + \sup_{\hat{t} \in [0,t]} |10x(\hat{t})|\big) \cdot h. \end{aligned} \tag{47}$$

Therefore

$$\begin{aligned} &\|x(t) - x(t-h)\|_\tau \\ &\quad \le \left\| L^{-1}\big(\sup_{\hat{t} \in [0,t]} |u(\hat{t})| + \sup_{\hat{t} \in [0,t]} |10x(\hat{t})|\big) \right\|_\tau \cdot h \\ &\quad \le L^{-1}\big(11 \max\{\|u\|_\tau, \|x\|_\tau\}\big) \cdot h \\ &\quad \le 2.2 \max\{\|u\|_\tau, \|x\|_\tau\} \cdot h. \end{aligned} \tag{48}$$

Hence

$$\|I - \mathfrak{T}\| = \sup_{\tau, \|u_1\|_\tau \neq 0} \frac{\|x(t) - x(t-h)\|_\tau}{\max\{\|u_1\|_\tau, \|x\|_\tau\}} \le 2.2\, h \tag{49}$$

Consequently, $\vec{\delta}(\mathcal{M}, \mathcal{M}_1) \le 2.2\, h$. On the other hand, let $u(t) = 1$ on $[0, h]$. It is Trivial that $(Pu)(t) = 0.1(1 - e^{-10t})$. For any $w \in \mathcal{M}_1$, we have $w_h = \left[\begin{smallmatrix} * \\ 0 \end{smallmatrix}\right]$ which is implied by the time delay in $P_1$. Therefore

$$\begin{aligned} \vec{\delta}(\mathcal{M}, \mathcal{M}_1) &= \sup_{u_1, y_1 \neq 0} \frac{\left\| (\mathfrak{T} - I)\left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right] \right\|}{\left\| \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right] \right\|} \\ &\ge \sup_{u_1, y_1 \neq 0} \frac{\left\| \left[\begin{smallmatrix} * \\ 0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} u_1 \\ Pu_1 \end{smallmatrix}\right] \right\|_h}{\max\{\|u_1\|_h, \|Pu_1\|_h\}} \\ &= \frac{\max\{\|* - u_1\|_h, \|Pu_1\|_h\}}{\max\{\|u_1\|_h, \|Pu_1\|_h\}} \\ &\ge \frac{\|Pu_1\|_h}{\max\{\|u_1\|_h, \|Pu_1\|_h\}} \\ &= 0.1(1 - e^{-10h}) \end{aligned} \tag{50}$$

Consequently

$$0.1(1 - e^{-10h}) \le \vec{\delta}(P, P_1) \le 2.2\, h \tag{51}$$

*II. The suggested method:*

Since $P$ is autonomous, $\gamma(P) = \gamma(P_1)$. Using Theorem 4.1, $\vec{\delta}(P, P_1) = 2\gamma(P)$. Using (43), $\vec{\delta}(P, P_1) \le 1.278$. It is clear that for $h > 0.58$ the suggested method provides smaller upper bound than the direct method.

## V. Conclusion

In this paper, we have considered the computation of the gap metric and the corresponding robust stability margin. Our results are applicable to a class of a nonlinear systems which satisfy a given inequality. The suggested methods have computational advantage compared to previous work in the sense that they are applicable to wider range of nonlinear systems. Our methods are based on two inequalities derived for the gap metric and the stability margin with respect to the gain of the relevant systems. An example is provided to illustrate the results.

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