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On the Input-Output Stability of Nonlinear Systems: Large Gain Theorem

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***Abstract*— In this paper, minimum gain of an operator is introduced. Moreover, some of its properties are presented. It is proved that the minimum gain of a strictly proper, stable, LTI system is zero. It is also shown that the minimum gain of an operator fails to satisfy the triangular inequality. Finally, the so-called large gain theorem is stated and a new stability condition for feedback interconnection of nonlinear systems is derived.**

***Index Terms*— Nonlinear systems, induced operator norms, continuous-time systems, discrete-time systems, small-gain theorem**

I. Introduction

One of the well-accepted and widely-used methods to study stability of systems is the input-output approach. It was initiated by Popov, Zames, and Sandberg, in the 1960s [8] [11] [9]. So far, it has been a fruitful area which has resulted in many of the recent developments in control theory, such as robust control and small-gain based nonlinear stabilization techniques. The input-output stability theory considers systems as mappings from an input space of functions into an output space. In this theory, the well-behaved input and output signals are considered as members of input and output spaces. Therefore, if the "well-behaved" inputs produce well-behaved outputs, the system is called stable.

The main contribution of the input-output stability theory in control theory is through the well-known small-gain theorem. In this context, the most notable contributions have also been made by Zames and Sandberg, e.g. [11] [9]. The small gain theorem says that the feedback loop will be stable if the loop gain is less than one. This simple rule has been a basis for numerous stabilization techniques such as nonlinear $\mathcal{H}_\infty$ control [3].

Stability of systems, in its various forms, still continues to inspire researchers. Motivated by the classical small gain theorem, "nonlinear gain" small gain theorems are discussed in literature as [5] [10] [4]. The notion of non-uniform in time robust global asymptotic output stability introduced in [6] for a wide class of systems. A small-gain theorem for a wide class of feedback systems is proposed in [7]. In [2], it is shown that for an open loop unstable system which is closed loop stable the gain must exceed one.

In this paper, the minimum gain of a system has been studied. Although it has been showed that the minimum gain is not a norm on space of operators, a new stability condition has been derived for feedback systems based on the minimum gains of the open-loop systems.

The paper is organized as follows. In Section II we introduce the notation and present some preliminary results. In Section III, the minimum gain of an operator is defined and some of its properties are derived. In Section IV, the large gain theorem is stated. An example is also provided to illustrate the usage of the theorem.

II. Notation and Preliminaries

A. Notation

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. RHP stands for the portion of the complex plane with positive real part. $\mathbb{R}^n$ denotes the space of $n\times 1$ real vectors. For $M \in \mathbb{R}^{n\times m}$, M^T is the transpose of M; $\underline{\sigma}(M)$ is the minimum singular value of M; $\sigma(M)$ is a singular value of M; $\overline{\sigma}(M)$ is the maximum singular value of M. The Euclidean norm in $\mathbb{R}^n$ is denoted by $\|\cdot\|$. $I_{n\times n}$ denotes the $n\times n$ identity matrix. Let $\mathbf{B}^p(c,r)$ denote the open ball with center c and radius r with norm p, i.e. $\mathbf{B}^p(c,r) := \{x| \ \|x-c\|_p < r\}$. $\mathcal{L}_p^r$ denotes Lebesgue p-space of r-vector valued functions on $[0,\infty]$, with norm defined as $\|f\|_p := \left(\int_0^\infty \|f(t)\|^p\right)^{1/p}$ for $1 \le p < \infty$ and $\|f\|_\infty := \operatorname{ess\,sup}_{t\in\mathbb{R}} \|f(t)\|$. Usually r is a finite integer; we drop r and write $\mathcal{L}_p$ instead of $\mathcal{L}_p^r$, if no confusion will arise. Similarly, let ℓ_p denote the vector space of discrete-time signals with norm $\|\cdot\|_p$. Let $\mathcal{X}_p$ denote either $\mathcal{L}_p$ or ℓ_p and $\mathcal{X}$ denote $\mathcal{X}_p$ for any $0 < p \le \infty$. To distinguish among various norm notation, we indicate the space as a subscript for the norm, such as $\|\cdot\|_{\mathbb{R}^n}$ or $\|\cdot\|_{\mathcal{X}_p}$. Whenever the space is not mentioned, norms with t argument denote Euclidean norm at t and without t denote the $\mathcal{X}_p$ norm where $\mathcal{X}_p$ is a general space or can clearly be understood from the content. Let $\mathbf{T}_\tau$ denote the truncation operator: for $f(t)\,,\ 0 \le t < \infty,\ \mathbf{T}_\tau f(t) = f(t)$ on $[0,\tau]$, and zero otherwise. We also denote the truncation of $f(t)$ by $f_T(t) := \mathbf{T}_\tau f(t)$.

Let $\mathcal{U} := \mathcal{X}$ and $\mathcal{Y} := \mathcal{X}$ denote input and output signal spaces, respectively. A nonlinear time-varying system can be thought of as a possibly unbounded operator $H : \mathcal{D}_h \rightarrow$

$\mathcal{Y}$ where $\mathcal{D}_h \subseteq \mathcal{U}$. The action of H on any $u \in \mathcal{D}_h$ is denoted by Hu. A system H is called to be stable if $\mathcal{D}_h = \mathcal{U}$. For an operator $H : \mathcal{U} \to \mathcal{Y}$, let $\gamma_p(H)$ stand for the induced norm (gain) of the operator defined as

$$\gamma_p(H) := \sup_{0 \neq u \in \mathcal{U}} \frac{\|Hu\|_T}{\|u\|_T} \tag{1}$$

where the supremum is taken over all $u \in \mathcal{U}$ and all T in $\mathbb{R}^+$ for which $u_T \neq 0$.

III. Minimum Gain of an Operator

Let $H : \mathcal{U} \to \mathcal{Y}$ denote an operator. We define the minimum gain of H as follows:

$$\nu(H) = \inf_{0 \neq u \in \mathcal{U}} \frac{\|(Hu)_T\|}{\|u_T\|} \tag{2}$$

where the infimum is taken over all $u \in \mathcal{U}$ and all T in $\mathbb{R}^+$ for which $u_T \neq 0$. It is trivial that the minimum gain of an operator is less or equal to its induced norm. It is also obvious that if a minimum gain of a system is infinite, then it is unstable. In other words, the minimum gain of a stable system is always finite. The converse is, however, not true.

Lemma 3.1: Let $M \in \mathbb{R}^{n \times n}$. Define $H : \mathcal{X}_2 \to \mathcal{X}_2$ as $Hx := Mx$, then

$$\nu(H) = \underline{\sigma}(M) \tag{3}$$

Proof: See Appendix I. ■

Lemma 3.2: Let $\Phi(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ ($\Phi(\cdot,\cdot) : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ in discrete time) and H be the operator defined as

$$H : \mathcal{X}_p \to \mathcal{X}_p \;; \quad Hx(t) := \Phi(t, x(t)) \tag{4}$$

Suppose there exists a constant μ_p such that

$$\mu_p \|x\|_p \leq \|\Phi(t,x)\|_p, \; \forall x \in \mathbb{R}^n, \; \forall t \geq 0 \tag{5}$$

then $\mu_p \leq \nu_p(H)$.

Proof: See Appendix II. ■

Example 3.1: Memory less Nonlinearities: Let $X = \mathcal{L}_\infty$, and consider nonlinear operators $H_1(u) = u^2$ and $H_2(.)$ defined by the graph in the plane shown in Fig. 1. We have

$$\nu(H_1) = \inf_{0 \neq u \in \mathcal{L}_\infty} \frac{\|(H_1 u)_T\|_{\mathcal{L}_\infty}}{\|u_T\|_{\mathcal{L}_\infty}} = \inf_{0 \neq u \in \mathcal{L}_\infty} |u| = 0 \tag{6}$$

The minimum gain $\nu(H_2)$ is easily determined from the slope of the graph of H_2.

$$\nu(H_2) = \inf_{0 \neq u \in \mathcal{L}_\infty} \frac{\|(H_2 u)_T\|_{\mathcal{L}_\infty}}{\|u_T\|_{\mathcal{L}_\infty}} = 0.5 \tag{7}$$

Lemma 3.3: Let $g(t)$ be the impulse response of a continuous-time, stable, LTI system. Let $G(s)$ denote the Laplace transform of $g(t)$. Furthermore, assume that there

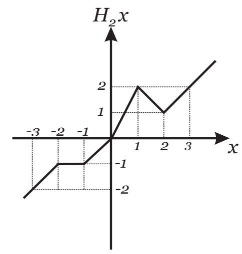


Fig. 1: H_2 in Example 3.1

exists a row in $G(s)$ where all elements are strictly proper, namely there is i such that for all j, $\lim_{s\to\infty} G_{ij}(s) = 0$. Let H stand for the convolution operator defined by

$$H(z(t)) = \int_0^t g(t-\tau) z(\tau) d\tau \tag{8}$$

We have

$$\nu(H) = 0 \tag{9}$$

Proof: See Appendix III. ■

Corollary 3.1: The minimum gain of a system with a strictly proper stable transfer function is zero.

Lemma 3.4: Let $g(t)$ be the impulse response of a continuous-time (discrete-time) LTI system. Let $G(s)\,(G(z))$ denote the Laplace transform (z-transform) of $g(t)$. Furthermore, assume that $G(s)\,(G(z))$ has at least one zero in the RHP (outside of the unit circle). Let H stand for the convolution operator defined by

$$H(z(t)) = \int_0^t g(t-\tau) z(\tau) d\tau \tag{10}$$

for continuous-time case and

$$H(z(t)) = \sum_{l=0}^{t} g(t-l) z(l) \tag{11}$$

for discrete-time one. We have

$$\nu(H) = 0 \tag{12}$$

Proof: See Appendix IV. ■

Lemma 3.5: Let $H : \mathcal{D}_h \subseteq \mathcal{U} \to \mathcal{Y}$ be a possibly unstable operator. Let $\mathcal{R}_h$ denote the range of H, namely $\mathcal{R}_h = \{y \in \mathcal{Y} : y = Hu \text{ for some } u \in \mathcal{D}_h\}$. Assume that H has a stable right inverse, i.e., there exists $H^{-1} : \mathcal{R}_h \to \mathcal{D}_h$ such that

$$H \cdot H^{-1} = I \tag{13}$$

and H^{-1} is stable. Moreover, assume that $\gamma(H^{-1}) < \infty$ Then

$$\nu(H) = \frac{1}{\gamma(H^{-1})} \tag{14}$$

Proof: See Appendix V. ■

Corollary 3.2: Unstable, bi-proper, LTI systems

1) Let $g(t)$ be the impulse response of a continues-time, unstable, bi-proper, LTI system. Let H stand for the convolution operator defined by

$$H(z(t)) = \int_0^t g(t-\tau)z(\tau)d\tau \tag{15}$$

Let $G(s)$ be the Laplace transform of $g(t)$. We have

$$\nu(H) = \left\|G^{-1}(s)\right\|_{\mathcal{H}_\infty}^{-1} \tag{16}$$

2) Let $g(t)$ be the impulse response of a discrete-time, unstable, strictly proper, LTI system. Let H denote the convolution operator defined by

$$H(z(t)) = \sum_{l=0}^{t} g(t-l)z(l) \tag{17}$$

Let $G(z)$ be the z-transform of $g(t)$. We have

$$\nu(H) = \left\|G^{-1}(z)\right\|_{\mathcal{H}_\infty}^{-1}. \tag{18}$$

Example 3.2: Let

$$G(s) = \frac{s+1}{s-1} \tag{19}$$

and $H : \mathcal{D}_h \subset \mathcal{L}_2 \to \mathcal{L}_2$ be an operator defined as (15). Equation (18) implies that

$$\nu(H) = \left\|G^{-1}(s)\right\|_{\mathcal{H}_\infty}^{-1} = 1. \tag{20}$$

For instance, let $u(t) := (1-2t)\,e^{-t}\,u_{-1}(t)$, where $u_{-1}(t)$ denotes the step function. We have $U(s) = \frac{s-1}{(s+1)^2}$ and consequently $Y(S) = \frac{1}{s+1}$ which shows that $y(t) = e^{-t}\,u_{-1}(t)$. This reveals that $\nu(H) \leq \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} = 1$. It is important to note that there is no input that satisfies $\frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} < 1$. This can be shown by contradiction. Assume there exists some input $\hat{u} \in \mathcal{L}_2^e$ such that $\frac{\|\hat{y}\|_{\mathcal{L}_2}}{\|\hat{u}\|_{\mathcal{L}_2}} < 1$ where $\hat{y}$ is the corresponding output. We have $\|\hat{y}\| < \|\hat{u}\| < \infty$. On the other hand, $\hat{u} = G^{-1}\,\hat{y}$. Since $\left\|G^{-1}\right\|_{\mathcal{H}_\infty} = 1$ $\|\hat{u}\| \leq \|\hat{y}\|$ which is a contradiction.

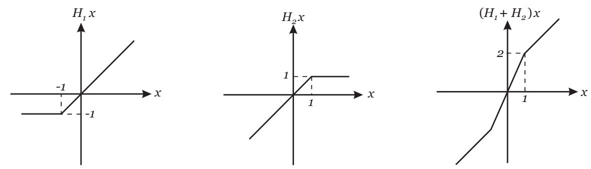
The minimum gain of operators satisfies the *positivity* and the *positive homogeneity* properties. To see this, we have

$$\nu(\cdot) \geq 0 \tag{21}$$

and

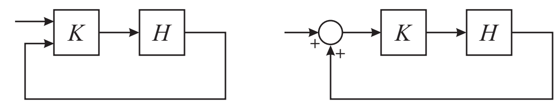
$$\begin{aligned} \nu(\lambda H) &= \inf_{0\neq u\in\mathcal{X}_e} \frac{\|\lambda H u\|}{\|u\|} \\ &= |\lambda| \inf_{0\neq u\in\mathcal{X}_e} \frac{\|Hu\|}{\|u\|} = |\lambda|\,\nu(H) \end{aligned} \tag{22}$$

However, it can be shown that it fails to satisfy the triangle inequality. For instance, suppose that H_1 and H_2 are memoryless nonlinearities depicted in Fig. 3. It is trivial that $\nu(H_1) = 0$, $\nu(H_2) = 0$ and $\nu(H_1 + H_2) = 1$. This shows that $\nu(H_1 + H_2) > \nu(H_1) + \nu(H_2)$. Consequently, the minimum gain of an operator is not a norm or even a semi-norm on the space of operators.



(a) H_1 (b) H_2 (c) $H_1 + H_2$

Fig. 2: The triangle inequality is not satisfied by $\nu(\cdot)$.



(a) (b)

Fig. 3: Stabilizable system

Lemma 3.6: Let $H : \mathcal{U} \to \mathcal{Y}$ denote an operator. Suppose that there exists a nonzero stable operator $K : \mathcal{R} \to \mathcal{U}$ such that $HK : \mathcal{R} \to \mathcal{Y}$ is stable, then $\nu(H) < \infty$.

Proof: Let $0 \neq r(t) \in \mathcal{R}$ such that $r \notin \text{Ker}(K)$, then $u(t) = K\,r(t) \in \mathcal{U}$, $u \neq 0$ and $y(t) = HK\,r(t) \in \mathcal{Y}$, implied by the stability of K and HK, respectively. Therefore $\|u\|_{\mathcal{U}} \neq 0$ and $\|u\|_{\mathcal{U}}, \|y\|_{\mathcal{Y}} < \infty$. Consequently, $\nu(H) \leq \frac{\|y\|_{\mathcal{Y}}}{\|u\|_{\mathcal{U}}} < \infty$. ■

Corollary 3.3: Any system that can be stabilized by a stable system with the mentioned properties in Lemma 3.6 and a structure as shown in either Fig. 3(a) or Fig. 3(b), has a finite minimum gain.

Proof: The corollary is based on Lemma 3.6 and the proof follows a similar routine as the proof of the lemma with defining a new $\mathcal{R}$ equals $\mathcal{R} \oplus \mathcal{Y}$ in 3(a) or $\mathcal{R} + \mathcal{Y}$ in 3(b). ■

Theorem 3.1: Sub-multiplicative property
Let $H_1, H_2 : \mathcal{X} \to \mathcal{X}$ be causal operators. Then

$$\nu(H_1 H_2) \leq \nu(H_1)\nu(H_2) \tag{23}$$

Proof: Let $u \in \mathcal{X}$, we have

$$\|H_1 H_2 u\| \geq \nu(H_1)\,\|H_2 u\| \geq \nu(H_1)\nu(H_2)\,\|u\| \tag{24}$$

Considering the fact that $\nu(H_1 H_2)$ is the infimum gain of the $H_1 H_2$, Inequality (24) implies (23). ■

IV. Large Gain Theorem

In this section, we concentrate on the feedback system shown in Fig. 4. Under mild conditions on H_1 and H_2 (e.g., the product of the instantaneous gains is less than one [1]), the feedback configuration is guaranteed to be well-posed. The equations describing this feedback system, to be known as the *Feedback Equations*, are:

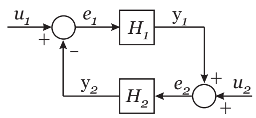


Fig. 4: The feedback system

$$
\begin{aligned}
e_1 &= u_1 - y_2 \\
e_2 &= u_2 + y_1 \\
y_1 &= H_1 e_1 \\
y_2 &= H_2 e_2
\end{aligned}
\tag{25}
$$

Theorem 4.1: Consider the feedback interconnection described by (25) and shown in Fig. 4. Then, if $1 < \nu(H_1)\nu(H_2) < \infty$, the feedback system is input-output-stable.

Proof: To show stability of the feedback interconnection, we must show that $u_1, u_2 \in \mathcal{X}$ imply that e_1, e_2, y_1 and y_2 are also in $\mathcal{X}$. According to the definition of ν, we have

$$\nu(H_1)\,\|e_{1T}\| \le \|y_{1T}\| \tag{26}$$

$$\nu(H_2)\,\|e_{2T}\| \le \|y_{2T}\| \tag{27}$$

On the other hand,

$$y_{1T} = e_{2T} - u_{2T} \tag{28}$$

$$y_{2T} = u_{1T} - e_{1T} \tag{29}$$

Thus,

$$\|y_{1T}\| \le \|e_{2T}\| + \|u_{2T}\| \tag{30}$$

$$\|y_{2T}\| \le \|e_{1T}\| + \|u_{1T}\| \tag{31}$$

Substituting (26) and (27) in (30) and (31), respectively,

$$\nu(H_1)\,\|e_{1T}\| \le \|e_{2T}\| + \|u_{2T}\| \tag{32}$$

$$\nu(H_2)\,\|e_{2T}\| \le \|e_{1T}\| + \|u_{1T}\| \tag{33}$$

Using (27) and (31), Equation (32) implies that

$$
\begin{aligned}
\nu(H_2)\nu(H_1)\,\|e_{1T}\| &\le \nu(H_2)\,\|e_{2T}\| + \nu(H_2)\,\|u_{2T}\| \\
&\le \|y_{2T}\| + \nu(H_2)\,\|u_{2T}\| \\
&\le \|e_{1T}\| + \|u_{1T}\| + \nu(H_2)\,\|u_{2T}\|
\end{aligned}
\tag{34}
$$

Since $\nu(H_1)\nu(H_2) > 1$,

$$\|e_{1T}\| \le \frac{1}{\nu(H_1)\nu(H_2) - 1}\left(\|u_{1T}\| + \nu(H_2)\,\|u_{2T}\|\right) \tag{35}$$

Similarly,

$$\|e_{2T}\| \le \frac{1}{\nu(H_1)\nu(H_2) - 1}\left(\nu(H_1)\,\|u_{1T}\| + \|u_{2T}\|\right) \tag{36}$$

Moreover, substituting (36) and (35) in (30) and (31), respectively,

$$\|y_{1T}\| \le \frac{\nu(H_1)}{\nu(H_1)\nu(H_2) - 1}\left(\|u_{1T}\| + \nu(H_2)\,\|u_{2T}\|\right) \tag{37}$$

and

$$\|y_{2T}\| \le \frac{\nu(H_2)}{\nu(H_1)\nu(H_2) - 1}\left(\nu(H_1)\,\|u_{1T}\| + \|u_{2T}\|\right) \tag{38}$$

Hence, the norms of $\|e_{1T}\|$, $\|1_{2T}\|$, $\|y_{1T}\|$ and $\|y_{2T}\|$ are bounded. If, in addition, u_1, $u_2 \in \mathcal{X}$, then (35-38) must also be satisfied if T approaches ∞. Therefore,

$$\|e_1\| \le \frac{1}{\nu(H_1)\nu(H_2) - 1}\left(\|u_1\| + \nu(H_2)\,\|u_2\|\right) \tag{39}$$

$$\|e_2\| \le \frac{1}{\nu(H_1)\nu(H_2) - 1}\left(\nu(H_1)\,\|u_1\| + \|u_2\|\right) \tag{40}$$

$$\|y_1\| \le \frac{\nu(H_1)}{\nu(H_1)\nu(H_2) - 1}\left(\|u_1\| + \nu(H_2)\,\|u_2\|\right) \tag{41}$$

$$\|y_2\| \le \frac{\nu(H_2)}{\nu(H_1)\nu(H_2) - 1}\left(\nu(H_1)\,\|u_1\| + \|u_2\|\right) \tag{42}$$

Consequently, e_1, e_2, y_1 and y_2 are also in $\mathcal{X}$. ∎

Example 4.1: Let H_1 be the convolution operator defined by (8) where $g(t)$ is the impulse response of

$$G(s) = k\frac{s+1}{s-1}$$

where $k \in \mathbb{R}$. Let H_2 be a memoryless nonlinearity depicted in Fig. 1. As shown in Example 3.2, $\nu(H_1/k) = 1$ which implies that $\nu(H_1) = |k|$. On the other hand, we have $\nu(H_2) = 0.5$. Consequently $\nu(H_1)\nu(H_2) = 0.5|k|$. The large gain theorem, namely Theorem 4.1, guarantees that the feedback system is stable if $|k| > 2$.

V. Conclusion

The minimum gain of an operator as well as some of its properties are introduced. These properties are useful in the computation of the minimum gain of a system. For instance, it is shown that the minimum gains of strictly proper, stable, LTI systems are zero. When it comes to the metric properties, the minimum gain of an operator fails to satisfy the triangular inequality which implies that it is not a metric or a norm in the space of operators. Finally, the so-called large gain theorem is stated and proved. This theorem implies a new stability condition for feedback interconnection of nonlinear systems. An example is provided to illustrate the derived stability condition.

APPENDIX I
PROOF OF LEMMA 3.1

The proofs for the continuous-time and discrete-time cases are the same and only the first one is given here. We use the following property of the smallest singular value of matrices (e.g. [12] pp. 21):

$$\underline{\sigma}(M) = \min_{\|x\|=1} \|Mx\| = \min_{x\neq 0} \frac{\|Mx\|}{\|x\|} \tag{43}$$

Let $M = U\Sigma V^T$ be the SVD of M, where $V = [v_1, v_2, \cdots, v_n] \in \mathbb{R}^{n\times n}$, $U, \Sigma \in \mathbb{R}^{n\times n}$. It is well-known that v_n is the minimizer of (43), e.g. [12]. Let $x \in \mathcal{L}_2$, we have

$$\begin{aligned} \|Mx\|^2 &= \int_0^\infty \|Mx(t)\|_2^2 \, dt \\ &\geq \int_0^\infty \underline{\sigma}(M)^2 \|x(t)\|_2^2 \, dt \\ &= \underline{\sigma}(M)^2 \int_0^\infty \|x(t)\|_2^2 \, dt = \underline{\sigma}(M)^2 \|x\|^2 \end{aligned} \tag{44}$$

which shows that $\underline{\sigma}(M)$ is a lower bound for $\nu(H)$. To show that it is the greatest lower bound, let $x(t) = \frac{v_n}{\|v_n\|} e^{-t}$. We have

$$\|x\|^2 = \int_0^\infty \left\| \frac{v_n}{\|v_n\|} e^{-t} \right\|^2 dt = \int_0^\infty \left\| e^{-t} \right\|^2 dt = {}^1\!/_2 \tag{45}$$

and

$$\begin{aligned} \|Mx\|^2 &= \int_0^\infty \left\| M \frac{v_n}{\|v_n\|} e^{-t} \right\|^2 dt \\ &= \int_0^\infty \|Mv_n\|^2 \frac{e^{-2t}}{\|v_n\|^2} dt \\ &= \int_0^\infty \|\underline{\sigma}(M) v_n\|^2 \frac{e^{-2t}}{\|v_n\|^2} dt \\ &= \|\underline{\sigma}(M)\|^2 \int_0^\infty e^{-2t} \, dt = {}^1\!/_2 \|\underline{\sigma}(M)\|^2 \end{aligned} \tag{46}$$

Equations (45) and (46) imply that $\nu(H)$ is equal to $\underline{\sigma}(M)$ for some input. This completes the proof.

APPENDIX II
PROOF OF LEMMA 3.2

Let $x \in \mathcal{L}_p$, for $p \neq \infty$,

$$\begin{aligned} \|Hx\|_{\mathcal{L}_p}^p &= \int_0^\infty \|\Phi(t, x(t))\|^p \, dt \geq \int_0^\infty \mu_p^p \|x(t)\|_p \, dt \\ &= \mu_p^p \int_0^\infty \|x(t)\|^p \, dt = \mu_p^p \|x\|_{\mathcal{L}_p}^p \end{aligned} \tag{47}$$

For $p = \infty$,

$$\begin{aligned} \|Hx\|_{\mathcal{L}_\infty} &= \sup_t \|\Phi(t, x(t))\| \geq \sup_t \mu_p \|x(t)\| \\ &= \mu_p \sup_t \|x(t)\| = \mu_p \|x\|_{\mathcal{L}_\infty}^p \end{aligned} \tag{48}$$

Equations (47) and (48) imply that μ_p is a lower bound for $\nu(H)$. This completes the proof.

APPENDIX III
PROOF OF LEMMA 3.3

Let $\hat{x}(t) = [\hat{x}_1(t) \; \hat{x}_2(t) \; \cdots \; \hat{x}_n(t)]^T$,

$$\hat{x}_k(t) = \begin{cases} \sin(\omega t) & k = i, \\ 0 & \text{otherwise.} \end{cases}$$

where i corresponds to the strictly proper row in $G(s)$ and $\omega \geq \pi$. Let

$$x(t) := \hat{x}(t) - \hat{x}\left(t - \left\lfloor \frac{\omega}{\pi} \right\rfloor \frac{\pi}{\omega}\right) \tag{49}$$

where $\lfloor r \rfloor$ denotes the floor function of a real number r, which is the largest integer less than or equal to r, namely $\forall r \in \mathbb{R}\, ; \lfloor r \rfloor := \sup\{n \in \mathbb{Z} | n \leq r\}$. It is trivial that

$$x(t) = \begin{cases} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \sin(\omega t) \\ \vdots \\ 0 \end{bmatrix} \; \text{ith row} & 0 \leq t \leq \left\lfloor \frac{\omega}{\pi} \right\rfloor, \\ 0 & t > \left\lfloor \frac{\omega}{\pi} \right\rfloor. \end{cases}$$

and

$$\|x(t)\| = \left| \hat{x}_i(t) - \hat{x}_i\left(t - \left\lfloor \frac{\omega}{\pi} \right\rfloor \frac{\pi}{\omega}\right) \right|$$

Thus,

$$\|x\|_{\mathcal{L}_\infty} = \sup_t |\sin(\omega t)| = 1 \tag{50}$$

$$\begin{aligned} \|x\|_{\mathcal{L}_2}^2 &= \int_0^{\lfloor \frac{\omega}{\pi} \rfloor \frac{\pi}{\omega}} |\sin(\omega t)|^2 \, dt \\ &= {}^1\!/_2 \left(t - \frac{\sin(2\omega t)}{2\omega}\right)\Big|_0^{\lfloor \frac{\omega}{\pi} \rfloor \frac{\pi}{\omega}} \\ &= \left\lfloor \frac{\omega}{\pi} \right\rfloor \frac{\pi}{2\omega} - \frac{\sin\left(2\pi \lfloor \frac{\omega}{\pi} \rfloor\right)}{4\omega} \end{aligned} \tag{51}$$

$$\|x\|_{\mathcal{L}_1} = \int_0^{\lfloor \frac{\omega}{\pi} \rfloor \frac{\pi}{\omega}} |\sin(\omega t)| \, dt \tag{52}$$

To calculate (52), consider the graph of $|\hat{x}(t)|$ depicted in Fig. 5. The number of peaks is $\left\lfloor \frac{\omega}{\pi} \right\rfloor$. Moreover,

$$S = \int_0^{\frac{\pi}{\omega}} \sin(\omega t) \, dt = \frac{2}{\omega} \tag{53}$$

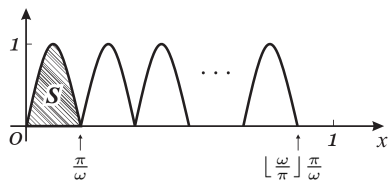


Fig. 5: $|\hat{x}(t)|$

Consequently,

$$\|x\|_{\mathcal{L}_1} = \left\lfloor \frac{\omega}{\pi} \right\rfloor S = \left\lfloor \frac{\omega}{\pi} \right\rfloor \frac{2}{\omega} \tag{54}$$

To calculate the norm of the output $\|y\|$, we can first find the response of the system to input $\hat{x}(t)$, namely $\hat{y}(t)$, and then obtain the output using $y(t) = \hat{y}(t) - \hat{y}\left(t - \left\lfloor \frac{\omega}{\pi} \right\rfloor \frac{\pi}{\omega}\right)$ implied by the linearity property of the system and (49). If we let $\omega \to \infty$, the response of the system to $\hat{x}(t)$ approaches to zero. The reason is that the amplitude of all elements of the i-th row of $G(s)$ approaches to zero at high frequencies. Therefore, $\lim_{w\to\infty} \|\hat{y}(t)\| = 0$ and consequently

$$\lim_{w\to\infty} \|y\| = 0 \tag{55}$$

On the other hand, (51) and (54) imply

$$\lim_{w\to\infty} \|x\|_{\mathcal{L}_2} = {}^1\!/_2, \lim_{w\to\infty} \|x\|_{\mathcal{L}_1} = \frac{2}{\pi} \tag{56}$$

Equations (50), (55) and (56) imply

$$\nu_1(H) = 0, \nu_2(H) = 0, \nu_\infty(H) = 0 \tag{57}$$

Appendix IV
Proof of lemma 3.4

The proofs for the continuous-time and discrete-time cases are the same and only the first one is given here.

Let s_0 be the RHP zero of $G(s)$, namely there exists w such that $G(s_0)w = 0$. If $\sigma_0 + i\omega_0 = s_0 \in \mathbb{C}$, trivially s_0^* is also a RHP zero of $G(s)$. Let

$$u(t) = \begin{cases} w\, e^{s_0 t} & \text{if } s_0 \in \mathbb{R}, \\ w\, e^{\sigma_0 t} \sin(\omega_0 t) & \text{if } s_0 \in \mathbb{C}. \end{cases} \tag{58}$$

Consequently,

$$U(s) = \begin{cases} w \cdot \frac{1}{s - s_0} & \text{if } s_0 \in \mathbb{R}. \\ w \cdot \frac{\omega_0}{(s-\sigma_0)^2 + \omega_0^2} & \text{if } s_0 \in \mathbb{C}, \end{cases} \tag{59}$$

We have

$$Y(s) = \begin{cases} G(s) \cdot w \cdot \frac{1}{s - s_0} & \text{if } s_0 \in \mathbb{R}. \\ G(s) \cdot w \cdot \frac{\omega_0}{(s-\sigma_0)^2 + \omega_0^2} & \text{if } s_0 \in \mathbb{C}, \end{cases} \tag{60}$$

Since $G(s)$ is assumed to be stable, $Y(s)$ is a stable signal. It is important to note that $Y(s)$ does not have a pole at s_0. The reason is that the pole at s_0 is canceled by the zero of $G(s)$ at s_0. Since all poles of $Y(s)$ are in LHP, $y(t)$ is a decaying signal. On the other hand, $u(t)$ is an unstable signal, rising by time. If we truncate both $u(t)$ and $y(t)$ at T, which is chosen sufficiently large, the corresponding gain of the system will be small. By increasing T, the gain can be decreased as much as desired. Therefore, $\nu(H) = 0$.

Appendix V
Proof of lemma 3.5

Let $y(t) := Hu(t)$, which implies that $u(t) = H^{-1}y(t)$. Therefore

$$\begin{aligned} \nu(H) &= \inf_{u\in\mathcal{U}} \frac{\|y_T\|}{\|u_T\|} = \inf_{u\in\mathcal{D}_h} \frac{\|y_T\|}{\|u_T\|} = \inf_{u\in\mathcal{D}_h} \frac{1}{\frac{\|u_T\|}{\|y_T\|}} \\ &= \frac{1}{\sup\limits_{u\in\mathcal{D}_h} \frac{\|u_T\|}{\|y_T\|}} = \frac{1}{\sup\limits_{u\in\mathcal{D}_h} \frac{\|H^{-1}y_T\|}{\|y_T\|}} \\ &= \frac{1}{\sup\limits_{y\in\mathcal{R}_h} \frac{\|H^{-1}y_T\|}{\|y_T\|}} = \frac{1}{\gamma(H^{-1})} \end{aligned} \tag{61}$$

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