2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

On Finite-Time Stability of State Dependent Impulsive Dynamical Systems

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***Abstract*— This paper extends the finite-time stability problem to state dependent impulsive dynamical systems. For this class of hybrid systems, the state jumps when the trajectory reaches a *resetting set*, which is a subset of the state space. A sufficient condition for finite-time stability of state dependent impulsive dynamical systems is provided. Moreover, $\mathcal{S}$ − procedure arguments are exploited to obtain a formulation of this sufficient condition which is numerically tractable by means of Differential Linear Matrix Inequalities (DLMIs). Such a formulation may be in general more conservative, for this reason a procedure which allows to automate its verification, without introduce conservatism, is given both for second order systems, and when the resetting set is *ellispoidal*.**

I. Introduction

The concept of finite-time stability (FTS) dates back to the Sixties, when it was introduced in the control literature [1], [2]. A system is said to be finite-time stable if, given a bound on the initial condition, its state does not exceed a certain threshold during a specified time interval. It is important to recall that FTS and Lyapunov Asymptotic Stability (LAS) are independent concepts; indeed a system can be FTS but not LAS, and vice versa. While LAS deals with the behavior of a system within a sufficiently long (in principle infinite) time interval, FTS is a more practical concept, useful to study the behavior of the system within a finite (possibly short) interval, and therefore it finds application whenever it is desired that the state variables do not exceed a given threshold (for example to avoid saturations or the excitation of nonlinear dynamics) during the transients.

In [3], [4], [5] sufficient conditions for FTS and finite-time stabilization of continuous-time linear systems have been provided; such conditions are based on the solution of a feasibility problem involving either Linear Matrix Inequalities (LMIs [6]) or Differential Linear Matrix Inequalities (DLMIs [7]). The former approach is less demanding from the computational point of view, while the latter is less conservative.

The increasing interest that the researchers have devoted in the last decade to the theory and application of hybrid systems represents a natural stimulus to the extension of the FTS concept to such context, which is the objective of the present work. Indeed, in this paper, we will focus on a class of hybrid systems, namely *state dependent* impulsive dynamical systems [8], where the state jumps occur when the trajectory reaches an assigned subset of the state space, the so-called *resetting set*. Classical examples which fall in this category of systems are the bouncing ball, whose velocity jumps from positive to negative values when the ball hits the ground and the automatic gear-box in cruise control (for more details and further examples see [9]).

Many results concerning the classical Lyapunov asymptotic stability for hybrid systems have been proposed in the literature (see for instance the monographs [9], [10], [8] and references therein). In this paper, as in [11] and [12], we exploit $\mathcal{S}$-procedure arguments, in order to end up with numerically tractable analysis conditions formulated as DLMIs. The main result of the paper is a sufficient condition which guarantees the FTS of a given state dependent impulsive system. Moreover it is shown that, either in presence of second order systems or when the resetting set is *ellipsoidal*, the $\mathcal{S}$-procedure does not introduce conservatism in the FTS analysis.

The paper is structured as follows. The next section presents the notation, the class of hybrid systems we deal with, and some preliminary results exploited throughout the paper. The main results are given in Section III. In Section IV the conservatism introduced by the $\mathcal{S}$-procedure is discussed, and two procedures are presented to reduce such conservatism. The applicability of the proposed results is illustrated through a numerical example. Some conclusions and plans for future works are given in Section V.

II. Preliminaries

The notation used throughout the paper is presented in this section, together with the FTS problem statement for the class of state dependent impulsive systems. Preliminary results on quadratic forms, which will be exploited in the sequel of the paper, are also provided at the end of the section.

A. Problem Statement

Let us consider the state dependent impulsive dynamical linear system described by

$$\dot{x}(t) = A(t)x(t)\,, \quad x(0) = x_0\,, \quad x(t) \in \mathbb{R}^n \setminus \bigcup_{k=1}^{N} \mathcal{S}_k \tag{1a}$$

$$x(t^+) = A_{d,k}x(t), \quad x(t) \in \mathcal{S}_k\,, \quad k = 1, \ldots, N \tag{1b}$$

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where $A(\cdot):\mathbb{R}_0^+\mapsto\mathbb{R}^{n\times n}$, $A_{d,k}\in\mathbb{R}^{n\times n}$, $k=1,\dots,N$. The sets $\mathcal{S}_k\subseteq\mathbb{R}^n$, $k=1,\dots,N$, are connected and closed pairwise disjoint sets (i.e. $\mathcal{S}_i\cap\mathcal{S}_j=\emptyset,\forall\, i\neq j$), such that $0\notin\mathcal{S}_k$. We refer to the differential equation (1a) as the *continuous-time dynamics*, to the difference equations (1b) as the *resetting laws*, and to the sets $\mathcal{S}_k$ as the *resetting sets* [8].

Let us denote by $x(\cdot)$ the solution of the impulsive dynamical system (1a)–(1b). We assume that the resetting laws keep $x(\cdot)$ away from $\mathcal{S}_k$, therefore no trajectory can intersect the interior of $\mathcal{S}_k$, $k=1,\dots,N$.

Definition 1 (FTS of impulsive systems): Given a positive scalar T, a positive definite matrix R, a positive definite matrix-valued function $\Gamma(\cdot)$ defined over $[0,T]$, with $\Gamma(0)<R$, system (1) is said to be finite-time stable with respect to $\big(T,R,\Gamma(\cdot)\big)$ if

$$x_0^TRx_0\le 1\Rightarrow x(t)^T\Gamma(t)x(t)<1\quad\forall\, t\in[0,T]\,. \qquad (2)$$

$\diamondsuit$

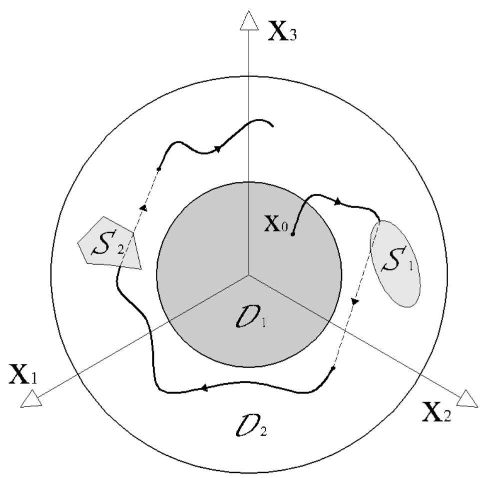

Fig. 1. Example of evolution of a state dependent impulsive dynamical system.

Fig. 1 shows an example of finite-time stable trajectory of a state dependent impulsive system. The trajectory starts inside the sphere $\mathcal{D}_1$ defined by a positive definite matrix R, and remains inside the sphere $\mathcal{D}_2$ defined by a positive definite matrix Γ, $\forall\, t\in[0,T]$. When the trajectory reaches one of the two resetting sets $\mathcal{S}_1$ or $\mathcal{S}_2$, the system state jumps.

B. Some Useful Definitions

The following definitions will be useful throughout the paper.

Definition 2 (Conical Hull [13], p. 28): Given a set $\mathcal{S}\subseteq\mathbb{R}^n$, the set

$$\mathrm{cone}(\mathcal{S}):=\{\lambda_1x_1+\cdots+\lambda_kx_k\,:\,\{x_1,\dots,x_k\}\subseteq\mathcal{S},\lambda_i\ge 0\}$$

is said to be the *conical hull* of $\mathcal{S}$ [13]. The notation $\mathrm{cone}(-\mathcal{S})$ denotes the set

$$\mathrm{cone}(-\mathcal{S}):=\{\lambda_1x_1+\cdots+\lambda_kx_k:\{x_1,\dots,x_k\}\subseteq\mathcal{S},\lambda_i\le 0\}\,.$$

$\diamondsuit$

Definition 3 (Projection with respect to the origin): Consider a hyper-surface $H\subset\mathbb{R}^n$ and a set $\mathcal{S}\subseteq\mathbb{R}^n$. The projection of $\mathcal{S}$ on H with respect to the origin is defined as

$$\mathcal{S}_H:=\{y\in H:y=\lambda x,\,\lambda\in\mathbb{R},\,x\in\mathcal{S}\}\,.$$

$\diamondsuit$

The projection with respect to the origin is a perspective projection [14], where the point of perspective is the origin of the state space, and the projection's surface is the hyper-surface H.

Definition 4 (Convex hull [13], p. 4): Given a set of points $K=\{x^{(1)},\dots,x^{(k)}\}\subset\mathbb{R}^n$, the *convex hull* of K is defined as

$$\mathrm{conv}(K):=\\ :=\left\{\lambda_1x^{(1)}+\cdots+\lambda_kx^{(k)}\,:\,\lambda_i\ge 0,\,\sum_{i=1}^k\lambda_i=1\right\},$$

i.e. as the smallest convex set containing K. $\diamondsuit$

Definition 5 (Chebyshev center [15]): The Chebyshev center of a set $\mathcal{S}\subseteq\mathbb{R}^n$ is

$$0_{\mathcal{S}}:=\arg\min_{x\in\mathbb{R}^n}\left(\max_{\theta\in\mathcal{S}}\|x-\theta)\|_\infty\right).$$

$\diamondsuit$

C. Preliminary results on quadratic forms

As it will be shown later, the main result of the paper requires to check whether, given a connected and closed set $\mathcal{S}\subseteq\mathbb{R}^n$ and a symmetric matrix $Q_0\in\mathbb{R}^{n\times n}$, the inequality

$$x^TQ_0x<0\,,\quad x\in\mathcal{S}\setminus\{0\}\,, \qquad (3)$$

is satisfied.

In the following, our goal is to find some numerically tractable conditions which guarantee the satisfaction of (3). Exploiting $\mathcal{S}$ − procedure arguments ([6], p. 24), it is readily seen that Q_0 satisfies (3) if the following feasibility problem admits a solution.

Problem 1: Given a connected and closed set $\mathcal{S}\subset\mathbb{R}^n$, a symmetric matrix $Q_0\in\mathbb{R}^{n\times n}$ and symmetric matrices $Q_i\in\mathbb{R}^{n\times n}$ satisfying

$$x^TQ_ix\le 0\,,\quad x\in\mathcal{S}\,,\quad i=1,\dots,p\,, \qquad (4a)$$

find nonnegative scalars c_i , $i=1,\dots,p$, such that

$$Q_0-\sum_{i=1}^pc_iQ_i<0\,. \qquad (4b)$$

$\diamondsuit$

Remark 1: The usefulness of Problem 1 relies in the fact that it can be recast in the LMIs framework, where the coefficients c_i are the optimization variables of the LMI (4b). Clearly, one needs a method to choose the matrices Q_i; in the next section we provide a procedure to build a suitable set of matrices Q_i, which can be exploited when the set $\mathcal{S}$ satisfies some assumptions. ◊

As mentioned above, if Problem 1 admits a feasible solution, then (3) is satisfied. In general, the converse is not true. Therefore it makes sense to investigate under which conditions solving Problem 1 is equivalent to check condition (3); the answer is given by the following lemma.

Lemma 1: Given a connected and closed set $\mathcal{S} \subset \mathbb{R}^n$ and a symmetric matrix $Q_0 \in \mathbb{R}^{n\times n}$, assume there exists a symmetric matrix $\bar{Q} \in \mathbb{R}^{n\times n}$ such that

$$x^T\bar{Q}x \le 0 \quad \forall\, x \in \big(\mathrm{cone}(\mathcal{S}) \cup \mathrm{cone}(-\mathcal{S})\big) \setminus \{0\} \tag{5a}$$

$$x^T\bar{Q}x > 0 \quad \forall\, x \in \mathbb{R}^n \setminus \big(\mathrm{cone}(\mathcal{S}) \cup \mathrm{cone}(-\mathcal{S})\big) \tag{5b}$$

$$\exists\, \tilde{x} \in \mathcal{S} \setminus \{0\} : \quad \tilde{x}^T\bar{Q}\tilde{x} < 0\,; \tag{5c}$$

then condition (3) the feasibility Problem 1 with $p = 1$ and $Q_1 = \bar{Q}$.

Proof: The proof is trivial once recognized that:

1) $x^TQ_0x < 0$ for all $x \in \mathcal{S}$ iff $x^TQ_0x < 0$ for all $x \in \mathrm{cone}(\mathcal{S}) \cup \mathrm{cone}(-\mathcal{S})$ (see Lemma 2 in the Appendix).
2) solving Problem 2 with $p = 1$ and $Q_1 = \bar{Q}$, is equivalent to applying lossless $\mathcal{S}$-procedure, since $\bar{Q}$ satisfies (15) of Lemma 3 reported in the Appendix.

■

In Section IV we shall see that when the set $\mathcal{S}$ satisfies certain assumptions, the hypotheses of Lemma 1 are fulfilled and the approach via Problem 1 does not add conservatism in the FTS analysis.

III. Main Results

The following theorem gives a sufficient condition for FTS of system (1).

Theorem 1: System (1) is FTS with respect to $\big(T, R, \Gamma(\cdot)\big)$ if the following coupled differential/difference Lyapunov inequalities with terminal and initial conditions,

$$\dot{P}(t) + A(t)^TP(t) + P(t)A(t) < 0\,, \tag{6a}$$

$$x^T\left(A_{d,k}^TP(t)A_{d,k} - P(t)\right)x < 0\,, \qquad \forall\, x \in \mathcal{S}_k\,, \quad k = 1,\dots,N \tag{6b}$$

$$P(t) \ge \Gamma(t)\,, \tag{6c}$$

$$P(0) < R\,, \tag{6d}$$

admit a continuously differentiable symmetric solution $P(\cdot)$ over the interval $[0, T]$.

Proof: Let $V(t, x) = x^TP(t)x$. If $x \notin \bigcup_{k=1}^N \mathcal{S}_k$, then the derivative of V along the trajectories of system (1a) yields

$$\dot{V}(t, x) = x^T\left(\dot{P}(t) + A(t)^TP(t) + P(t)A(t)\right)x\,,$$

which is negative definite by virtue of (6a).

At the discontinuity points ($x \in \mathcal{S}_k$) we have

$$V(t^+, x) - V(t, x) = x^T\left(A_{d,k}P(t^+)A_{d,k} - P(t)\right)x\,,$$

which is negative definite in view of (6b).

We can conclude that $V(t, x)$ is strictly decreasing along the trajectories of system (1a)–(1b); hence, given x_0 such that $x_0^TRx_0 \le 1$, we have, for all $t \in [0, T]$,

$$\begin{aligned} x(t)^T\Gamma(t)x(t) &\le x(t)^TP(t)x(t) \quad \text{by (6c)} \\ &< x(0)^TP(0)x(0) \\ &< x(0)^TRx(0) \le 1 \quad \text{by (6d)}\,. \end{aligned}$$

■

Note that, for a given k and t, condition (6b) is equal to (3) if we let $Q_0 = A_{d,k}^TP(t)A_{d,k} - P(t)$ and $\mathcal{S} = \mathcal{S}_k$. Therefore, by exploiting the machinery introduced in Section II-C, we can relax inequality (6b) and replace it with (see Problem 1)

$$A_{d,k}^TP(t)A_{d,k} - P(t) - \sum_{i=1}^{p} c_{i,k}(t)Q_{i,k} < 0$$

where $Q_{i,k}$ are given symmetric matrices satisfying

$$x^TQ_{i,k}x \le 0\ ,\, x \in \mathcal{S}_k\,, i = 1,\dots,p_k\,,$$

and $c_{i,k}(t) \ge 0$, $t \in [0, T]$, for $i = 1,\dots,p$.

On the basis of this consideration, we can immediately derive the following corollary of Theorem 1.

Theorem 2: Given a set of symmetric matrices $Q_{i,k}$, $i = 1,\dots,p_k$, $k = 1,\dots,N$, satisfying

$$x^TQ_{i,k}x \le 0\,, \quad x \in \mathcal{S}_k\,, i = 1,\dots,p_k\,, k = 1,\dots,N\,, \tag{7}$$

assume there exist a continuously differentiable symmetric matrix function $P(\cdot)$ and nonnegative scalar functions $c_{i,k}(\cdot)$, $i = 1,\dots,p_k$, $k = 1,\dots,N$, such that, for all $t \in [0, T]$,

$$\dot{P}(t) + A(t)^TP(t) + P(t)A(t) < 0\,, \tag{8a}$$

$$A_{d,k}^TP(t)A_{d,k} - P(t) - \sum_{i=1}^{p_k} c_{i,k}(t)Q_{i,k} < 0 \qquad k = 1,\dots,N \tag{8b}$$

$$P(t) \ge \Gamma(t) \tag{8c}$$

$$P(0) < R\,, \tag{8d}$$

then system (1a)–(1b) is FTS with respect to $\big(T, R, \Gamma(\cdot)\big)$.

Remark 2: In view of the results given in Theorem 2, it is now possible to clarify the usefulness of the formulation introduced in Problem 1. Such formulation, indeed, allows to replace condition (6b) with condition (8b). Note that the former requires to solve a number of time-varying inequalities, depending on the matrices $A_{d,k}$, over the sets $\mathcal{S}_k$; the latter, instead, is just a set of LMIs, therefore it can be easily solved.

For the sake of simplicity, in the following we will assume that there is only one resetting set $\mathcal{S}$, i.e. $N = 1$.

IV. Analysis of some cases of interest

Theorem 2 may introduce conservatism with respect to Theorem 1 since, in general, the $\mathcal{S}$-procedure is lossy. However, if there exists a symmetric matrix-valued function $Q(t)$ which satisfies the conditions (5) for each $t \in [0,T]$, then the $\mathcal{S}$-procedure is lossless and Theorem 2 is equivalent to Theorem 1.

In the sequel we will discuss two cases where Theorem 2 does not introduce conservatism: resetting sets in $\mathbb{R}^2$, and ellipsoidal resetting sets; we prove that the conservatism can be eliminated in both cases, except for the following trivial cases:

1) $\mathcal{S} \subseteq \mathbb{R}^n$ lies on a hyperplane which intersect the origin;
2) $\mathcal{S} \subseteq \mathbb{R}^n$ has dimension less than $n-1$.

The definition of ellipsoidal resetting set is based on the following constructive geometrical procedure.

Procedure 1 (Construction of $\mathcal{E}_H$): Given a connected and closed set $\mathcal{S} \subset \mathbb{R}^n$, construct the set $\mathcal{E}_H$ as follows:

1) denote with $\mathcal{S}_s$ the projection, with respect to the origin, of $\mathcal{S}$ on the unit sphere $x^T x = 1$;
2) denote with 0_s the Chebyshev center of $\mathcal{S}_s$;
3) denote with H the hyper-plane of dimension $n-1$ orthogonal to the line that joins the origin to 0_s, and such that $0_s \in H$;
4) $\mathcal{E}_{\mathcal{H}}$ is the projection, with respect to the origin, of $\mathcal{S}$ on the hyper-plane H.

An example of construction of the set $\mathcal{E}_H$ is shown in Fig. 2. ◊

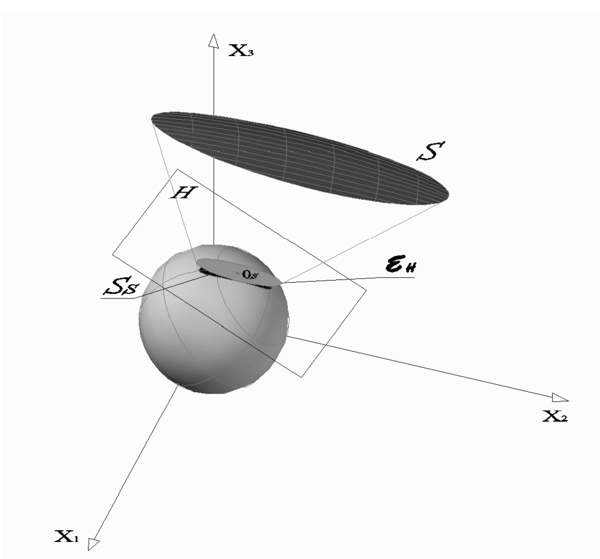

Fig. 2. Construction of the set $\mathcal{E}_H$.

Definition 6 (Ellipsoidal resetting set): Consider a *non-trivial* resetting set $\mathcal{S} \subset \mathbb{R}^n$ and construct the set $\mathcal{E}_H$ using Procedure 1. If $\mathcal{E}_{\mathcal{H}}$ is an hyper-ellipsoid of dimension $n-1$, then $\mathcal{S}$ is called *ellipsoidal resetting set*. ◊

Remark 3: Since $\mathcal{E}_H$ is constructed using two projections with respect to the origin it follows that

$$\operatorname{cone}(\mathcal{S}) = \operatorname{cone}(\mathcal{E}_H)\,.$$

◊

A. Resetting set $\mathcal{S}$ in $\mathbb{R}^2$

The following theorem provides a necessary and sufficient condition which enables to find a symmetric matrix $Q \in \mathbb{R}^{2\times 2}$ that verifies conditions (5).

Theorem 3: Every *non-trivial* resetting set $\mathcal{S}$ in $\mathbb{R}^2$ admits a symmetric matrix $Q \in \mathbb{R}^{2\times 2}$ that verifies conditions (5).

Proof: To prove our statement, we provide a procedure to calculate a matrix Q satisfying conditions (5).

Let $s_1,\, s_2 \in \mathcal{S}$ such that, said $\bar{\mathcal{S}} = \operatorname{conv}(\{s_1\,,\, s_2\})$, we have

$$\operatorname{cone}(\bar{\mathcal{S}}) = \operatorname{cone}(\mathcal{S})\,.$$

Then, taking into account Lemma 1, condition (5) can be equivalently evaluated on the set $\bar{\mathcal{S}}$. In particular, considering the properties of the quadratic forms, it is easy to verify that such condition can be replaced by the following

$$x^T Q x < 0 \quad \forall\, x \in \operatorname{int}(\bar{\mathcal{S}}) \tag{9a}$$

$$x^T Q x = 0 \quad \text{for } x = s_1,\, s_2 \tag{9b}$$

$$x^T Q x > 0 \quad \forall\, x \in H \setminus \bar{\mathcal{S}}. \tag{9c}$$

where H is the hyperplane on which $\bar{\mathcal{S}}$ lies, and $\operatorname{int}(\bar{\mathcal{S}})$ denotes the interior of the set $\bar{\mathcal{S}}$. Letting $s_m = \frac{s_1+s_2}{2}$, a symmetric matrix $Q \in R^{2\times 2}$ such that

$$s_1^T Q s_1 = 0\,, \quad s_2^T Q s_2 = 0\,, \quad s_m^T Q s_m < 0\,.$$

verifies conditions (9). ■

B. Ellipsoidal resetting sets $\mathcal{S}$

The following theorem provides a sufficient condition to find a matrix $Q \in \mathbb{R}^{n\times n}$ that verifies conditions (5).

Theorem 4: If $\mathcal{S}$ is an ellipsoidal resetting set, then there exists a matrix $Q \in \mathbb{R}^{n\times n}$ that verifies conditions (5).

Proof: If $\mathcal{S}$ is an ellipsoidal resetting set then (see Remark 3)

$$\operatorname{cone}(\mathcal{S}) = \operatorname{cone}(\mathcal{E}_H)\,.$$

Taking into account Lemma 1, it follows that conditions (5) can be equivalently evaluated on the set $\mathcal{E}_H$. In particular, considering the properties of the quadratic forms, it is easy to verify that such conditions can be replaced by the following

$$x^T Q x < 0 \quad \forall\, x \in \operatorname{int}(\mathcal{E}_H) \tag{10a}$$

$$x^T Q x = 0 \quad \forall\, x \in \partial\mathcal{E}_H \tag{10b}$$

$$x^T Q x > 0 \quad \forall\, x \in H \setminus \mathcal{E}_H. \tag{10c}$$

To conclude the proof we need to show that the assumption of ellipsoidal set $\mathcal{E}_H$ is sufficient to find a matrix Q which verifies conditions (10).

In the sequel of the proof we will make the following assumptions:

- 0_s is on the n-th coordinated axis , i.e.

$$0_s = \begin{pmatrix} 0 & \dots & 0 & r \end{pmatrix}^T, \quad r \in \mathbb{R}.$$

- The hyper-plane H is orthogonal to the n-th coordinated axis.

As a matter of fact, it is always possible, by means of opportune rotations, to satisfy these assumptions.

In view of the assumptions made above, it is possible to describe the set $\partial\mathcal{E}_H$ by the system of equations

$$\frac{x_1^2}{a_1^2} + \cdots + \frac{x_{n-1}^2}{a_{n-1}^2} = 1\,,$$
$$x_n = r\,,$$

where $a_i \geq 0$, $i = 1, \dots, n-1$. Hence, it is straightforward to check that the matrix

$$Q = \mathrm{diag}\begin{pmatrix} \frac{1}{a_1^2} & \cdots & \frac{1}{a_{n-1}^2} & -\frac{1}{r^2} \end{pmatrix},$$

satisfies conditions (10). ■

The following example illustrates the effectiveness of the proposed procedure.

Example 1 (Impulsive system in $\mathbb{R}^2$): Let consider the following second order impulsive system

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -0.25 \end{pmatrix},$$
$$A_{d,1} = \begin{pmatrix} 1.6 & -1.4 \\ -1.4 & 3.2 \end{pmatrix}, \quad A_{d,2} = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix},$$

where the two resetting sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are given by

$$\mathcal{S}_1 = \mathrm{conv}\left(\begin{pmatrix} 0.5 \\ 0.2 \end{pmatrix}, \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix}\right),$$
$$\mathcal{S}_2 = \mathrm{conv}\left(\begin{pmatrix} -0.7 \\ 0.5 \end{pmatrix}, \begin{pmatrix} -0.2 \\ 0.5 \end{pmatrix}\right).$$

Note that the two discrete dynamics $A_{d,1}$ and $A_{d,2}$ are not Schur stable.

We want to analyze the FTS for such impulsive system, for

$$T = 1\,\mathrm{s}\,, \quad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad R = \begin{pmatrix} 5 & 0 \\ 0 & 2.5 \end{pmatrix}.$$

Theorem 3 assures that for each of the considered resetting sets in $\mathbb{R}^2$ there exist a matrix Q which verifies the conditions (5). Applying the procedure proposed in the proof of Theorem 3, the following matrices have been found:

$$Q_1 = \begin{pmatrix} -0.2222 & 0.3889 \\ 0.3889 & -0.5556 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} -0.16 & -0.144 \\ -0.144 & -0.0896 \end{pmatrix}.$$

Using matrices Q_1 and Q_2, Theorem 2 admits a solution, therefore the considered system is FTS with respect to (T, R, Γ). ♢

V. Conclusions

An extension of the finite-time stability concept to a class of hybrid systems has been presented in this paper. In particular a sufficient condition for FTS of state dependent impulsive dynamical systems has been provided. A DLMIs formulation of this condition for FTS has been provided as well, in order to check it in a numerically tractable way. Such a formulation has been obtained exploiting $\mathcal{S}$-procedure arguments, and it may be in general more conservative than the original sufficient condition. In general, it also requires the definition of a set of specific symmetric matrices for each resetting set, which is not a straightforward task. To deal with these problems, a procedure which allows to automate the building of the symmetric matrices, without introduce conservatism, is provided both when the resetting sets are ellispoidal, and when dealing with second order systems.

Appendix

The two results presented in this appendix are needed to prove Lemma 1 in Section II.

Lemma 2: Consider a nonempty, connected and closed set $\mathcal{S} \subseteq \mathbb{R}^n$ and a symmetric matrix $Q_0 \in \mathbb{R}^{n\times n}$; then (3) is satisfied if and only if

$$x^T Q_0 x < 0\,, \quad \forall\, x \in \big(\mathrm{cone}(\mathcal{S}) \cup \mathrm{cone}(-\mathcal{S})\big) \setminus \{0\}\,. \quad (11)$$

Proof: (11) $\Longrightarrow$ (3). Trivial, since $\mathcal{S} \subseteq \mathrm{cone}(\mathcal{S}) \cup \mathrm{cone}(-\mathcal{S})$.

(3)$\Longrightarrow$(11). Any point $\bar{x} \in \mathrm{cone}(\mathcal{S}) \setminus \{0\}$ can be written as

$$\bar{x} = \lambda_1 x_1 + \cdots + \lambda_k x_k\,,$$

where $\{x_1, \dots, x_k\} \subseteq \mathcal{S}$, and $\lambda_i \geq 0$. Since $\mathcal{S}$ is a connected set, it follows that there exists a scalar $\tilde{\lambda} \geq 0$ and a vector $\tilde{x} \in \mathcal{S}$ such that $\bar{x} = \tilde{\lambda}\tilde{x}$. Therefore

$$\bar{x}^T Q_0 \bar{x} = \tilde{\lambda}^2 \tilde{x}^T Q_0 \tilde{x} < 0\,.$$

A similar statement can be made for every point $\bar{x} \in \mathrm{cone}(-\mathcal{S}) \setminus \{0\}$. The implication easily follows. ■

From Lemma 2 it follows that, when

$$\mathrm{cone}(\mathcal{S}) \cup \mathrm{cone}(-\mathcal{S}) = \mathbb{R}^n\,, \quad (12)$$

the satisfaction of (3) is equivalent to require the negative definiteness of Q_0. Note that condition (12) is verified whenever the set $\mathcal{S}$ surrounds the origin.

Lemma 3 ($\mathcal{S}$-Procedure [16]): Let $Q_0, Q_1, \dots, Q_p \in \mathbb{R}^{n\times n}$ be $p+1$ symmetric matrices. Consider the following condition on $Q_0, Q_1, \dots, Q_p$

$$x^T Q_0 x < 0 \quad \forall x : x \neq 0 \wedge x^T Q_i x \leq 0\,, \quad (13)$$
$$i = 1, \dots, p\,.$$

It is obvious that if

$$\exists\, c_1 \geq 0, \dots, c_p \geq 0 : \, Q_0 - \sum_{i=1}^{p} c_i Q_i < 0\,, \quad (14)$$

then condition (13) holds. It is not a trivial fact that when $p = 1$, the converse holds, provided that

$$\exists\ \tilde{x} \quad \text{such that} \quad \tilde{x}^T Q \tilde{x} < 0\,. \tag{15}$$

■

Proof: Let $p = 1$, we will prove that if (13) and (15) hold, then it is always possible to find a constant $c_1 \geq 0$ such that (14) is verified.

Define the mapping $\varphi(x) :\ \mathbb{R}^n \ \mapsto\ \mathbb{R}^2$ as follows:

$$\varphi(x) = \left(x^T Q_1 x, x^T Q_0 x\right)\,, \quad x \in \mathbb{R}^n\,.$$

It can be easily shown that the set $\varphi(\mathbb{R}^n)$ is convex.

If

$$F = \left\{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \leq 0\,,\, z_2 > 0\right\}\,,$$

then condition (13) implies that the set $\varphi(\mathbb{R}^n)$ does not intersect the second quadrant, that is $\varphi(\mathbb{R}^n) \cap \mathrm{cl}(F) = \emptyset$[1].

Since $\varphi(\mathbb{R}^n)$ is convex, then the sets $\varphi(\mathbb{R}^n)$ and $\mathrm{cl}(F)$ do not have common interior points. It follows that $\varphi(\mathbb{R}^n)$ and $\mathrm{cl}(F)$ are separated, i.e. exist λ_1, $\lambda_2 \in \mathbb{R}$ not all zero such that

$$\lambda_1 z_1 + \lambda_2 z_2 \leq 0 \quad \forall\ (z_1, z_2) \in \mathrm{cl}\left(\varphi(\mathbb{R}^n)\right)\,, \tag{16a}$$

$$\lambda_1 z_1 + \lambda_2 z_2 \geq 0 \quad \forall\ (z_1, z_2) \in \mathrm{cl}(F)\,. \tag{16b}$$

Since $(0, 1) \in \mathrm{cl}(F)$ then $\lambda_2 \geq 0$. Similarly $\lambda_1 \leq 0$. Moreover $\lambda_2 \neq 0$, otherwise $\lambda_1 \tilde{x}^T Q_1 \tilde{x} \leq 0$ with $\lambda_1 < 0$, which contradicts (15).

Thus $\lambda_2 > 0$ and $\lambda_1 \leq 0$, and (13) follows from (16a) setting $c_1 = -\lambda_1/\lambda_2$. ■

If condition (15) holds the $\mathcal{S}$-procedure is said to be *lossless*, otherwise it is said to be *lossy* [16]. Hence condition (15) must be satisfied, in order to do not introduce conservatism when applying the $\mathcal{S}$-procedure to check (3).

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[1] $\mathrm{cl}(F)$ denotes the closure of a set F.