2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

On Robust Control Algorithms for Nonlinear Network Consensus Protocols

Qing Hui, Wassim M. Haddad, and Sanjay P. Bhat

Abstract— **Even though many consensus protocol algorithms have been developed over the last several years in the literature, robustness properties of these algorithms involving nonlinear dynamics have been largely ignored. Robustness here refers to sensitivity of the control algorithm achieving semistability and consensus in the face of model uncertainty. In this paper, we develop robust control algorithms for network consensus protocols with information model uncertainty of a specified structure. In particular, we construct homogeneous control protocol functions that scale in a consistent fashion with respect to a scaling operation on an underlying space with the additional property that the protocol functions can be written as a sum of functions, each homogeneous with respect to a fixed scaling operation, that retain system semistability and consensus.**

I. Introduction

Due to advances in embedded computational resources over the last several years, a considerable research effort has been devoted to the control of networks and control over networks. Network systems involve distributed decision-making for coordination of networks of dynamic agents involving information flow enabling enhanced operational effectiveness via cooperative control in autonomous systems. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles (UAV's) and autonomous underwater vehicles (AUV's) for combat, surveillance, and reconnaissance; distributed reconfigurable sensor networks for managing power levels of wireless networks; air and ground transportation systems for air traffic control and payload transport and traffic management; swarms of air and space vehicle formations for command and control between heterogeneous air and space vehicles; and congestion control in communication networks for routing the flow of information through a network.

To enable the applications for these multiagent systems, cooperative control tasks such as formation control, rendezvous, flocking, cyclic pursuit, cohesion, separation, alignment, and consensus need to be developed [1–3]. To realize these tasks, individual agents need to share information of the system objectives as well as the dynamical network. In particular, in many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. Information consensus over dynamic information-exchange topologies guarantees agreement between agents for a given coordination task. Distributed consensus algorithms involve neighbor-to-neighbor interaction between agents wherein agents update their information state based on the information states of the neighboring agents. A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial state as well.

In systems possessing a continuum of equilibria, *semistability*, and not asymptotic stability is the relevant notion of stability [4], [5]. Semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. Semistability thus implies Lyapunov stability, and is implied by asymptotic stability. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable. References [2], [3] build on the results of [4], [5] and give semistable stabilization results for nonlinear network dynamical systems.

Even though many consensus protocol algorithms have been developed over the last several years in the literature (see [1–3], [6] and the numerous references therein), and some robustness issues have been considered [1], [7], [8], robustness properties of these algorithms involving nonlinear dynamics have been largely ignored. Robustness here refers to sensitivity of the control algorithm achieving semistability and consensus in the face of model uncertainty. In this paper, we build on the results of [2] to develop robust control algorithms for network consensus protocols with information model uncertainty of a specified structure. In particular, we construct homogeneous control protocol functions that scale in a consistent fashion with respect to a scaling operation on an underlying space with the additional property that the protocol functions can be written as a sum of functions, each homogeneous with respect to a fixed scaling operation, that retain system semistability and consensus.

II. Mathematical Preliminaries

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^{\#}$ denotes the group generalized inverse, and "$\circ$" denotes the composition operator. For $A \in \mathbb{R}^{m\times n}$ we write $\operatorname{rank} A$ to denote the rank of A. Furthermore, $\partial\mathcal{S}$ and $\overline{\mathcal{S}}$ denote the boundary and the closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. Finally, we write $\|\cdot\|$ for the Euclidean vector norm and $\operatorname{dist}(p, \mathcal{M})$ for the smallest distance from a point p to the set $\mathcal{M}$, that is, $\operatorname{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p - x\|$.

In this paper, we consider nonlinear dynamical systems of

This research was supported in part by the Air Force Office of Scientific Research under Grant FA9550-06-1-0240.

Q. Hui and W. M. Haddad are with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA (qing_hui@ae.gatech.edu), (wm.haddad@aerospace.gatech.edu).

S. P. Bhat is with the Department of Aerospace Engineering, Indian Institute of Technology, Powai, Mumbai 400076, India (bhat@aero.iitb.ac.in).

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

the form

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \in \mathcal{I}_{x_0}, \tag{1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \in \mathcal{I}_{x_0}$, is the system state vector, $\mathcal{D}$ is an open set, $f : \mathcal{D} \to \mathbb{R}^n$ is continuous on $\mathcal{D}$, $f^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) = 0\}$ is nonempty, and $\mathcal{I}_{x_0} = [0, \tau_{x_0})$, $0 \leq \tau_{x_0} \leq \infty$, is the maximal interval of existence for the solution $x(\cdot)$ of (1). A continuously differentiable function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is said to be a *solution* of (1) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$ if x satisfies (1) for all $t \in \mathcal{I}_{x_0}$. The continuity of f implies that, for every $x_0 \in \mathcal{D}$, there exist $\tau_0 < 0 < \tau_1$ and a solution $x(\cdot)$ of (1) defined on (τ_0, τ_1) such that $x(0) = x_0$. A solution x is said to be *right maximally defined* if x cannot be extended on the right (either uniquely or nonuniquely) to a solution of (1). Here, we assume that for every initial condition $x_0 \in \mathcal{D}$, (1) has a unique right maximally defined solution, and this unique solution is defined on $[0, \infty)$. Furthermore, we assume that $f(\cdot)$ is locally Lipschitz continuous on $\mathcal{D}\backslash f^{-1}(0)$. Note that the local Lipschitzness of $f(\cdot)$ on $\mathcal{D}\backslash f^{-1}(0)$ implies local uniqueness in forward and backward time for nonequilibrium initial states.

Under these assumptions on f, the solutions of (1) define a continuous *global semiflow* on $\mathcal{D}$, that is, $s : [0, \infty) \times \mathcal{D} \to \mathcal{D}$ is a jointly continuous function satisfying the *consistency property* $s(0, x) = x$ and the *semi-group property* $s(t, s(\tau, x)) = s(t+\tau, x)$ for every $x \in \mathcal{D}$ and $t, \tau \in [0, \infty)$. Given $t \in [0, \infty)$ we denote the *flow* $s(t, \cdot) : \mathcal{D} \to \mathcal{D}$ of (1) by $s_t(x_0)$ or s_t. Likewise, given $x \in \mathcal{D}$ we denote the *solution curve* or *trajectory* $s(\cdot, x) : \overline{\mathbb{R}}_+ \to \mathcal{D}$ of (1) by $s^x(t)$ or s^x.

A set $\mathcal{M} \subset \mathbb{R}^n$ is *positively invariant* if $s_t(\mathcal{M}) \subseteq \mathcal{M}$ for all $t \geq 0$. The set $\mathcal{M}$ is *negatively invariant* if, for every $z \in \mathcal{M}$ and every $t \geq 0$, there exists $x \in \mathcal{M}$ such that $s(t, x) = z$ and $s(\tau, x) \in \mathcal{M}$ for all $\tau \in [0, t]$. Finally, the set $\mathcal{M}$ is *invariant* if $s_t(\mathcal{M}) = \mathcal{M}$ for all $t \geq 0$. Note that a set is invariant if and only if it is positively and negatively invariant.

Definition 2.1 ([4]): An equilibrium point $x \in \mathcal{D}$ of (1) is *Lyapunov stable under* f if for every open subset $\mathcal{N}_\varepsilon$ of $\mathcal{D}$ containing x, there exists an open subset $\mathcal{N}_\delta$ of $\mathcal{D}$ containing x such that $s_t(\mathcal{N}_\delta) \subset \mathcal{N}_\varepsilon$ for all $t \geq 0$. An equilibrium point $x \in \mathcal{D}$ of (1) is *semistable under* f if it is Lyapunov stable under f and there exists an open subset $\mathcal{U}$ of $\mathcal{D}$ containing x such that for all initial conditions in $\mathcal{U}$, the trajectory of (1) converges to a Lyapunov stable equilibrium point, that is, $\lim_{t\to\infty} s(t, x) = y$, where $y \in \mathcal{D}$ is a Lyapunov stable equilibrium point of (1) and $x \in \mathcal{U}$. If, in addition, $\mathcal{U} = \mathcal{D} = \mathbb{R}^n$, then an equilibrium point $x \in \mathcal{D}$ of (1) is a *globally semistable equilibrium*. The system (1) is said to be *semistable under* f if every equilibrium point of (1) is semistable. Finally, (1) is said to be *globally semistable under* f if (1) is semistable under f and $\mathcal{U} = \mathcal{D} = \mathbb{R}^n$.

Definition 2.2: The *domain of semistability* is the set of points $x_0 \in \mathcal{D}$ such that if $x(t)$ is a solution to (1) with $x(0) = x_0$, $t \geq 0$, then $x(t)$ converges to a Lyapunov stable equilibrium point in $\mathcal{D}$.

Note that if (1) is semistable, then its domain of semistability contains the set of equilibria in its interior. Next, we present alternative equivalent characterizations of semistability of (1).

Lemma 2.1 ([9]): Consider the nonlinear dynamical system (1). Then the following statements are equivalent:

i) The system (1) is semistable.

ii) For each $x_e \in f^{-1}(0)$, there exist class $\mathcal{K}$ and $\mathcal{L}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, respectively, and $\delta = \delta(x_e) > 0$, such that if $\|x_0 - x_e\| < \delta$, then $\|x(t) - x_e\| \leq \alpha(\|x_0 - x_e\|)$, $t \geq 0$, and $\mathrm{dist}(x(t), f^{-1}(0)) \leq \beta(t)$, $t \geq 0$.

iii) For each $x_e \in f^{-1}(0)$, there exist class $\mathcal{K}$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, a class $\mathcal{L}$ function $\beta(\cdot)$, and $\delta = \delta(x_e) > 0$, such that if $\|x_0 - x_e\| < \delta$, then $\mathrm{dist}(x(t), f^{-1}(0)) \leq \alpha_1(\|x(t) - x_e\|)\beta(t) \leq \alpha_2(\|x_0 - x_e\|)\beta(t)$, $t \geq 0$.

Given a continuous function $V : \mathcal{D} \to \mathbb{R}$, the *upper right Dini derivative* of V along the solution of (1) is defined by

$$\dot{V}(s(t, x)) \triangleq \limsup_{h \to 0^+} \frac{1}{h}[V(s(t+h, x)) - V(s(t, x))]. \tag{2}$$

It is easy to see that $\dot{V}(x_e) = 0$ for every $x_e \in f^{-1}(0)$. In addition, note that $\dot{V}(x) = \dot{V}(s(0, x))$. Finally, if $V(\cdot)$ is continuously differentiable, then $\dot{V}(x) = V'(x)f(x)$.

In the sequel, we will need to consider a complete vector field ν on $\mathbb{R}^n$ such that the solutions of the differential equation $\dot{y}(t) = \nu(y(t))$ define a continuous *global flow* $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$, where $\nu^{-1}(0) = f^{-1}(0)$. For each $\tau \in \mathbb{R}$, the map $\psi_\tau(\cdot) = \psi(\tau, \cdot)$ is a homeomorphism and $\psi_\tau^{-1} = \psi_{-\tau}$. We define a function $V : \mathbb{R}^n \to \mathbb{R}$ to be *homogeneous of degree* $l \in \mathbb{R}$ *with respect to* ν if and only if

$$(V \circ \psi_\tau)(x) = e^{l\tau}V(x), \quad \tau \in \mathbb{R}, \quad x \in \mathbb{R}^n. \tag{3}$$

Note that it follows from (3) that $V(x) = 0$ if $x \in \nu^{-1}(0)$. Our assumptions imply that every connected component of $\mathbb{R}^n \backslash f^{-1}(0)$ is invariant under ν.

The following proposition provides a useful comparison between positive definite homogeneous functions with respect to an equilibrium set.

Proposition 2.1: Assume $V_1(\cdot)$ and $V_2(\cdot)$ are continuous real-valued functions on $\mathbb{R}^n$, homogeneous with respect to ν of degrees $l_1 > 0$ and $l_2 > 0$, respectively, and $V_1(\cdot)$ satisfies $V_1(x) = 0$ for $x \in \nu^{-1}(0)$ and $V_1(x) > 0$ for $x \in \mathbb{R}^n \backslash \nu^{-1}(0)$. Then for each $x_e \in \nu^{-1}(0)$ and each bounded open neighborhood $\mathcal{D}_0$ containing x_e, there exist $c_1 = c_1(\mathcal{D}_0) \in \mathbb{R}$ and $c_2 = c_2(\mathcal{D}_0) \in \mathbb{R}$, where $c_2 \geq c_1$, such that

$$c_1(V_1(x))^{\frac{l_2}{l_1}} \leq V_2(x) \leq c_2(V_1(x))^{\frac{l_2}{l_1}}, \quad x \in \mathcal{D}_0. \tag{4}$$

If, in addition, $V_2(x) = 0$ for $x \in \nu^{-1}(0)$ and $V_2(x) < 0$ for $x \in \mathbb{R}^n \backslash \nu^{-1}(0)$, then $c_2 \geq c_1 > 0$.

The *Lie derivative* of a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to ν is given by $L_\nu V(x) \triangleq \lim_{t\to 0^+} \frac{1}{t}[V(\psi(t, x)) - V(x)]$, whenever the limit on the right-hand side exists. If V is a continuous homogeneous function of degree $l > 0$, then $L_\nu V$ is defined everywhere and satisfies $L_\nu V = lV$. We assume that the vector field ν is a *semi-Euler vector field*, that is, the dynamical system

$$\dot{y}(t) = -\nu(y(t)), \quad y(0) = y_0, \quad t \geq 0, \tag{5}$$

is globally semistable. Thus, for each $x \in \mathbb{R}^n$, $\lim_{\tau\to\infty} \psi(-\tau, x) = x^* \in \nu^{-1}(0)$, and for each $x_e \in \nu^{-1}(0)$, there exists $z \in \mathbb{R}^n$ such that $x_e = \lim_{\tau\to\infty} \psi(-\tau, z)$. If $\nu^{-1}(0) = \{0\}$, then the semi-Euler vector field becomes the *Euler vector field* given in [10]. Finally, we say that the vector field f is *homogeneous of degree* $k \in \mathbb{R}$ *with respect to* ν if and only if $\nu^{-1}(0) = f^{-1}(0)$ and, for every $t \in \overline{\mathbb{R}}_+$ and $\tau \in \mathbb{R}$,

$$s_t \circ \psi_\tau = \psi_\tau \circ s_{e^{k\tau}t}. \tag{6}$$

Note that if $V : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous function of degree l such that $L_f V(x)$ is defined everywhere, then

$L_f V(x)$ is a homogeneous function of degree $l+k$. Finally, note that if ν and f are continuously differentiable in a neighborhood of $x \in \mathbb{R}^n$, then (6) holds at x for sufficiently small t and τ if and only if $[\nu, f](x) = kf(x)$ in a neighborhood of $x \in \mathbb{R}^n$, where the Lie bracket $[\nu, f]$ of ν and f can be computed using $[\nu, f] = \frac{\partial f}{\partial x}\nu - \frac{\partial \nu}{\partial x} f$.

III. Semistability and Homogeneous Dynamical Systems

Homogeneity of dynamical systems is a property whereby system vector fields scale in relation to a scaling operation or *dilation* on the state space. In this section, we present a robustness result of a vector field that can be written as a sum of several vector fields, each of which is homogeneous with respect to a certain fixed dilation. First, however, we present a result that shows that a semistable homogeneous system admits a homogeneous Lyapunov function. This is a weaker version of Theorem 6.2 of [10] which considers asymptotically stable homogeneous systems.

Theorem 3.1 ([2]): Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree $k \in \mathbb{R}$ with respect to ν and (1) is semistable under f. Then for every $l > \max\{-k, 0\}$, there exists a continuous nonnegative function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ that is homogeneous of degree l with respect to ν, continuously differentiable on $\mathbb{R}^n \backslash f^{-1}(0)$, $V^{-1}(0) = f^{-1}(0)$, and $V'(x)f(x) < 0$ for $x \in \mathbb{R}^n \backslash f^{-1}(0)$.

Next, we state the main theorem of this section involving a robustness result of a vector field that can be written as a sum of several vector fields.

Theorem 3.2: Let $f = g_1 + \cdots + g_p$, where, for each $i = 1, \ldots, p$, the vector field g_i is continuous, homogeneous of degree m_i with respect to ν, and $m_1 < m_2 < \cdots < m_p$. If every equilibrium point in $g_1^{-1}(0)$ is semistable under g_1 and is Lyapunov stable under f, then every equilibrium point in $g_1^{-1}(0)$ is semistable under f.

Proof. Let every point in $g_1^{-1}(0)$ be a semistable equilibrium under g_1. Choose $l > \max\{-m_1, 0\}$. Then it follows from Theorem 3.1 that there exists a continuous homogeneous function $V : \mathbb{R}^n \to \mathbb{R}$ of degree l such that $V(x) = 0$ for $x \in g_1^{-1}(0)$, $V(x) > 0$ for $x \in \mathbb{R}^n \backslash g_1^{-1}(0)$, and $L_{g_1} V$ satisfies $L_{g_1} V(x) = 0$ for $x \in g_1^{-1}(0)$ and $L_{g_1} V(x) < 0$ for $x \in \mathbb{R}^n \backslash g_1^{-1}(0)$. For each $i \in \{1, \ldots, p\}$, $L_{g_i} V$ is continuous and homogeneous of degree $l + m_i > 0$ with respect to ν. Let $x_\mathrm{e} \in g_1^{-1}(0)$ and $\mathcal{U}$ be a bounded neighborhood of x_e. Then it follows from Proposition 2.1 and Theorem 3.1 that there exist $c_1 > 0$, $c_2, \ldots, c_p \in \mathbb{R}$ such that

$$L_{g_i} V(x) \leq -c_i (V(x))^{\frac{l+m_i}{l}}, \quad x \in \mathcal{U}, \quad i = 1, \ldots, p. \quad (7)$$

Hence, for every $x \in \mathcal{U}$,

$$\begin{aligned} L_f V(x) &\leq -\sum_{i=1}^{p} c_i (V(x))^{\frac{l+m_i}{l}} \\ &= (V(x))^{\frac{l+m_1}{l}} (-c_1 + U(x)), \end{aligned} \quad (8)$$

where $U(x) \triangleq -\sum_{i=2}^{p} c_i (V(x))^{\frac{m_i - m_1}{l}}$.

Since $m_i - m_1 > 0$ for every $i \geq 2$, it follows that the function $U(\cdot)$, which takes the value 0 at the set $g_1^{-1}(0) \cap \mathcal{U}$, is continuous. Hence, for $x_\mathrm{e} \in g_1^{-1}(0)$, there exists an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of x_e such that $U(x) < c_1/2$. Now, it follows from (8) that $L_f V(x) \leq -\frac{c_1}{2}(V(x))^{\frac{l+m_1}{l}}$, $x \in \mathcal{V}$. Since x_e is Lyapunov stable, it follows that one can find a bounded neighborhood $\mathcal{W}$ of x_e such that solutions in $\mathcal{W}$ remain in $\mathcal{V}$. Take an initial condition in $\mathcal{W}$. Since the solution is bounded (remains in $\mathcal{U}$), it follows from the Krasovskii-LaSalle invariance theorem that this solution converges to its compact positive limit set in $f^{-1}(0)$. Since all points in $f^{-1}(0)$ are Lyapunov stable, it follows from Proposition 5.4 of [4] that the positive limit set is a singleton involving a Lyapunov stable equilibrium in $f^{-1}(0)$. Since x_e was chosen arbitrarily, it follows that all equilibria in $g_1^{-1}(0)$ are semistable. □

IV. Robust Control Algorithms for Network Consensus Protocols

In this section, we apply the results of Section III to develop robust control algorithms for the consensus problem in dynamical networks [1]. The information consensus problem appears frequently in coordination of multiagent systems and involves finding a dynamic algorithm that enables a group of agents in a network to agree upon certain quantities of interest with directed information flow. In this paper, we use undirected and directed graphs to represent a nonlinear dynamical network and present solutions to the consensus problem for nonlinear networks with both graph *topologies* (or information flows) [1]. Specifically, let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a *directed graph* (or digraph) denoting the dynamical network (or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1, \ldots, q\}$ involving a finite nonempty set denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and an *adjacency matrix* $\mathcal{A} \in \mathbb{R}^{q \times q}$ such that $\mathcal{A}_{(i,j)} = 1$, $i, j = 1, \ldots, q$, if $(j, i) \in \mathcal{E}$, and 0 otherwise. The edge $(j, i) \in \mathcal{E}$ denotes that agent $\mathcal{G}_j$ can obtain information from agent $\mathcal{G}_i$, but not necessarily vice versa. Moreover, we assume that $\mathcal{A}_{(i,i)} = 0$ for all $i \in \mathcal{V}$. A *graph* or *undirected graph* $\mathfrak{G}$ associated with the adjacency matrix $\mathcal{A} \in \mathbb{R}^{q \times q}$ is a directed graph for which the *arc set* is symmetric, that is, $\mathcal{A} = \mathcal{A}^\mathrm{T}$. A graph $\mathfrak{G}$ is *balanced* if $\sum_{j=1}^{q} \mathcal{A}_{(i,j)} = \sum_{j=1}^{q} \mathcal{A}_{(j,i)}$ for all $i = 1, \ldots, q$. Finally, we denote the *value* of the node $i \in \{1, \ldots, q\}$ at time t by $x_i(t) \in \mathbb{R}$. The consensus problem involves the design of a dynamic algorithm that guarantees information state equipartition, that is, $\lim_{t \to \infty} x_i(t) = \alpha \in \mathbb{R}$ for $i = 1, \ldots, q$.

The consensus problem is a dynamic graph involving the trajectories of the dynamical network characterized by the multiagent dynamical system $\mathcal{G}$ given by

$$\dot{x}_i(t) = \sum_{j=1,\, j \neq i}^{q} \phi_{ij}(x_i(t), x_j(t)), \quad (9)$$

where $x_i(0) = x_{i0}$, $t \geq 0$, $i = 1, \ldots, q$, or, in vector form,

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (10)$$

where $x(t) \triangleq [x_1(t), \ldots, x_q(t)]^\mathrm{T}$, $t \geq 0$, and $f = [f_1, \ldots, f_q]^\mathrm{T} : \mathcal{D} \to \mathbb{R}^q$ is such that $f_i(x) = \sum_{j=1,\, j \neq i}^{q} \phi_{ij}(x_i, x_j)$, where $\mathcal{D} \subseteq \mathbb{R}^q$ is open. Here, $x_i(t)$, $t \geq 0$, represents an *information state* and $f_i(t) = u_i(t)$ is a distributed consensus algorithm involving neighbor-to-neighbor interaction between agents. This nonlinear model is proposed in [11] and is called a *power balance equation*. Here, however, we address a more general model in that $\phi_{ij}(\cdot, \cdot)$ has no special structure and x need not be constrained to the nonnegative orthant of the state space. For the statement of the main results of this section the following definition is needed.

Definition 4.1 ([12]): A directed graph $\mathfrak{G}$ is *strongly connected* if for any ordered pair of vertices (i,j), $i \neq j$, there exists a *path* (i.e., sequence of arcs) leading from i to j.

Recall that $\mathcal{A} \in \mathbb{R}^{q\times q}$ is *irreducible*, that is, there does not exist a permutation matrix such that $\mathcal{A}$ is cogredient to a lower-block triangular matrix, if and only if $\mathfrak{G}$ is strongly connected (see Theorem 2.7 of [12]).

Assumption 1: For the *connectivity matrix* $\mathcal{C} \in \mathbb{R}^{q\times q}$ associated with the multiagent dynamical system $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)} = \begin{cases} 0, & \text{if } \phi_{ij}(x) \equiv 0, \\ 1, & \text{otherwise}, \end{cases} \quad (11)$$

for $i \neq j$, $i,j = 1,\ldots,q$, and $\mathcal{C}_{(i,i)} = -\sum_{k=1,\,k\neq i}^{q} \mathcal{C}_{(i,k)}$, $i = 1,\ldots,q$, $\operatorname{rank}\mathcal{C} = q-1$, and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\phi_{ij}(x_i,x_j) = 0$ if and only if $x_i = x_j$.

Assumption 2: For $i,j = 1,\ldots,q$, $(x_i - x_j)\phi_{ij}(x_i,x_j) \leq 0$, $x_i, x_j \in \mathbb{R}$.

For further details on Assumptions 1 and 2, see [11]. For the statement of the next result, let $\mathbf{e} \in \mathbb{R}^q$ denote the ones vector of order q, that is, $\mathbf{e} \triangleq [1,\ldots,1]^{\mathrm{T}}$.

Theorem 4.1 ([2]): Consider the multiagent dynamical system (10) and assume that Assumptions 1 and 2 hold. Then the following statements hold:

i) Assume that $\phi_{ij}(x_i,x_j) = -\phi_{ji}(x_j,x_i)$ for all $i,j = 1,\ldots,q$, $i \neq j$. Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (10). Furthermore, $x(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_0$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_0$ is a semistable equilibrium state.

ii) Let $\phi_{ij}(x_i,x_j) = \mathcal{C}_{(i,j)}[\sigma(x_j) - \sigma(x_i)]$ for all $i,j = 1,\ldots,q$, $i \neq j$, where $\sigma(0) = 0$ and $\sigma(\cdot)$ is strictly increasing, and assume that $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$. Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (10). Furthermore, $x(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_0$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_0$ is a semistable equilibrium state.

Theorem 4.1 implies that the steady-state value of the information state in each agent $\mathcal{G}_i$ of the multiagent dynamical system $\mathcal{G}$ is equal, that is, the steady-state value of the multiagent dynamical system $\mathcal{G}$ given by $x_\infty = \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_0 = \left[\frac{1}{q}\sum_{i=1}^{q} x_{i0}\right]\mathbf{e}$ is uniformly distributed over all multiagents of $\mathcal{G}$. This phenomenon is known as *equipartition of energy* [11] in system thermodynamics and *information consensus* or *protocol agreement* [1] in cooperative network dynamical systems.

Next, consider q continuous-time integrator agents with dynamics

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad (12)$$

where for each $i \in \{1,\ldots,q\}$, $x_i(t) \in \mathbb{R}$ denotes the information state and $u_i(t) \in \mathbb{R}$ denotes the information control input for all $t \geq 0$. The consensus protocol is given by

$$u_i(t) = f_i(x(t)) = \sum_{j=1, j\neq i}^{q} \phi_{ij}(x_i(t), x_j(t)), \quad (13)$$

where $\phi_{ij}(\cdot,\cdot)$ satisfies the conditions in Theorem 4.1. Note that (12) and (13) describes an interconnected network where information states are updated using a distributed controller involving neighbor-to-neighbor interaction between agents. We assume that the vector field $f = [f_1,\ldots,f_q]$ is homogeneous of degree $k \in \mathbb{R}$ with respect to ν. Finally, consider the generalized (or perturbed) consensus protocol architecture

$$\dot{z}_i(t) = \sum_{j=1, j\neq i}^{q} \phi_{ij}(z_i(t), z_j(t)) + \Delta_i(z),$$
$$z_i(0) = z_{i0}, \quad i = 1,\ldots,q, \quad t \geq 0, \quad (14)$$

where $\Delta = [\Delta_1,\ldots,\Delta_q]^{\mathrm{T}} : \mathbb{R}^q \to \mathbb{R}$ is a continuous function such that Δ is homogeneous of degree $l \in \mathbb{R}$ with respect to ν and (14) possesses unique solutions in forward time for initial conditions in $\mathbb{R}^q \backslash \{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$.

Theorem 4.2: Consider the nominal consensus protocol (12) and (13), and the generalized consensus protocol (14). If $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\} = \Delta^{-1}(0)$, every equilibrium point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$ is a Lyapunov stable equilibrium of (14), and $k < l$, then every equilibrium point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$ is a semistable equilibrium of (12) and (13), and (14).

Proof. It follows from Proposition 5.1 of [2] that for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is an equilibrium point of (12) and (13). Next, it follows from Theorem 4.1 that $\alpha\mathbf{e}$ is a semistable equilibrium state of (12) and (13). Now, the result is a direct consequence of Theorem 3.2. □

As a special case of Theorem 4.2, consider the nominal linear consensus protocol given by

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[x_j(t) - x_i(t)],$$
$$x_i(0) = x_{i0}, \quad i = 1,\ldots,q, \quad t \geq 0, \quad (15)$$

where for each $i \in \{1,\ldots,q\}$, $x_i \in \mathbb{R}$, $\mathcal{C}$ satisfies Assumption 1, and $\mathcal{C}^{\mathrm{T}} = \mathcal{C}$. Next, consider the generalized consensus protocol given by

$$\dot{z}_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[z_j(t) - z_i(t)] + \sum_{j=1, j\neq i}^{q} \delta_{ij}(z_j(t) - z_i(t)), \quad (16)$$

where $z_i(0) = z_{i0}$, $i = 1,\ldots,q$, $t \geq 0$, and assume $\Delta = [\Delta_1,\ldots,\Delta_q]^{\mathrm{T}}$, $\Delta_i = \sum_{j=1,j\neq i}^{q} \delta_{ij}(z_j(t) - z_i(t))$, is homogeneous of degree $l > 0$ with respect to $\nu(x) = -\sum_{i=1}^{q}\left[\sum_{j=1,j\neq i}^{q}(x_j - x_i)\right]\frac{\partial}{\partial x_i}$, $i,j = 1,\ldots,q$, $i \neq j$. Furthermore, assume $\delta_{ij} : \mathbb{R} \to \mathbb{R}$ satisfies $\delta_{ij} \equiv 0$ if $\mathcal{C}_{(i,j)} = 0$, $\delta_{ij}(\lambda z) = \lambda^{1+r}\delta_{ij}(z)$ for all $\lambda > 0$ and for some $r > 0$, and $\delta_{ij}(z) = -\delta_{ji}(-z)$ for $z \in \mathbb{R}$ and $i,j = 1,\ldots,q$, $i \neq j$.

Lemma 4.1: The vector field of (15) is homogeneous of degree $k = 0$ with respect to the semi-Euler vector field $\nu(x) = -\sum_{i=1}^{q}\left[\sum_{j=1,j\neq i}^{q}(x_j - x_i)\right]\frac{\partial}{\partial x_i}$.

Corollary 4.1: Consider the linear nominal consensus protocol (15) and the generalized nonlinear consensus protocol (16). Then every equilibrium point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$ is a semistable equilibrium of (15) and (16). Furthermore, $z(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}z_0$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}z_0$ is a semistable equilibrium state.

Proof. It follows from *i)* of Theorem 4.1 that $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, is a semistable equilibrium of (15). Next, it follows from Lemma 4.1 that the right-hand side of (15) is homogeneous of degree $k = 0$ with respect to the semi-Euler vector field $\nu(x) = -\sum_{i=1}^{q}\left[\sum_{j=1,j\neq i}^{q}(x_j - x_i)\right]\frac{\partial}{\partial x_i}$. To show that every point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$ is a Lyapunov stable equilibrium of (16), consider the Lyapunov function candidate given by $V(z - \alpha\mathbf{e}) = \frac{1}{2}\|z - \alpha\mathbf{e}\|^2$. Then it follows that

$$\begin{aligned}\dot{V}(z-\alpha\mathbf{e}) &= (z-\alpha\mathbf{e})^{\mathrm{T}}\dot{z}\\ &= \sum_{i=1}^{q}(z_i-\alpha)\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}[z_j-z_i]\\ &\quad+\sum_{i=1}^{q}(z_i-\alpha)\sum_{j=1,j\neq i}^{q}\delta_{ij}(z_j-z_i)\\ &= -\sum_{i=1}^{q}\sum_{j=i+1}^{q-1}\mathcal{C}_{(i,j)}[z_i-z_j]^2\\ &\quad+\sum_{i=1}^{q}\sum_{j=i+1}^{q-1}\mathcal{C}_{(i,j)}[z_i-z_j]\delta_{ij}(z_j-z_i),\\ &\qquad z\in\mathbb{R}^q. \qquad (17)\end{aligned}$$

Next, since, by homogeneity of δ_{ij}, $\delta_{ij}(\cdot)$ is such that $\lim_{z\to 0}\delta_{ij}(z)/z = 0$, it follows that for every $\gamma > 0$, there exists $\varepsilon_{ij} > 0$ such that $|\delta_{ij}(z)| \le \gamma|z|$ for all $|z| < \varepsilon_{ij}$. Hence,

$$\begin{aligned}&\sum_{i=1}^{q}\sum_{j=i+1}^{q-1}\mathcal{C}_{(i,j)}[z_i-z_j]\delta_{ij}(z_j-z_i)\\ &\le\sum_{i=1}^{q}\sum_{j=i+1}^{q-1}\gamma\mathcal{C}_{(i,j)}[z_i-z_j]^2,\quad |z_i-z_j|<\varepsilon_{ij}. \qquad (18)\end{aligned}$$

Now, choosing $\gamma \le 1$, it follows from (17) and (18) that

$$\begin{aligned}\dot{V}(z-\alpha\mathbf{e}) &\le -\sum_{i=1}^{q}\sum_{j=i+1}^{q-1}(1-\gamma)\mathcal{C}_{(i,j)}[z_i-z_j]^2\\ &\le 0,\quad |z_i-z_j|<\varepsilon_{ij}, \qquad (19)\end{aligned}$$

which establishes Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$. Now, the result follows from Theorem 4.2. □

It is important to note that Corollary 4.1 still holds for the case where the generalized consensus protocol has a hierarchical structure of the form

$$\dot{z}(t) = \mathcal{C}z(t) + \sum_{i=1}^{p} g_i(z(t)), \quad z(0) = z_0, \quad t \ge 0, \qquad (20)$$

where for each $i \in \{1,\ldots,q\}$, $g_i(z)$ is homogeneous of degree $l_i > 0$ with respect to $\nu(x) = -\sum_{i=1}^{q}[\sum_{j=1,j\neq i}^{q}(x_j - x_i)]\frac{\partial}{\partial x_i}$ and $l_1 < \cdots < l_p$. As an application of Corollary 4.1, consider the Kuramoto model [13] given by

$$\begin{aligned}&\dot{x}_1(t) = \sin(x_2(t) - x_1(t)),\ x_1(0) = x_{10},\ t \ge 0, \qquad (21)\\ &\dot{x}_2(t) = \sin(x_1(t) - x_2(t)),\ x_2(0) = x_{20}. \qquad (22)\end{aligned}$$

Note that for sufficiently small x, $\sin x$ can be approximated by $x - x^3/3! + \cdots + (-1)^{p-1}x^{2p-1}/(2p-1)!$, where p is a positive integer. The truncated system associated with (21) and (22) is given by

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1 - \frac{1}{3!}(x_2-x_1)^3 + \cdots\\ &\quad+\frac{(-1)^{p-1}}{(2p-1)!}(x_2-x_1)^{2p-1}, \qquad (23)\\ \dot{x}_2 &= x_1 - x_2 - \frac{1}{3!}(x_1-x_2)^3 + \cdots\\ &\quad+\frac{(-1)^{p-1}}{(2p-1)!}(x_1-x_2)^{2p-1}, \qquad (24)\end{aligned}$$

or, equivalently,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \sum_{i=1}^{p-1} g_i(x_1,x_2), \qquad (25)$$

where for $i = 1,\ldots,p-1$,

$$g_i(x_1,x_2) \triangleq \frac{(-1)^i}{(2i+1)!}\begin{bmatrix}(x_2-x_1)^{2i+1}\\ (x_1-x_2)^{2i+1}\end{bmatrix}. \qquad (26)$$

It can be easily shown that all the conditions of Corollary 4.1 hold for (25). Hence, it follows from Corollary 4.1 that every equilibrium point in $\{\alpha[1,1]^{\mathrm{T}} : \alpha \in \mathbb{R}\}$ is a local semistable equilibrium of (23) and (24), which implies that the equilibrium set $\{\alpha[1,1]^{\mathrm{T}} : \alpha \in \mathbb{R}\}$ of (23) and (24) has the same stability properties as the linear nominal system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \qquad (27)$$

Note that Corollary 4.1 deals with the undirected graph $\mathfrak{G} = (\mathcal{V},\mathcal{E},\mathcal{A})$, where $\mathcal{A}$ is a symmetric adjacency matrix. Next, we consider the case where $\mathfrak{G}$ is a directed graph. The following lemma is needed for the next result.

Lemma 4.2: Let $A \in \mathbb{R}^{q\times q}$ and $A_{\mathrm{d}i} \in \mathbb{R}^{q\times q}$, $i = 1,\ldots,n_{\mathrm{d}}$, be given by

$$A_{(i,j)} = \begin{cases}\mathcal{C}_{(i,i)}, & i = j,\\ 0, & i \neq j,\end{cases}\quad A_{\mathrm{d}(i,j)} = \begin{cases}0, & i = j,\\ \mathcal{C}_{(i,j)}, & i \neq j,\end{cases}\quad i,j = 1,\ldots,q, \qquad (28)$$

where $A_{\mathrm{d}} \triangleq \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}$. Assume that $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$. Then there exist nonnegative definite matrices $Q_i \in \mathbb{R}^{q\times q}$, $i = 1,\ldots,n_{\mathrm{d}}$, such that

$$2A + \sum_{i=1}^{n_{\mathrm{d}}}(Q_i + A_{\mathrm{d}i}^{\mathrm{T}}Q_i^{\#}A_{\mathrm{d}i}) \le 0. \qquad (29)$$

Theorem 4.3: Consider the linear nominal consensus protocol (15), where $\mathcal{C}$ satisfies Assumption 1 and $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$, and the generalized nonlinear consensus protocol given by

$$\begin{aligned}\dot{z}_i(t) &= \sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}[z_j(t)-z_i(t)]\\ &\quad+\sum_{j=1,j\neq i}^{q}\mathcal{H}_{(i,j)}[\sigma(z_j(t))-\sigma(z_i(t))],\\ &\quad z_i(0) = z_{i0},\quad i = 1,\ldots,q,\quad t\ge 0, \qquad (30)\end{aligned}$$

where $\sigma(\cdot)$ satisfies $\sigma(0) = 0$ and $\sigma(\cdot)$ is strictly increasing. Furthermore, assume the matrix $\mathcal{H} = [\mathcal{H}_{(i,j)}]$ satisfies Assumption 1, $\mathcal{H}^{\mathrm{T}}\mathbf{e} = 0$, $\mathcal{H}_{(i,j)} = 0$ whenever $\mathcal{C}_{(i,j)} = 0$, $i,j = 1,\ldots,q$, $i \neq j$, and $\mathcal{H} = \mathcal{C} - \mathcal{L}$, where $\mathcal{L}^{\mathrm{T}} = \mathcal{L} \in \mathbb{R}^{q\times q}$. Then every equilibrium point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$

is a semistable equilibrium of (15) and (30). Furthermore, $z(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}z_0$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}z_0$ is a semistable equilibrium state.

Proof. It follows from ii) of Theorem 4.1 that $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, is a semistable equilibrium of (15). Next, note that (30) can be rewritten as

$$\begin{aligned}\dot{z}_i(t) &= \sum_{j=1, j\neq i}^{q} \mathcal{H}_{(i,j)}[(z_j(t)+\sigma(z_j(t))) - (z_i(t) \\ &\quad +\sigma(z_i(t)))] + \sum_{j=1,j\neq i}^{q}\mathcal{L}_{(i,j)}[\sigma(z_j(t)) \\ &\quad -\sigma(z_i(t))],\ z_i(0)=z_{i0},\ i=1,\ldots,q,\ t\geq 0.\end{aligned}$$

Define $\hat{\sigma} : \mathbb{R}^q \to \mathbb{R}^q$ by $\hat{\sigma}(z) \triangleq [\sigma(z_1), \ldots, \sigma(z_q)]^{\mathrm{T}}$. Now, it follows from Lemma 4.2 that there exist nonnegative definite matrices $Q_i \in \mathbb{R}^{q\times q}$, $i = 1, \ldots, q$, such that

$$2C + \sum_{i=1}^{q}(Q_i + C_{\mathrm{d}i}^{\mathrm{T}}Q_i^{\#}C_{\mathrm{d}i}) \leq 0, \tag{31}$$

where $C \in \mathbb{R}^{q\times q}$ and $C_{\mathrm{d}i} \in \mathbb{R}^{q\times q}$, $i = 1, \ldots, q$, are given by

$$C_{(i,j)} = \begin{cases} \mathcal{H}_{(i,i)}, & i=j, \\ 0, & i\neq j, \end{cases} \quad C_{\mathrm{d}(i,j)} = \begin{cases} 0, & i=j, \\ \mathcal{H}_{(i,j)}, & i\neq j, \end{cases} \quad i,j=1,\ldots,q, \tag{32}$$

where $C_{\mathrm{d}} \triangleq \sum_{i=1}^{q} C_{\mathrm{d}i}$.

To show that every point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$ is Lyapunov stable, consider the Lyapunov function candidate given by

$$V(z-\alpha\mathbf{e}) = \|z-\alpha\mathbf{e}\|^2 + 2\sum_{i=1}^{q}\int_{\alpha}^{z_i}[\sigma(\theta)-\sigma(\alpha)]\mathrm{d}\theta. \tag{33}$$

Now, the derivative of $V(z - \alpha\mathbf{e})$ along the trajectories of (30) is given by

$$\begin{aligned}&\dot{V}(z-\alpha\mathbf{e}) \\ &\quad = 2[z-\alpha+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]^{\mathrm{T}}C[z-\alpha+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})] \\ &\qquad +2\sum_{i=1}^{q}[z-\alpha+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]^{\mathrm{T}}C_{\mathrm{d}i} \\ &\qquad \cdot[z-\alpha+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})] \\ &\qquad +2\sum_{i=1}^{q}[z_i-\alpha+\sigma(z_i)-\sigma(\alpha)] \\ &\qquad \cdot\sum_{j=1,j\neq i}^{q}\mathcal{L}_{(i,j)}[\sigma(z_j)-\sigma(z_i)] \\ &\quad \leq -\sum_{i=1}^{q}(-Q_i[z-\alpha+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})] \\ &\qquad +C_{\mathrm{d}i}[z-\alpha+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})])^{\mathrm{T}}Q_i^{\#} \\ &\qquad \cdot(-Q_i[z-\alpha+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})] \\ &\qquad +C_{\mathrm{d}i}[z-\alpha+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]) \\ &\qquad -\sum_{i=1}^{q}[z-\alpha+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]^{\mathrm{T}}C_{\mathrm{d}i}^{\mathrm{T}}Q_i^{\#}C_{\mathrm{d}i} \\ &\qquad \cdot[z-\alpha+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})] \\ &\qquad -2\sum_{i=1}^{q}\sum_{j=i+1}^{q-1}\mathcal{L}_{(i,j)}(z_i-z_j)[\sigma(z_i)-\sigma(z_j)] \\ &\qquad -2\sum_{i=1}^{q}\sum_{j=i+1}^{q-1}\mathcal{L}_{(i,j)}[\sigma(z_i)-\sigma(z_j)]^2 \\ &\quad \leq 0, \quad z\in\mathbb{R}^q,\end{aligned} \tag{34}$$

which establishes Lyapunov stability of $\alpha\mathbf{e}$.

Next, let $\mathcal{R} \triangleq \{x \in \mathbb{R}^q : -Q_i[x+\hat{\sigma}(x)]+C_{\mathrm{d}i}[x+\hat{\sigma}(x)] = 0, i = 1, \ldots, q\}$. Then it follows from the Krasovskii-LaSalle invariant set theorem that $x(t) \to \mathcal{M}$ as $t \to \infty$, where $\mathcal{M}$ denotes the largest invariant set contained in $\mathcal{R}$. Now, since $C + \sum_{i=1}^{q} Q_i = 0$, it follows that $\mathcal{R} \subseteq \hat{\mathcal{R}} \triangleq \{x \in \mathbb{R}^q : C\hat{\sigma}(x) + \sum_{i=1}^{q} C_{\mathrm{d}i}\hat{\sigma}(x) = 0\}$. Hence, since $C + \sum_{i=1}^{q} C_{\mathrm{d}i} = \mathcal{H}$, $\operatorname{rank}\mathcal{H} = q - 1$, and $\mathcal{H}\mathbf{e} = 0$, it follows that the largest invariant set $\hat{\mathcal{M}}$ contained in $\hat{\mathcal{R}}$ is given by $\hat{\mathcal{M}} = \{x \in \mathbb{R}^q : x = \alpha\mathbf{e}, \alpha \in \mathbb{R}\}$. Furthermore, since $\hat{\mathcal{M}} \subseteq \mathcal{R} \subseteq \hat{\mathcal{R}}$, it follows that $\mathcal{M} = \hat{\mathcal{M}}$. Next, note that $\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{M}) = 0$. Let $x_{\mathrm{e}} \in \mathcal{M}$. Choosing x_0 sufficiently close to x_{e}, it follows from Lyapunov stability of x_{e} that trajectories of (30) starting sufficiently close to x_{e} are bounded, and hence, the positive limit set of (30) is nonempty. Since every point in $\mathcal{M}$ is Lyapunov stable, it follows from Proposition 5.4 of [4] that $\lim_{t\to\infty} x(t) = x^*$, where $x^* \in \mathcal{M}$ is Lyapunov stable. Hence, it follows that every equilibrium point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$ is a semistable equilibrium of (15) and (30). □

REFERENCES

[1] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, pp. 1520–1533, 2004.

[2] Q. Hui, W. M. Haddad, and S. P. Bhat, "Finite-time semistability theory with applications to consensus protocols in dynamical networks," in *Proc. Amer. Control Conf.*, New York, NY, 2007, pp. 2411–2416.

[3] Q. Hui and W. M. Haddad, "Continuous and hybrid distributed control for multiagent coordination: Consensus, flocking, and cyclic pursuit," in *Proc. Amer. Control Conf.*, New York, NY, 2007, pp. 2576–2581.

[4] S. P. Bhat and D. S. Bernstein, "Nontangency-based Lyapunov tests for convergence and stability in systems having a continuum of equilibra," *SIAM J. Control Optim.*, vol. 42, pp. 1745–1775, 2003.

[5] ——, "Arc-length-based Lyapunov tests for convergence and stability in systems having a continuum of equilibria," in *Proc. Amer. Control Conf.*, Denver, CO, 2003, pp. 2961–2966.

[6] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 50, pp. 169–182, 2005.

[7] P.-A. Bliman and G. Ferrari-Trecate, "Average consensus problems in networks of agents with delayed communications," in *Proc. IEEE Conf. Decision and Control*, Seville, Spain, 2005, pp. 7066–7071.

[8] V. Gupta, C. Langbort, and R. M. Murray, "On the robustness of distributed algorithms," in *Proc. IEEE Conf. Decision and Control*, San Diego, CA, 2006, pp. 3473–3478.

[9] Q. Hui, W. M. Haddad, and S. P. Bhat, "Lyapunov and converse Lyapunov theorems for semistability," in *Proc. IEEE Conf. Decision and Control*, New Orleans, LA, 2007, pp. 5870–5874.

[10] S. P. Bhat and D. S. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Math. Control Signals Syst.*, vol. 17, pp. 101–127, 2005.

[11] W. M. Haddad, V. Chellaboina, and S. G. Nersesov, *Thermodynamics: A Dynamical Systems Approach*. Princeton, NJ: Princeton Univ. Press, 2005.

[12] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. New York: Academic, 1979.

[13] S. H. Strogatz, "From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators," *Physica D*, vol. 143, pp. 1–20, 2000.