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A Comparison Theorem for Cooperative Control of Nonlinear Systems

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Abstract— **Asymptotic cooperative stability is studied in the paper, and explicit conditions are found for heterogeneous nonlinear systems to reach a consensus. Specifically, a new comparison theorem is proposed for concluding both cooperative stability and Lyapunov stability, and it is in terms of vector nonlinear differential inequalities (on Lyapunov function components). It is unique that the proposed result admits both heterogeneous dynamics of nonlinear systems and intermittent unpredictable changes in their associated sensing/communication network. Its proof is done using a combination of Lyapunov argument (in terms of the Lyapunov function components) and topology-dependent argument (in terms of structural properties of reducible matrices). Consequently, the proposed result does not impose any of the following assumptions required in the existing results: the knowledge of a successful Lyapunov function, system dynamics being convex, nonsmooth analysis, fixed or certain types of communication patterns, quasi-monotone property on differential inequalities. If the systems under consideration are all linear, the theorem reduces to the necessary and sufficient condition of cooperative controllability obtained using the matrix-theoretical approach, and the inequalities become equalities. For nonlinear systems, the proposed cooperative stability conditions are straightforward to verify. Several types of nonlinear systems are used as examples to illustrate application potentials of the comparison theorem in both cooperative stability analysis and cooperative control design.**

I. INTRODUCTION

In layman's language, a group of systems are called to be cooperative if the future behavior of a specific system corresponds in a certain way the behaviors of those systems in its sensing/communication range, and the corresponding feedback control is said to be cooperative control. The most distinctive feature of cooperative control design is that feedbacks are intermittently available and the changes cannot be predicted. As such, stability analysis must be done for the general case of uncertain changes. The basic setting of cooperative control is the consensus problem in which dynamical systems are desired to reach a common consensus value. To ensure convergence, certain connectivity condition over time on the sensing/communication network among the dynamical systems would be needed.

For linear dynamical systems, the consensus problem is essentially solved. It is shown in [2] that, for linear first-order integrator systems, the nearest neighboring rule [16] solves the consensus problem provided that their communication topology is characterized by an undirected and connected graph. This graph-based condition is relaxed in [13], [4] so that the changing network topologies over repeated intervals correspond to a directed graph either with strong connectivity or of a spanning tree. For heterogeneous linear systems, it is shown using a matrix-theoretical approach [11], [10] that output-feedback cooperative control can be designed if and only if the corresponding sensing/communication matrix sequence is sequentially complete.

For cooperative control of nonlinear systems, there are several results to deal with the following specific cases. Fixed patterns of information feedback or dynamics coupling among systems are qualitatively analyzed to study dynamical circuit networks [1], to design decentralized control [14], and to synchronize coupled oscillators with linear coupling [19], [18]. For discrete systems with convex dynamics, stability analysis is done using the combination of graph theory and discrete set-valued Lyapunov functions for time-varying topological patterns [7], and this result is also extended to continuous-time coupled nonlinear systems [5]. Should the communication pattern be time varying but bidirectional, Lyapunov function can be found to design cooperative control for nonlinear systems [12]. In [9], cooperative control of nonlinear systems is designed by employing state transformation and by extending the matrix theoretical approach through the use of Lyapunov function components, but the result reported involves nonlinear transformations and is limited to the case that sensing/communication matrix sequence is lower triangularly complete.

In this paper, we consider the most general case that the systems are heterogeneous and nonlinear and that their corresponding sensing/communication matrix sequence is sequentially complete while arbitrary otherwise. The major challenge of applying the standard nonlinear analysis methodologies is that successful Lyapunov function can only be found by a backward procedure [3], [8] and hence is too difficult to be found due to the combination of uncertain topological changes and nonlinear dynamics. To overcome this inherent difficulty, we choose to extend the approach of employing Lyapunov function components [9] by establishing a new comparison theorem. The theorem is proven using a combined Lyapunov and topology-dependent argument so that the resulting vector nonlinear differential inequalities admit heterogeneous dynamics of nonlinear systems as well as unknown intermittently changing sensing/communication network and that Lyapunov function of the overall system is not needed.

It is worth noting that the existing comparison theorem on vector differential inequalities [17] for concluding asymptotic stability requires the so-called quasi-monotone property. The proposed comparison theorem does not require such an assumption, which enriches the comparison theory. Compared

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to the existing results on consensus of nonlinear systems, the proposed theorem does not require that system dynamics be convex, does not involve any use of nonsmooth analysis, does not require any state transformation, and does not limit sensing/communication network to any specific patterns. And, it provides a general set of nonlinear inequalities that can be easily checked. Several examples are used to illustrate its applications in both analysis and design.

II. Problem Statement

Consider the following nonlinear heterogeneous systems:

$$\dot{z}_\mu = f_\mu(z_\mu) + g_\mu(z_\mu)u_\mu, \quad w_\mu = h_\mu(z_\mu), \tag{1}$$

where $\mu = 1, \cdots, q$; $z_\mu \in \Re^{n_\mu}$, $u_\mu \in \Re^m$ and $w_\mu \in \Re^m$ are the state, the control, the output of the μth system; and $n = n_1 + \cdots + n_q$.

The distinctive feature in the design of cooperative control u_i is that feedback from the sensing/communication network keeps changing intermittently and the changes are not known apriori. In the general case of a dynamically changing environment, information exchanges among the systems are captured by sensing/communication matrix $S(t)$, where

$$S(t) = \begin{bmatrix} 1 & s_{12}(t) & \cdots & s_{1q}(t) \\ s_{21}(t) & 1 & \cdots & s_{2q}(t) \\ \vdots & \vdots & \vdots & \vdots \\ s_{q1}(t) & s_{q2}(t) & \cdots & 1 \end{bmatrix} \tag{2}$$

$s_{ii} \equiv 1$, $s_{ij}(t) = 1$ if output $w_j(t)$ from the jth dynamical system is known to the ith system at time t, and $s_{ij}(t) = 0$ if otherwise. Over time, binary changes of $S(t)$ occur at an infinite sequence of time instants, denoted by $\{t_k : k \in \aleph\}$, and $S(t)$ is piecewise constant as $S(t) = S(t_k)$ for all $t \in [t_k, t_{k+1})$, where $\aleph \triangleq \{0, 1, \cdots, \infty\}$.

Time sequence $\{t_k : k \in \aleph\}$ and the values of $S(t_k)$ may not be known apriori and should be treated as uncertainties and handled in cooperative control design. At time t, the ith system gets whatever feedback information received and hence the binary values of $s_{ij}(t)$ (for $j = 1, \cdots, q$) become known. Accordingly, the following nonlinear control can be implemented:

$$u_i = U_i(s_{i1}(t)w_1(t), \cdots, s_{iq}(t)w_q(t)), \tag{3}$$

in which $s_{ij}(t)$ are necessarily included. In case that only the relative feedback information available, the cooperative control must be of form

$$u_i = U_i(s_{i1}(t)[w_1(t) - w_i(t)], \cdots, s_{iq}(t)[w_q(t) - w_i(t)]). \tag{4}$$

The fundamental problem studied in this paper is to analytically determine a set of stability conditions for systems (1) under cooperative control (3) or (4). The stability of our interest is whether systems (1) are cooperative in the sense that all their state variables reach the same value of consensus. In what follows, systems in (1) are said to be *cooperatively stable* if, for every $\epsilon > 0$, there exist non-empty set Ω_0 and constants $\delta > 0$ and $c \in \Re$ such that $z_\mu(t_0) \in \Omega_0$ and $\|z_\mu(t_0) - c\mathbf{1}\| \leq \delta$ imply $\|z_\mu(t) - c\mathbf{1}\| \leq \epsilon$ for all $t \geq t_0$ and for all μ, where $\mathbf{1}$ is the column vector of 1s. The systems are said to be *asymptotically cooperatively stable* if they are cooperatively stable and if $\lim_{t\to\infty} z_\mu(t) = c\mathbf{1}$. *

To ensure that the states remain bounded, Lyapunov stability is also of interest. If the systems are all asymptotically stable as $z_\mu \to 0$, the consensus at the origin is reached. Since asymptotic stability can be viewed as a special case of asymptotic cooperative stability and since existing results are available to check asymptotic stability, we will study in this paper the general cooperative stability problem in which limit c is not fixed.

III. Stability Result of Linear Cooperative Systems

For heterogeneous linear systems under linear cooperative control, the cooperative system becomes [11]:

$$\dot{x} = [-I + D(t)]x, \quad x(0) \text{ given}, \quad t \geq 0, \tag{5}$$

where $x = [\, z_1^T \ \cdots \ z_q^T \,]^T$, and $D(t)$ is the matrix combining sensing/communication matrix $S(t)$ with the dynamics of individual systems as well as linear cooperative control laws. For instance, if $n_i = 1$ and the average protocol is used,

$$d_{ij}(t) = \frac{s_{ij}(t)}{s_{i1}(t) + \cdots + s_{iq}(t)}.$$

In example 2 to be presented later, $n_q > 1$ and the corresponding matrix $D(t)$ is provided. A nonnegative and piecewise constant matrix such as $D(t)$ (or $S(t)$) has a canonical form $E_\searrow$ in the following lower-triangular expression [6]:

$$\mathcal{T}^T D \mathcal{T} = \begin{bmatrix} E_{11} & 0 & \cdots & 0 \\ E_{21} & E_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_{p1} & E_{p2} & \cdots & E_{pp} \end{bmatrix} \triangleq E_\searrow, \tag{6}$$

where $\mathcal{T}$ is a permutation matrix, $1 \leq p \leq n$, $E_{ii} \in \Re^{r_i \times r_i}$ are square and irreducible, and $r_1 + \cdots + r_p = n$. The structure reveals connectivity or grouping properties of the systems. If $p = 1$, $D(t)$ is said to be *irreducible*, and it is known that the corresponding graph is *strongly connected*. On the other hand, matrix $D(t)$ is said to be *reducible* if $p > 1$ in (6), which is the more general case. For any $p \geq 1$, matrix $D(t)$ is said to be *lower triangularly complete* and the corresponding graph has at least one *globally reachable node* if, in (6) and for every $i > 1$, there exists at least one $j < i$ such that $E_{ij} \neq 0$. Physically, matrix $D(t)$ being irreducible means all the systems are connected together as one group; and matrix $D(t)$ being lower triangularly complete means that those systems corresponding to block E_{11} act as the instantaneous leaders and the rest of systems follow the leaders.

Asymptotic cooperative stability of system (5) is determined by cumulative connectivity property of the network.

*In many of existing literature, systems in (1) and with $n_1 = \cdots = n_q$ are said to reach a consensus if $\lim_{t\to\infty} z_\mu(t) = c'$ for all μ and for $c' \in \Re^{n_1}$. In that case, the concept of cooperative stability can be applied to $z_{\mu j}$ for fixed j and for all $\mu = 1, \cdots, q$.

Mathematically, cumulative connectivity over consecutive time intervals is described by the following binary product: for any given subsequence $\{k'_v : v \in \aleph\}$ of $\aleph$,

$$S_{[t_{k'_v}, t_{k'_{v+1}}]} \overset{\triangle}{=} S(t_{k'_{v+1}}) \bigwedge S(t_{k'_{v+1}-1}) \bigwedge \cdots \bigwedge S(t_{k'_v}), \quad (7)$$

where $\bigwedge$ denotes the operation of generating a binary product of two binary matrices. Then, sensor/communication matrix sequence $\{S(t_k), k \in \aleph\}$ of (2) is said to be *sequentially complete* if an infinitely-long subsequence $\{k'_v : v \in \aleph\}$ exists such that $S_{[t_{k'_v}, t_{k'_{v+1}}]}$ is lower triangularly complete. In other words, sequence $\{S(t_k), k \in \aleph\}$ of (2) is sequentially complete if $S_{[t_k, \infty]}$ is lower triangularly complete for all t_k. Again, this regularity condition on infinite number of switchings is equivalent to the connectivity condition that, staring from any t_k, the union of all the future graphs has at least one globally reachable node.

It can be easily shown by counter examples that, if $S(t)$ is not sequentially complete, cooperative asymptotic stability cannot be achieved. It is also shown in [11] that, if the systems are all controllable, matrix $D(t)$ is non-negative, piecewise constant and row-stochastic and has the same sequential completeness property as matrix $S(t)$ by first mapping the systems into a canonical form and then properly choosing linear cooperative laws. This leads us to make the following assumptions without loss of any generality.

Assumption 1: Sequence $\{S(t_k), k \in \aleph\}$ is sequentially complete.

Assumption 2: Matrix $D(t)$ is non-negative, piecewise constant and row-stochastic; and, whenever its element $d_{ij}(t) \neq 0$, it is uniformly bounded from below by a positive constant. Furthermore, the sequence $\{D(t_k), k \in \aleph\}$ is sequentially complete if sequence $\{S(t_k), k \in \aleph\}$ is sequentially complete.

Since $S(t)$ is not known apriori in control design, neither is $D(t)$. The following theorem on asymptotic cooperative stability summarizes the results in [11]. Note that system (5) has equilibrium points of $c\mathbf{1}$ for all $c \in \Re$. Its Lyapunov stability can be shown using Lyapunov function $V(x) = x^T x$, but the cooperative stability problem of system (5) is not trivial or similar to the standard asymptotic stability problem.

Theorem 1: [11] Given a collection of controllable linear systems and their sensing/communication network, neighboring feedback control laws can be chosen to yield closed-loop linear system (5) such that assumption 2 holds. Furthermore, system (5) is asymptotically cooperatively stable if and only if assumption 1 holds.

Proof of theorem 1 is done by using the general matrix sequence solution of linear time-varying but piecewise-constant systems and by studying its convergence. Hence, the proof itself cannot be extended to nonlinear systems. Nonetheless, the result contained in theorem 1 has wider implications. Specifically, consider the two functions of x_i^2 and $(x_\mu - x_k)^2$ for any $i, \mu, k \in \{1, \cdots, n\}$. While the quantities are not Lyapunov functions, they can be viewed as Lyapunov function components and used to reveal or conclude stability properties. It follows from (5) that

$$\frac{d}{dt} x_i^2 = -2x_i^2 + 2\sum_{l=1}^{n} d_{il}(t) x_i x_l, \quad (8)$$

and that

$$\begin{aligned} &\frac{d}{dt}(x_\mu - x_k)^2 \\ &= -2(x_\mu - x_k)^2 + 2\sum_{l=1}^{n}(x_\mu - x_k)[d_{\mu l}(t) - d_{kl}(t)]x_l. \quad (9) \end{aligned}$$

According to theorem 1, x_i^2 are uniformly bounded, and $(x_\mu - x_k)^2$ converges to zero. A natural question arising is whether Lyapunov stability and asymptotic cooperative stability can be directly concluded from equalities (8) and (9). Though not trivial, the answer to this question should be affirmative since the equalities are equivalent to dynamics of system (5). A more interesting and important question is whether the same stability results can be concluded for nonlinear systems based on inequalities similar to the above equalities. The affirmative answer to this question is the subject of our next section.

IV. Comparison Theorem for Nonlinear Cooperative Systems

In what follows, a scalar function $\alpha(s)$ is said to be strictly monotone increasing (or decreasing) if $\alpha(s_1) < \alpha(s_2)$ (or $\alpha(s_1) > \alpha(s_2)$) for any $s_1 < s_2$; and the function $\alpha(s)$ is said to be strictly increasing (or decreasing) over an interval $[s_1, s_2]$ if $\alpha(s_1) < \alpha(s_2)$ (or $\alpha(s_1) > \alpha(s_2)$) and if, for any $[s'_1, s'_2] \subset [s_1, s_2]$, $\alpha(s'_1) \le \alpha(s'_2)$ (or $\alpha(s'_1) \ge \alpha(s'_2)$). Then, we have the following theorem on stability and cooperative stability of nonlinear cooperative systems. Comparing (8) and (11) as well as (9) and (12), we know that, as a new addition to the comparison theory [17], the theorem can be referred to as comparison theorem for cooperative control.

Definition: Scalar function $\mathcal{E} : \Re \to \Re_+$ is said to be a Lyapunov function component if $\mathcal{E}(s)$ is differentiable and positive definite with respect to s and it is also strictly monotone increasing and radially-unbounded with respect to $|s|$.

Theorem 2: Consider the closed-loop networked-connected nonlinear system

$$\dot{y} = \mathcal{F}(y, D(t)), \quad y \in \Re^n, \quad (10)$$

which corresponds to system (10) under neighboring feedback control laws (3) or (4). Suppose that, through the cooperative control design, assumption 2 holds and the following two inequalities are satisfied along trajectories of system (10):

(i) For Lyapunov function components $V_i(\cdot)$ with $i = 1, \cdots, n$,

$$\begin{aligned} \frac{d}{dt} V_i(y_i) \;\; &\le \;\; -\xi_i(|y_i|) - 2|\eta_i(y_i)|^2 \\ &\quad + 2\sum_{l=1}^{n} d_{il}(t)\eta_i(y_i)\eta_i(y_l), \quad (11) \end{aligned}$$

where $\xi_i(\cdot) \geq 0$, and $\eta_i(\cdot)$ are function with $\eta_i(0) = 0$.

(ii) For any pair of indices (μ, k) and for Lyapunov function component $L_{\mu,k}(\cdot)$,

$$\frac{d}{dt}L_{\mu,k}(y_\mu - y_k)$$
$$\leq -\xi'_{\mu,k}(|y_\mu - y_k|) - 2|\eta'_{\mu,k}(y_\mu) - \eta'_{\mu,k}(y_k)|^2 + 2\sum_{l=1}^{n}$$
$$[\eta'_{\mu,k}(y_\mu) - \eta'_{\mu,k}(y_k)][d_{\mu l}(t) - d_{kl}(t)]\eta'_{\mu,k}(y_l), \quad (12)$$

where $\xi'_{\mu,k}(\cdot) \geq 0$, and $\eta'_{\mu,k}(\cdot)$ are strictly monotone increasing function satisfying $\eta'_{\mu,k}(0) = 0$.

Then, system (10) is Lyapunov stable and, under assumption 1, it is also asymptotically cooperatively stable.

Proof of theorem 2 is included in the appendix. In essence, inequality (11) renders Lyapunov stability and, under assumption 1, inequality (12) ensures asymptotic cooperative stability. Several comments are worth noting here. First, local stability results can be concluded using theorem 2 if inequalities (11) and (12) are satisfied locally in a compact set containing the origin. Second, theorem 2 includes theorem 1 as the special case of $\eta_i(s) = \eta'_{\mu,k}(s) = s$, $\xi_i(\cdot) = \xi'_{\mu,k}(\cdot) = 0$, and $V_i(s) = L_{\mu,k}(s) = s^2$. Third, while Lyapunov function components $V_i(\cdot)$ and $L_{\mu,k}(\cdot)$ can always be chosen, it is usually too difficult to find or assume a differentiable Lyapunov function because of nonlinear dynamics and of time-varying sensing/communication topology whose changes are sequentially complete but otherwise unknown apriori. Fourth, despite the unpredictable changes in $S(t)$ and hence in $D(t)$, the two inequalities in theorem 2 can be checked, and they can also be used to guide a cooperative control design. Fifth, no assumption is made about convexity of the solutions of nonlinear systems, and stability proof is done using the new and topology-dependent Lyapunov argument. The right hand sides of (11) and (12) are not negative semi-definite in general, nonetheless asymptotic cooperative stability is concluded in general for the overall system. And, there is no need to use nonsmooth analysis. Sixth, the existing comparison theorem on vector differential inequalities (theorem 1.5.1 on page 22 of [17]) requires the so-called quasi-monotone property, such a property does not hold for cooperative systems, nor is it required by theorem 2. Finally, it has been shown in theorem 1 that $S(t)$ being sequentially complete is necessary for concluding cooperative stability, while the two inequalities in theorem 2 can be relaxed. One such relaxation is given by the following corollary, and its proof is analogous and hence is left to the readers.

Corollary 1: Theorem 2 holds if inequality (11) is replaced by either

$$\frac{d}{dt}V_i(y_i) \leq -\xi_i(|y_i|) + 2\eta_i(y_i)\sum_{l=1}^{n} d_{il}(t)[\beta_{i,l}(y_l) - \beta_{i,l}(y_i)], \quad (13)$$

or

$$\frac{d}{dt}V_i(y_i) \leq -\xi_i(|y_i|) + 2\eta_i(y_i)\sum_{l=1}^{n} d_{il}(t)\beta_{i,l}(y_l - y_i), \quad (14)$$

and/or if inequality (12) is replaced by either

$$\frac{d}{dt}L_{\mu,k}(y_\mu - y_k)$$
$$\leq -\xi'_{\mu,k}(|y_\mu - y_k|) + 2[\eta'_{\mu,k}(y_\mu) - \eta'_{\mu,k}(y_k)]\sum_{l=1}^{n} d_{\mu l}(t)$$
$$\times[\beta'_{\mu,k,l}(y_l) - \beta'_{\mu,k,l}(y_\mu)] - 2[\eta'_{\mu,k}(y_\mu) - \eta'_{\mu,k}(y_k)]$$
$$\times\sum_{l=1}^{n} d_{kl}(t)[\beta''_{\mu,k,l}(y_l) - \beta''_{\mu,k,l}(y_k)], \quad (15)$$

or

$$\frac{d}{dt}L_{\mu,k}(y_\mu - y_k)$$
$$\leq \quad -\xi'_{\mu,k}(|y_\mu - y_k|) + 2\eta'_{\mu,k}(y_\mu - y_k)\sum_{l=1}^{n}$$
$$[d_{\mu l}(t)\beta'_{\mu,k,l}(y_l - y_\mu) - d_{kl}(t)\beta''_{\mu,k,l}(y_l - y_k)], \quad (16)$$

where $\beta_{i,l}(\cdot)$, $\beta'_{\mu,k,l}(\cdot)$, and $\beta''_{\mu,k,l}(\cdot)$ are also scalar functions that are strictly monotone increasing functions and pass through the origin.

Compared to those in theorem 2, the inequalities in corollary 1 are more general and also easier to be used. In particular, corollary 1 includes theorem 2 as the special case that $\beta_{i,l}(s) = \eta_i(s)$ and $\beta'_{\mu,k,l}(s) = \beta''_{\mu,k,l}(s) = \eta'_{\mu,k}(s)$. Using theorem 2 and corollary 1, systematical designs of cooperative control can be done for several classes of nonlinear systems, but the details are beyond the scope of this paper. In what follows, several examples are included to illustrate applications of theorem 2 and corollary 1, and matrix sequence of $S(t)$ is assumed to be sequentially complete. Due to space limitation, simulation results of these examples are not included but will be presented at the conference.

Example 1: Consider the following version of Kuramoto model [15]:

$$\dot{\theta}_\mu = \sum_{j=1}^{q} e_{\mu j}(t)\sin(\theta_j - \theta_\mu),$$

where $\mu = 1, \cdots, q$, and

$$e_{\mu j}(t) = \frac{s_{\mu j}(t)}{\sum_{i=1}^{q} s_{\mu i}(t)}. \quad (17)$$

It is easy to check that matrix $D(t) = E(t) = [e_{\mu j}] \in \Re^{q\times q}$ satisfies assumption 2. In this case, it is easy to establish inequalities (14) and (16) using $V_i(s) = L_{\mu,k}(s) = 1 - \cos(s)$. Hence, the overall system is locally asymptotically cooperatively stable.

Similarly, global asymptotic cooperative stability can be concluded using $V_i(s) = L_{\mu,k}(s) = s^2$ for the more general class of systems in the form

$$\dot{\theta}_\mu = \sum_{j=1}^{q} e_{\mu j}(t)\gamma_{\mu j}(\theta_j - \theta_\mu), \quad (18)$$

where $\gamma_{\mu j}(\cdot)$ are strictly monotone increasing and $\gamma_{\mu j}(0) = 0$. Note that equation (18) can also be viewed as a nonlinear version of Vicsek's model [16]. △

Example 2: As another generalization of Vicsek's model [16], consider the case that 2-D heading of a particle is adjusted according to the average of particles' velocity projections along one of the primary axis. Then, dynamic equations are

$$\dot{\theta}_\mu = -\tan(\theta_\mu) + \frac{1}{\cos(\theta_\mu)} \sum_{j=1}^{q} e_{\mu j}(t) \sin(\theta_j),$$

where $\mu = 1, \cdots, q$, and $e_{\mu j}(t)$ is given by (17). It is straightforward to verify using $V_i(s) = L_{\mu,k}(s) = s^2$ that inequalities (11) and (12) hold locally. Thus, the overall system is asymptotically cooperatively stable in the sense that $\sin(\theta_j) = c$ for some $c \in \Re$ and for all initial conditions.

More generally, inequalities (13) and (15) can be established and (either local or global) asymptotical cooperative stability can be claimed for the following class of systems

$$\dot{\theta}_\mu = -\gamma(\theta_\mu) + \sum_{j=1}^{q} e_{\mu j}(t) \gamma(\theta_j),$$

where $\gamma(\cdot)$ is any (either locally or globally) strictly monotone increasing function with $\gamma(0) = 0$. △

Example 3: Consider the following heterogeneous systems:

$$\begin{cases} \dot{x}_i = u_i, \quad i = 1, \cdots, q-1, \\ \dot{x}_{q1} = \gamma_1(x_{q2}) - \gamma_1(x_{q1}), \\ \dot{x}_{q2} = \gamma_2(x_{q3}) - \gamma_2(x_{q2}), \\ \dot{x}_{q3} = \gamma_3(x_{q1}) - \gamma_3(x_{q3}) + u_q, \end{cases}$$

where $u_j \in \Re$ for all j, and $\gamma_i(\cdot)$ are strictly monotone increasing functions with $\gamma_i(0) = 0$. For this group of systems, nonlinear output-feedback cooperative control can be chosen to be

$$u_\mu = \sum_{j=1}^{q} e_{\mu j}(t) [\gamma_3(y_j) - \gamma_3(y_\mu)],$$

where $\mu = 1, \cdots, q$, $e_{\mu j}(t)$ is given by (17), $y_i = x_i$ for $i = 1, \cdots, q-1$, and $y_q = x_{q1}$. The matrix corresponding to the overall system is

$$D(t) = \begin{bmatrix} e_{11}(t) & \cdots & e_{1(q-1)} & e_{1q} & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ e_{(q-1)1}(t) & \cdots & e_{(q-1)(q-1)} & e_{(q-1)q} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ e_{q1}(t) & \cdots & e_{q(q-1)} & e_{qq} & 0 & 0 \end{bmatrix},$$

and it is shown in [11] that the above matrix satisfies assumption 2. Inequalities (14) and (16) can be established using $V_i(s) = L_{\mu,k}(s) = s^2$, according to which asymptotical cooperative stability can be shown.

Analogously, for the group of systems

$$\begin{cases} \dot{x}_i = u_i, \quad i = 1, \cdots, q-1, \\ \dot{x}_{q1} = \gamma_1(x_{q2} - x_{q1}), \\ \dot{x}_{q2} = \gamma_2(x_{q3} - x_{q2}), \\ \dot{x}_{q3} = \gamma_3(x_{q1} - x_{q3}) + u_q, \end{cases}$$

nonlinear cooperative control can be chosen to be

$$u_\mu = \sum_{j=1}^{q} e_{\mu j}(t) \gamma_3(y_j - y_\mu).$$

The above control is also output-feedback and only requires relative measurements. Upon verifying that inequalities (13) and (15) hold (under many choices of $V_i(s)$ and $L_{\mu,k}(s)$), asymptotical cooperative stability can be concluded.

It is apparent that cooperative stability of the above systems with $\gamma_1(s) = s^3$ cannot be analyzed using linearization around the origin and that the systems with $\gamma_2(s) = s^{1/3}$ do not have a linearized version in any neighborhood of the origin. △

V. Conclusion

In this paper, asymptotic cooperative stability of heterogeneous and nonlinear systems is investigated. A new comparison theorem is presented to conclude cooperative stability, and its proof fully explores properties of both system dynamics in terms of their Lyapunov function components and their associated sensing/communication network. It is shown by illustrative examples that the proposed new stability result can easily be used to establish cooperative stability or to carry out cooperative control design.

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VI. Appendix

Proof of theorem 2: Let $\Omega = \{1, \cdots, n\}$ be the set of indices on state variables. We can define the following three subsets of indices: at any instant time t,

$$\left\{ \begin{array}{l} \Omega_{max}(t) = \{i \in \Omega \,:\, y_i = y_{max}\} \\ \Omega_{mid}(t) = \{i \in \Omega \,:\, y_{min} < y_i < y_{max}\} \\ \Omega_{min}(t) = \{i \in \Omega \,:\, y_i = y_{min}\} \end{array} \right. ,$$

where

$$y_{max}(t) = \max_{j \in \Omega} y_j(t), \quad y_{min}(t) = \min_{j \in \Omega} y_j(t).$$

It is apparent that, unless $y_i = y_j$ for all i and j, $y_{min} < y_{max}$ and set Ω is partitioned into the three mutually disjoint subsets of Ω_{max}, Ω_{mid} and Ω_{min}. Defining another index set

$$\Omega_{mag}(t) = \{i \in \Omega \,:\, |y_i| = y_{mag}\}\,, \quad y_{mag}(t) = \max_{j \in \Omega} |y_j(t)|,$$

we know that, if $i \in \Omega_{mag}(t)$, either $i \in \Omega_{max}(t)$ or $i \in \Omega_{min}(t)$ but not both unless y_j are identical for all $j \in \Omega$. For each state variable y_i, we can define the set of its neighbors as

$$\Theta_i(t) = \{j \in \Omega \,:\, j \neq i \quad \text{and} \quad d_{ij} > 0\}\,.$$

In addition, we can define the set of its neighbors with distinct values as

$$\Theta_i'(t) = \{j \in \Omega \,:\, j \neq i, \quad d_{ij} > 0, \quad \text{and} \quad y_j \neq y_i\}\,.$$

Then, for any $i \in \Omega_{mag}$, define

$$\Theta_{i,mag}'(t) = \{j \in \Omega : \, d_{ij} > 0 \text{ and } j \notin \Omega_{mag}\}\,.$$

Finally, let us define the maximum relative distance as

$$\delta_{max}(t) = \max_{\mu, k \in \Omega} |y_\mu(t) - y_k(t)|.$$

It is obvious that $\delta_{max}(t) = y_{max}(t) - y_{min}(t)$.

The proof is completed by establishing the following six claims. The first claim deals with Lyapunov stability, and the rest are about convergence and asymptotic cooperative stability. It is worth noting that, if $y_{min} = y_{max}$ at some instant of time t, $\Omega = \Omega_{max} = \Omega_{min}$ while Ω_{mid} is empty and that, by claim 3, the system is already asymptotically cooperative stable. Thus, in the analysis after claim 3, we can assume without loss of any generality that $y_{min} < y_{max}$.

Claim 1: Lyapunov stability. To show Lyapunov stability, it is sufficient to demonstrate that the maximum magnitude of all the state variables does not increase over time. Suppose without loss of any generality that, at time t, $i^* \in \Omega_{mag}(t)$. It follows from (11), from the definition of $\Omega_{mag}(t)$, from $\eta_i(\cdot)$ being strictly monotone increasing and passing through the origin, and from $D(t)$ being non-negative and row stochastic that

$$\begin{aligned} & \frac{d}{dt} V_{i^*}(y_{i^*}) \\ \leq \; & -2|\eta_{i^*}(y_{i^*})|^2 + 2\sum_{l=1}^{n} d_{i^*l}(t) |\eta_{i^*}(y_{i^*})| \cdot |\eta_{i^*}(y_l)| \\ \leq \; & 0, \end{aligned} \qquad (19)$$

from which, by the monotone-increasing property of $V_i(\cdot)$ with respect to magnitude of its argument, $y_{mag}(t)$ is known to be non-increasing over time.

Claim 2: $y_{mag}(t)$ is strictly decreasing over any time interval if, for all $i \in \Omega_{mag}(t)$, the corresponding index sets $\Theta'_{i,mag}(t)$ do not remain empty for the entire interval. The claim is established by showing that, if $|y_i| = y_{mag}$ and $\Theta'_{i,mag}(t)$ is nonempty, $|y_i|$ is strictly decreasing at time t. By definition, we know from $\Theta'_{i,mag}(t)$ being non-empty that there exists some $j \neq i$ such that $d_{ij} > 0$ and $j \notin \Omega_{mag}$. In this case, (19) becomes a strict inequality and hence $|y_i|$ is strictly decreasing. Furthermore, $y_{mag}(t)$ is strictly monotone decreasing if $\Theta'_{i,mag}(t)$ are nonempty for all t and for all $i \in \Omega_{mag}$. In addition, since $d_{ij}(t)$ is uniformly bounded away from zero whenever $d_{ij}(t) \neq 0$, it follows from (19) that, if $y_{mag}(t)$ is decreasing, the decreasing is uniformly with respect to time.

Claim 3: Maximum distance $\delta_{max}(t)$ is non-increasing. It follows from (12) and from $D(t)$ being row-stochastic that

$$\begin{aligned} & \frac{d}{dt} L_{\mu,k}(y_\mu - y_k) \\ \leq \; & -\xi'_{\mu,k}(|y_\mu - y_k|) - 2|\eta'_{\mu,k}(y_\mu) - \eta'_{\mu,k}(y_k)|^2 + 2\sum_{l=1}^{n} [\\ & \eta'_{\mu,k}(y_\mu) - \eta'_{\mu,k}(y_k)][d_{\mu l}(t) - d_{kl}(t)][\eta'_{\mu,k}(y_l) - \eta'_{\mu,k}(y_k)] \\ = \; & -\xi'_{\mu,k}(|y_\mu - y_k|) - 2|\eta'_{\mu,k}(y_\mu) - \eta'_{\mu,k}(y_k)|^2 + 2[\eta'_{\mu,k}(y_\mu) \\ & -\eta'_{\mu,k}(y_k)] \sum_{l=1}^{n} d_{\mu l}(t)[\eta'_{\mu,k}(y_l) - \eta'_{\mu,k}(y_k)] - 2[\eta'_{\mu,k}(y_\mu) \\ & -\eta'_{\mu,k}(y_k)] \sum_{l=1}^{n} d_{kl}(t)[\eta'_{\mu,k}(y_l) - \eta'_{\mu,k}(y_k)]. \end{aligned} \qquad (20)$$

Recall the property of $\eta'_{\mu,k}(\cdot)$ and note that, for any $\mu^* \in \Omega_{max}$ and $k^* \in \Omega_{min}$,

$$\eta'_{\mu^*,k^*}(y_{\mu^*}) - \eta'_{\mu^*,k^*}(y_{k^*}) = \max_{\mu,k \in \Omega} |\eta'_{\mu^*,k^*}(y_\mu) - \eta'_{\mu^*,k^*}(y_k)| \geq 0,$$

and that, for all $l \in \Omega$,

$$\eta'_{\mu^*,k^*}(y_l) - \eta'_{\mu^*,k^*}(y_{k^*}) \geq 0.$$

Since matrix $D(t)$ is row stochastic,

$$\begin{aligned} 0 \;\leq\; & [\eta'_{\mu^*,k^*}(y_{\mu^*}) - \eta'_{\mu^*,k^*}(y_{k^*})] \sum_{l=1}^{n} d_{\mu^* l}(t)[\eta'_{\mu^*,k^*}(y_l) \\ & -\eta'_{\mu^*,k^*}(y_{k^*})] \\ \leq\; & |\eta'_{\mu^*,k^*}(y_{\mu^*}) - \eta'_{\mu^*,k^*}(y_{k^*})|^2. \end{aligned}$$

Therefore, we know that

$$-2[\eta'_{\mu^*,k^*}(y_{\mu^*}) - \eta'_{\mu^*,k^*}(y_{k^*})] \times \sum_{l=1}^{n} d_{k^*l}(t)[\eta'_{\mu^*,k^*}(y_l) - \eta'_{\mu^*,k^*}(y_{k^*})] \leq 0, \quad (21)$$

and

$$\begin{aligned} & 2[\eta'_{\mu^*,k^*}(y_{\mu^*}) - \eta'_{\mu^*,k^*}(y_{k^*})] \\ & \quad \times \sum_{l=1}^{n} d_{\mu^*l}(t)[\eta'_{\mu^*,k^*}(y_l) - \eta'_{\mu^*,k^*}(y_{k^*})] \\ \leq \quad & 2|\eta'_{\mu^*,k^*}(y_{\mu^*}) - \eta'_{\mu^*,k^*}(y_{k^*})|^2. \end{aligned} \quad (22)$$

Substituting the above inequalities into (20) yields

$$\frac{d}{dt} L_{\mu^*,k^*}(y_{\mu^*} - y_{k^*}) \leq -\xi'_{\mu^*,k^*}(|y_{\mu^*} - y_{k^*}|) \leq 0,$$

from which $\delta_{max}(t)$ being non-increasing can be concluded.

Claim 4: δ_{max} is strictly monotone decreasing as long as $D(t)$ is lower triangularly complete, and the decreasing is uniformly with respect to time. It follows from the derivations in claim 3 that we need only show that at least one of inequalities (21) and (22) is a strict inequality. To prove this proposition by contradiction, let us assume that both (21) and (22) be equalities. It follows that, unless $y_{min} = y_{max}$,

$$\begin{aligned} & \sum_{l=1}^{n} d_{k^*l}(t)[\eta'_{\mu,k}(y_l) - \eta'_{\mu,k}(y_{k^*})] = 0 \\ \Longrightarrow \quad & d_{k^*l}(t) = 0 \text{ if } l \in \Omega_{mid} \cup \Omega_{max} \text{ and } k^* \in \Omega_{min}, \end{aligned}$$

and that

$$\begin{aligned} & \sum_{l=1}^{n} d_{\mu^*l}(t)[\eta'_{\mu,k}(y_l) - \eta'_{\mu,k}(y_{k^*})] = \eta'_{\mu,k}(y_{\mu^*}) - \eta'_{\mu,k}(y_{k^*}) \\ \Longrightarrow \; & d_{\mu^*l}(t) = 0 \text{ if } l \in \Omega_{mid} \cup \Omega_{min} \text{ and } \mu^* \in \Omega_{max}. \end{aligned}$$

Recall that, as long as $y_{min} < y_{max}$, index sets Ω_{min}, Ω_{mid} and Ω_{max} are mutually exclusive, and $\Omega_{min} \cup \Omega_{mid} \cup \Omega_{max} = \Omega$. This means that, unless $y_{min} = y_{max}$, there is permutation matrix $P(t)$ under which

$$P(t)D(t)P^T(t) = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \triangleq E(t), \quad (23)$$

where E_{ii} are square blocks, row indices of $E_{11} \in \Re^{n_{min} \times n_{min}}$ correspond to those in Ω_{min}, row indices of $E_{22} \in \Re^{n_{max} \times n_{max}}$ correspond to those in Ω_{max}, and row indices of $E_{33} \in \Re^{n_{mid} \times n_{mid}}$ correspond to those in Ω_{mid}. Note that $n_{min}(t) > 0$ and $n_{max}(t) > 0$ and that, if $n_{mid} = 0$, E_{33} is empty and matrix E becomes 2-block diagonal. Clearly, the structure of matrix $E(t)$ contradicts with the knowledge that $D(t)$ is lower triangularly complete. Hence, we know that at least one of inequalities (21) and (22) must be a strict inequality and hence δ_{max} is strictly monotone decreasing. Again, the decrease is uniform with respect to time since, whenever $d_{ij}(t) \neq 0$, $d_{ij}(t)$ is uniformly bounded away from zero.

Claim 5: δ_{max} is strictly decreasing over an infinite sequence of finite time intervals if $D(t)$ over time may not be lower triangularly complete but is sequentially complete. Assume that $\delta_{max}(t_0) > 0$. Then, the claim is established by showing that, given any time instant t_1, there exists a finite duration Δt such that

$$\delta_{max}(t_1 + \Delta t) < \delta_{max}(t_1), \quad (24)$$

where $\Delta t > 0$ depends upon changes of $D(t)$ over $[t_1, t_2)$ and the value of $\delta_{max}(t_1)$.

Consider index sets $\Omega_{max}(t_1)$ and $\Omega_{min}(t_1)$. It follows that $\delta_{max}(t_1) = y_{\mu^*}(t_1) - y_{k^*}(t_1)$, where $\mu^* \in \Omega_{max}(t_1)$ and $k^* \in \Omega_{min}(t_1)$. Evolution of $\delta_{max}(t)$ after $t = t_1$ has two possibilities. The first case is that, for every $\mu^* \in \Omega_{max}(t_1)$, there exists $k^* \in \Omega_{min}(t_1)$ such that index pair $\{\mu^*, k^*\}$ belongs to the same lower-triangularly-complete block in the lower triangular canonical form of $D(t_1)$. In this case, it follows from claims 4 and 3 that $\delta_{max}(t)$ is strictly decreasing at time $t = t_1$ and non-increasing afterwards. Therefore, we know that, for any $\Delta t > 0$, inequality (24) holds.

The second and more general case is that, at time $t = t_1$ as well as in a finite interval afterwards, some of the indices in $\Omega_{max}(t)$ correspond to different diagonal block in the lower triangular canonical form of $D(t)$ than those for all the indices in $\Omega_{min}(t)$. In this case, claim 4 is no longer applicable, while claim 3 states that $\delta_{max}(t)$ is non-increasing for all $t \geq t_1$. Nonetheless, the sequence of matrix $D(t)$ over time is sequentially complete and hence we know that, for any index $i \in \Omega_{mag}$, either $i \in \Omega_{max}(t)$ or $i \in \Omega_{min}(t)$, and set $\Theta'_{i,mag}$ cannot be nonempty except over some sub-intervals. It follows from claims 1 and 2 that $y_{mag}(t)$ is non-increasing over time and is also strictly monotone decreasing over all (possibly intermittent) time intervals with nonempty $\Theta'_{i,mag}$ and hence there exists a finite length Δt such that

$$y_{mag}(t_1 + \Delta t) < 0.5[y_{max}(t_1) - y_{min}(t_1)]. \quad (25)$$

Recalling $y_{mag}(t) = \max\{|y_{max}(t)|, |y_{min}(t)|\}$, we know from (25) that, for any $\mu \in \Omega_{max}(t_1 + \Delta t)$ and $k \in \Omega_{min}(t_1 + \delta t)$,

$$\begin{aligned} \delta_{max}(t_1 + \Delta t) &= y_{max}(t_1 + \Delta t) - y_{min}(t_1 + \Delta t) \\ &\leq 2y_{mag}(t_1 + \Delta t) \\ &< y_{max}(t_1) - y_{min}(t_1), \end{aligned}$$

which establishes inequality (24). In essence, while $\max\{|y_{max}(t)|, |y_{min}(t)|\}$ decreases, the value of $[y_{max}(t) - y_{min}(t)]$ could remain unchanged but only temporarily (and at latest till the time instant that $y_{max}(t) = -y_{min}(t)$), and afterwards $\delta_{max}(t)$ must decrease as $y_{mag}(t)$ does. Since t_1 is arbitrary, strictly decreasing of $\delta_{max}(t)$ over an infinite sequence of finite intervals is shown.

Claim 6: Asymptotic cooperative stability if $D(t)$ over time is sequentially complete. It is clear that claim 5 includes claim 4 as a special case. We know from claims 3 and 5 that δ_{max} is asymptotically convergent to zero. Hence, asymptotic cooperative stability is concluded. □